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NONEXISTENCE OF GLOBAL SOLUTIONS OF SOME NONLINEAR SPACE-NONLOCAL EVOLUTION EQUATIONS ON THE HEISENBERG GROUP

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Abstract. This article presents necessary conditions for the existence of weak solutions of the following space-nonlocal evolution equations on $\mathbb{H} \times (0, +\infty)$, where $\mathbb{H}$ is the Heisenberg group:

$$\frac{\partial^2 u}{\partial t^2} + (-\Delta_{\mathbb{H}})^{\alpha/2}|u|^m = |u|^p,$$
$$\frac{\partial u}{\partial t} + (-\Delta_{\mathbb{H}})^{\alpha/2}|u|^m = |u|^p,$$
$$\frac{\partial^2 u}{\partial t^2} + (-\Delta_{\mathbb{H}})^{\alpha/2}|u|^m + \frac{\partial u}{\partial t} = |u|^p,$$

$p \in \mathbb{R}$, $p > 1$, $m \in \mathbb{N}$. Moreover, the life span for each equation is estimated under some suitable conditions. Our method of proof is based on the test function method.

1. Introduction and preliminaries

In this article we are concerned with the nonexistence of global solutions of nonlinear wave equations, of nonlinear wave equations with linear damping, and of nonlinear parabolic equations with nonlocal diffusion posed on the Heisenberg group. We start with the wave equation

$$\frac{\partial^2 u}{\partial t^2} + (-\Delta_{\mathbb{H}})^{\alpha/2}|u|^m = |u|^p, \tag{1.1}$$

where $p \in \mathbb{R}$, $p > 1$, $m \in \mathbb{N}$, posed in $\mathbb{R}^{2N+1} \times \mathbb{R}_+ = \mathbb{R}_+^{2N+1,1}$, supplemented with the initial data

$$u(\eta, 0) = u_0(\eta), \quad \frac{\partial u}{\partial t}(\eta, 0) = u_1(\eta), \quad \eta = (x, y, \tau).$$

The operator $(-\Delta_{\mathbb{H}})^{\alpha/2}$ $(0 < \alpha < 2)$ accounts for anomalous diffusion (see below for the definition). Following the lines of the paper of Véron and Pohozaev [22] and using a variant of Cordoba-Cordoba's inequality [3] for the Heisenberg group proved in this paper, we find an exponent for (1.1) similar to Kato's exponent [9] but with the improvement on the data given in [22].

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In a second case, we prove a Fujita-Galaktionov type result for the heat equation

$$\frac{\partial u}{\partial t} + (-\Delta_{\mathbb{H}})^{\alpha/2}|u|^m = |u|^p, \tag{1.2}$$

where $p \in \mathbb{R}, p > 1$, posed in $\mathbb{R}^{2N+1,1}_+$, supplemented with the initial data $u(\eta,0) = u_0(\eta)$.

Next, we present the critical exponent for the wave equation with linear damping

$$\frac{\partial^2 u}{\partial t^2} + (-\Delta_{\mathbb{H}})^{\alpha/2}|u|^m + \frac{\partial u}{\partial t} = |u|^p, \tag{1.3}$$

where $p \in \mathbb{R}$, $p > 1$, posed in $\mathbb{R}^{2N+1,1}_+$, supplemented with the initial data $u(\eta,0) = u_0(\eta)$, $\frac{\partial u}{\partial t}(\eta,0) = u_1(\eta)$.

For the reader's convenience, let us briefly recall the definition and the basic properties of the Heisenberg group.

Heisenberg group. The Heisenberg group $\mathbb{H}$, whose points will be denoted by $\eta = (x, y, \tau)$, is the Lie group $(\mathbb{R}^{2N+1}, \circ)$ with the non-commutative group operation $\circ$ defined by

$$\eta \circ \tilde{\eta} = (x + \tilde{x}, y + \tilde{y}, \tau + \tilde{\tau} + 2(\langle x, \tilde{y}\rangle - \langle \tilde{x}, y\rangle)),$$

where $\langle \cdot, \cdot\rangle$ is the usual inner product in $\mathbb{R}^N$. The Laplacian $\Delta_{\mathbb{H}}$ over $\mathbb{H}$ is obtained from the vector fields $X_i = \frac{\partial}{\partial x_i} + 2y_i\frac{\partial}{\partial \tau}$ and $Y_i = \frac{\partial}{\partial y_i} - 2x_i\frac{\partial}{\partial \tau}$, as

$$\Delta_{\mathbb{H}} = \sum_{i=1}^{N}(X_i^2 + Y_i^2). \tag{1.4}$$

Observe that the vector field $T = \frac{\partial}{\partial \tau}$ does not appear in (1.4). This fact makes us presume a "loss of derivative" in the variable τ. The compensation comes from the relation

$$[X_i, Y_j] = -4T, \quad i, j \in \{1, 2, \cdots, N\}. \tag{1.5}$$

The relation (1.5) proves that $\mathbb{H}$ is a nilpotent Lie group of order 2. Incidently, (1.5) constitutes an abstract version of the canonical relations of commutation of Heisenberg between momentum and positions. Explicit computation gives the expression

$$\Delta_{\mathbb{H}} = \sum_{i=1}^{N}\Big(\frac{\partial^2}{\partial x_i^2} + \frac{\partial^2}{\partial y_i^2} + 4y_i\frac{\partial^2}{\partial x_i\partial \tau} - 4x_i\frac{\partial^2}{\partial y_i\partial \tau} + 4(x_i^2 + y_i^2)\frac{\partial^2}{\partial \tau^2}\Big). \tag{1.6}$$

A natural group of dilatations on $\mathbb{H}$ is given by

$$\delta_\lambda(\eta) = (\lambda x, \lambda y, \lambda^2\tau), \quad \lambda > 0,$$

whose Jacobian determinant is λ^Q, where

$$Q = 2N + 2 \tag{1.7}$$

is the homogeneous dimension of $\mathbb{H}$.

The natural distance from η to the origin is introduced by Folland and Stein, see [5]

$$|\eta|_{\mathbb{H}} = \Big(\tau^2 + \Big(\sum_{i=1}^{N}(x_i^2 + y_i^2)\Big)^2\Big)^{1/4}. \tag{1.8}$$

Fractional powers of sub-elliptic Laplacians. Here, we recall a result on fractional powers of sub-Laplacian in the Heisenberg group taken from [6]. Let $\mathcal{N}(t,x)$ be the fundamental solution of $\Delta_{\mathbb{H}} + \frac{\partial}{\partial t}$. For all $0 < \beta < 4$, the integral

$$R_\beta(x) = \frac{1}{\Gamma(\frac{\beta}{2})} \int_0^{+\infty} t^{\frac{\beta}{2}-1} \mathcal{N}(t,x)\, dt$$

converges absolutely for $x \neq 0$. If $\beta < 0$, $\beta \notin \{0,-2,-4,\dots\}$, then

$$\tilde{R}_\beta(x) = \frac{\frac{\beta}{2}}{\Gamma(\frac{\beta}{2})} \int_0^{+\infty} t^{\frac{\beta}{2}-1} \mathcal{N}(t,x)\, dt$$

defines a smooth function in $\mathbb{H} \setminus \{0\}$, since $t \mapsto \mathcal{N}(t,x)$ vanishes of infinite order as $t \to 0$ if $x \neq 0$. In addition, $\tilde{R}_\beta$ is positive and $\mathbb{H}$-homogeneous of degree $\beta - 4$.

Theorem 1.1. *For every $v \in \mathcal{S}(\mathbb{H})$, $(-\Delta_{\mathbb{H}})^s v \in L^2(\mathbb{H})$ and*

$$\begin{aligned}(-\Delta_{\mathbb{H}})^s v(x) &= \int_{\mathbb{H}} (v(x \circ y) - v(x) - \chi(y)\langle \nabla_{\mathbb{H}} v(x), y\rangle) \tilde{R}_{-2s}(y)\, dy \\ &= P.V \int_{\mathbb{H}} (v(y) - v(x)) \tilde{R}_{-2s}(y^{-1} \circ x)\, dy,\end{aligned}$$

where χ is the characteristic function of the unit ball $B_\rho(0,1)$, ($\rho(x) = R_{2-\alpha}^{\frac{-1}{2+\alpha}}(x)$, $0 < \alpha < 2$, ρ is an $\mathbb{H}$-homogeneous norm in $\mathbb{H}$ smooth outside the origin).

Before we present our results, let us dwell a while some literature concerning non-existence or blowing-up solutions to wave equations and parabolic equations posed in the Heisenberg group. A lot has been said on the non-existence or blowing-up solutions to the wave equation (1.1), with $\alpha = 2$ and $p = 1$, with a final result in [23], for the parabolic equation [7] (for the pioneering paper) and [1] (for the alternative proof that will be used here), for the wave equation with a linear damping [10, 21, 26] where the critical has been decided mainly in [21], when the equations are posed in the Euclidian space.

Concerning (1.1), (1.2) and (1.3) with $\alpha = 2$ and $p = 1$, posed on the Heisenberg group, a few papers appeared; we may cite [4, 8, 15, 16, 17, 22, 24, 25, 26]. As far as we know, no blowing-up solutions or non-existence results for (1.1), (1.2) and (1.3) have been published.

Evolution equation with fractional power of the Laplacian posed on the Heisenberg space are mentioned, for example, in [11, 12, 13, 19, 20].

Our method of proof relies on a method due to Baras and Pierre [1]; it had been remained dormant until Zhang [24, 25, 26] revived it. Later, this method has been successfully applied in a great number of situations by Mitidieri and Pohozaev [14].

2. Main results

Proposition 2.1. *Consider a convex function $F \in C^2(\mathbb{R})$. Assume that $\varphi \in C_0^\infty(\mathbb{R}^{2N+1})$. Then*

$$F'(\varphi)(-\Delta_{\mathbb{H}})^{\alpha/2}\varphi \geq (-\Delta_{\mathbb{H}})^{\alpha/2} F(\varphi) \tag{2.1}$$

holds point-wise. In particular, if $F(0) = 0$ and $\varphi \in C_0^\infty(\mathbb{R}^{2N+1})$, then

$$\int_{R^{2N+1}} F'(\varphi)(-\Delta_{\mathbb{H}})^{\alpha/2}\varphi\, d\eta \geq 0. \tag{2.2}$$

Proof. We have $F(\theta)-F(\varrho)-F'(\varrho)(\theta-\varrho)\geq 0$ for all θ,ϱ. We substitute then $\theta=\varphi(x-y)$, $\varrho=\varphi(x)$ and integrate against $\tilde{R}_\alpha(y^{-1}x)\,dy$ (recall that $\tilde{R}_\alpha(y^{-1}x)\geq 0$).

Let us mention that hereafter we will use inequality (2.1) for $F(\varphi)=\varphi^\sigma$, $\sigma\gg 1,\varphi\geq 0$; in this case it reads

$$\sigma\varphi^{\sigma-1}(-\Delta_{\mathbb{H}})^{\alpha/2}\varphi\geq(-\Delta_{\mathbb{H}})^{\alpha/2}\varphi^\sigma. \tag{2.3}$$

□

We need the following Lemma taken from [18].

Lemma 2.2 ([18, Lemma 3.1]). *Let $f\in L^1(\mathbb{R}^{2N+1})$ and $\int_{\mathbb{R}^{2N+1}}f d\eta\geq 0$. Then there exists a test function $0\leq\varphi\leq 1$ such that*

$$\int_{\mathbb{R}^{2N+1}}f\varphi\,d\eta\geq 0. \tag{2.4}$$

Let us set

$$\int_{\mathcal{Q}_T}=\int_0^T\int_{\mathbb{R}^{2N+1}},\quad \int_{\mathcal{Q}}=\int_0^\infty\int_{\mathbb{R}^{2N+1}}.$$

3. Wave equation (1.1)

Definition 3.1. A locally integrable function $u\in L^{\max\{p,m\}}_{\rm loc}(\mathbb{R}^{2N+1}\times(0,T))$ is called a local weak solution of (1.1) in $\mathbb{R}^{2N+1}\times(0,T)$ subject to the initial data $u_0,u_1\in L^1_{\rm loc}(\mathbb{R}^{2N+1})$ if the equality

$$\begin{aligned}&\int_{\mathcal{Q}_T}\Big(u\frac{\partial^2\varphi}{\partial t^2}+|u|^m(-\Delta_{\mathbb{H}})^{\alpha/2}\varphi\Big)\,d\eta\,dt\\ &=\int_{\mathcal{Q}_T}|u|^p\varphi\,d\eta\,dt-\int_{\mathbb{R}^{2N+1}}u_0(\eta)\frac{\partial\varphi}{\partial t}(\eta,0)\,d\eta+\int_{\mathbb{R}^{2N+1}}u_1(\eta)\varphi(\eta,0)\,d\eta\end{aligned} \tag{3.1}$$

is satisfied for any regular function

$$\varphi\in C^2((0,T];H^\alpha(\mathbb{R}^{2N+1}))\cap C^1([0,T];H^\alpha(\mathbb{R}^{2N+1})),$$

with $\varphi(\cdot,T)=0,\varphi\geq 0$, where $H^\alpha(\mathbb{R}^{2N+1})$ is the homogeneous Sobolev space of order α. The solution is called global if $T=+\infty$.

Theorem 3.2. *Let $1<m<p<p_{m,\alpha}:=\frac{2mQ+\alpha}{2Q-\alpha}$, and $\int_{\mathbb{R}^{2N+1}}u_1(\eta)\,d\eta>0$. Then,* (1.1) *does not have a nontrivial weak solution.*

Proof. The proof is by contradiction. For that, let u be a solution and φ be a smooth nonnegative test function such that

$$\begin{aligned}\mathcal{A}(\varphi)&:=\int_{\mathcal{Q}}\Big|\sigma(\sigma-1)\Big(\frac{\partial\varphi}{\partial t}\Big)^2+\sigma\varphi\frac{\partial^2\varphi}{\partial t^2}\Big|^{\frac{p}{p-1}}\varphi^{(\sigma-\frac{2p}{p-1})}\,d\eta\,dt<\infty,\\ \mathcal{B}(\varphi)&:=\int_{\mathcal{Q}}|(-\Delta_{\mathbb{H}})^{\alpha/2}\varphi|^{\frac{p}{p-m}}\varphi^{(\sigma-\frac{p}{p-m})}\,d\eta\,dt<\infty.\end{aligned} \tag{3.2}$$

Then, taking $\varphi^\sigma, \sigma \gg 1$ instead of φ in (3.1) and using inequality (2.3), we obtain

$$
\begin{aligned}
&\int_{\mathcal{Q}} |u|^p \varphi^\sigma \,d\eta\,dt + \int_{\mathbb{R}^{2N+1}} u_1(\eta)\varphi^\sigma(\eta,0)\,d\eta - \sigma \int_{\mathbb{R}^{2N+1}} u_0(\eta)\varphi^{\sigma-1}\frac{\partial\varphi}{\partial t}(\eta,0)\,d\eta \\
&\le \int_{\mathcal{Q}} \Big(u(\sigma(\sigma-1)\varphi^{\sigma-2}\Big(\frac{\partial\varphi}{\partial t}\Big)^2 + \sigma\varphi^{\sigma-1}\frac{\partial^2\varphi}{\partial t^2})\Big)\,d\eta\,dt \\
&\quad + \sigma\int_{\mathcal{Q}} |u|^m \varphi^{\sigma-1}(-\Delta_{\mathbb{H}})^{\alpha/2}\varphi\,d\eta\,dt \\
&\le \frac{1}{2}\int_{\mathcal{Q}} |u|^p\varphi^\sigma\,d\eta\,dt + C\big(\mathcal{A}(\varphi)+\mathcal{B}(\varphi)\big)
\end{aligned}
\tag{3.3}
$$

by means of the ε-Young's inequality $ab \le \varepsilon a^p + C(\varepsilon)b^{p'}$, $p+p'=pp'$, $a\ge 0$, $b\ge 0$. Choosing φ such that

$$
\frac{\partial\varphi}{\partial t}(\eta,0)=0,
\tag{3.4}
$$

from (3.3), (3.4), and using Lemma 2.2, we obtain

$$
\int_{\mathcal{Q}} |u|^p\varphi^\sigma\,d\eta\,dt \le C\big(\mathcal{A}(\varphi)+\mathcal{B}(\varphi)\big).
\tag{3.5}
$$

Set

$$
\varphi(\eta,t)=\varphi_1(\eta)\varphi_2(t)=\Phi\big(\frac{\tau^2+|x|^4+|y|^4}{R^4}\big)\Phi\big(\frac{t^2}{R^{2\rho}}\big),\quad \rho=\frac{\alpha(p-1)}{2(p-m)},
$$

where $R>0$, and $\Phi\in\mathcal{D}([0,+\infty[)$ is the standard cut-off function

$$
\Phi(r)=\begin{cases} 1, & 0\le r\le 1,\\ \searrow, & 1\le r\le 2,\\ 0, & r\ge 2.\end{cases}
$$

Set

$$
\Omega_1=\big\{\tilde\eta\in\mathbb{H}; 0\le\tilde\tau^2+|\tilde x|^4+|\tilde y|^4\le 2\big\},\quad \Omega_2=\big\{\tilde t; 0\le \tilde t^2\le 2\big\}.
$$

Note that

$$
\frac{\partial\varphi}{\partial t}(\eta,t)=2tR^{-2}\Phi\Big(\frac{\tau^2+|x|^4+|y|^4}{R^4}\Big)\Phi'\big(\frac{t^2}{R^{2\rho}}\big) \implies \frac{\partial\varphi}{\partial t}(\eta,0)=0;
$$

so equality (3.4) is satisfied as required. Moreover, using the scaled variables

$$
\tilde\tau=R^{-2}\tau,\quad \tilde x=R^{-1}x,\quad \tilde y=R^{-1}y,\quad \tilde t=R^{-\rho}t,
$$

we obtain the estimates

$$
\mathcal{A}(\varphi)\le C_1R^{\vartheta},\quad \mathcal{B}(\varphi)\le C_2R^{\vartheta},
\tag{3.6}
$$

with $\vartheta=-\frac{\alpha p}{p-m}+Q+\rho$; the constants C_1,C_2 are $\mathcal{A}(\varphi)$ and $\mathcal{B}(\varphi)$ evaluated on $\Omega_1\times\Omega_2$. Now, if

$$
-\frac{\alpha p}{p-m}+Q+\rho<0 \quad\iff\quad p<p_{\alpha,m},
$$

by letting $R\to\infty$ in (3.5), we obtain

$$
\int_{\mathcal{Q}} |u|^p\,d\eta\,dt=0 \quad\implies\quad u\equiv 0;
$$

this is a contradiction. □

Remark 3.3. For the equation

$$\frac{\partial^2 u}{\partial t^2} + (-\Delta_{\mathbb{H}})^{\alpha/2}|u| = |u|^p,$$

the found exponent is $p_{\alpha,1} := \frac{2Q+\alpha}{2Q-\alpha}$ which, in the limiting case $\alpha = 2$, gives $p_{2,1} = \frac{Q+1}{Q-1}$, the one obtained in [22].

Remark 3.4. For $u_0, u_1 \in C_0^\infty(\mathbb{R}^{2N+1})$, the solution exists locally and is very regular as it can be proved following the lines of [27]. So in this case, the nonexistence result we have proved is in fact a blow-up result, that is, the solutions blows-up at a finite time.

4. Parabolic equation (1.2)

Definition 4.1. A locally integrable function $u \in L^p_{\rm loc}(\mathbb{R}^{2N+1} \times (0,T))$ is called a local weak solution of the differential equation (1.2) in $\mathbb{R}^{2N+1} \times (0,T)$ subject to the initial data $u_0 \in L^1_{\rm loc}(\mathbb{R}^{2N+1})$ if the equality

$$\begin{aligned} &\int_{\mathcal{Q}_T} |u|^p \varphi \, d\eta \, dt + \int_{\mathbb{R}^{2N+1}} u_0(\eta)\varphi(\eta,0)\, d\eta \\ &= \int_{\mathcal{Q}_T} \Big(- u\frac{\partial \varphi}{\partial t} + |u|^m (-\Delta_{\mathbb{H}})^{\alpha/2}\varphi \Big)\, d\eta \, dt \end{aligned} \tag{4.1}$$

is satisfied for any regular function

$$\varphi \in C^1((0,T]; H^\alpha(\mathbb{R}^{2N+1})) \cap C([0,T]; H^\alpha(\mathbb{R}^{2N+1})),$$

with $\varphi(.,T) = 0, \varphi \geq 0$. The solution is called global if $T = +\infty$.

Theorem 4.2. *Let $1 < p < m + \frac{\alpha}{Q}$ and $\int_{\mathbb{R}^{2N+1}} u_0(\eta) \geq 0$. Then, (1.2) does not have a nontrivial global weak solution.*

Proof. The proof is similar to the previous one. Let u be a global weak solution and φ be a smooth nonnegative test function such that

$$\mathcal{E}(\varphi) = \int_{\mathcal{Q}} \Big(\big|\frac{\partial \varphi}{\partial t}\big|^{\frac{p}{p-1}} \varphi^{\sigma - \frac{p}{p-1}} + |(-\Delta_{\mathbb{H}})^{\alpha/2}\varphi|^{\frac{p}{p-m}} \varphi^{\sigma - \frac{p}{p-m}} \Big)\, d\eta \, dt < \infty. \tag{4.2}$$

Then, using inequality (2.3) in (4.1) with φ^σ as a test function and using ε-Young's inequality, we obtain

$$\begin{aligned} &\int_{\mathcal{Q}} |u|^p \varphi^\sigma \, d\eta \, dt + \int_{\mathbb{R}^{2N+1}} u_0(\eta)\varphi^\sigma(\eta,0)\, d\eta \\ &= \int_{\mathcal{Q}} \Big(- u\frac{\partial \varphi^\sigma}{\partial t} + |u|^m (-\Delta_{\mathbb{H}})^{\alpha/2}\varphi^\sigma \Big)\, d\eta \, dt \\ &\leq \sigma \int_{\mathcal{Q}} \Big(- u\varphi^{\sigma-1}\frac{\partial \varphi}{\partial t} + |u|^m \varphi^{\sigma-1}(-\Delta_{\mathbb{H}})^{\alpha/2}\varphi \Big)\, d\eta \, dt \\ &\leq \frac{1}{2}\int_{\mathcal{Q}} |u|^p \varphi \, d\eta \, dt + C\int_{\mathcal{Q}} \big|\frac{\partial \varphi}{\partial t}\big|^{\frac{p}{p-1}} \varphi^{\sigma-\frac{p}{p-1}}\, d\eta \, dt \\ &\quad + C\int_{\mathcal{Q}} |(-\Delta_{\mathbb{H}})^{\alpha/2}\varphi|^{\frac{p}{p-m}} \varphi^{\sigma - \frac{p}{p-m}}\, d\eta \, dt, \end{aligned} \tag{4.3}$$

so, using Lemma 2.2, we obtain

$$\begin{aligned}\int_{\mathcal{Q}} |u|^p\varphi^\sigma \,d\eta\,dt &\le C\int_{\mathcal{Q}} \big|\frac{\partial\varphi}{\partial t}\big|^{\frac{p}{p-1}}\varphi^{\sigma-\frac{p}{p-1}}\,d\eta\,dt \\ &\quad + C\int_{\mathcal{Q}} |(-\Delta_{\mathbb{H}})^{\alpha/2}\varphi|^{\frac{p}{p-m}}\varphi^{\sigma-\frac{p}{p-m}}\,d\eta\,dt,\end{aligned} \tag{4.4}$$

for some constant $C>0$. Setting

$$\varphi(\eta,t)=\Phi\Big(\frac{\tau^2+|x|^4+|y|^4}{R^4}\Big)\Phi\big(\frac{t}{R^\rho}\big),\quad \rho=\frac{\alpha(p-1)}{p-m}$$

and using the change of variables,

$$\tilde\tau=R^{-2}\tau,\quad \tilde x=R^{-1}x,\quad \tilde y=R^{-1}y,\quad \tilde t=R^{-\rho}t,$$

we obtain the estimates

$$\int_{\mathcal{Q}} \big|\frac{\partial\varphi}{\partial t}\big|^{\frac{p}{p-1}}\varphi^{\sigma-\frac{p}{p-1}}\,d\eta\,dt \le CR^{-\frac{\rho p}{p-1}+Q+\rho}, \tag{4.5}$$

$$\int_{\mathcal{Q}} |(-\Delta_{\mathbb{H}})^{\alpha/2}(\varphi)|^{\frac{p}{p-m}}\varphi^{\sigma-\frac{p}{p-m}}\,d\eta\,dt \le CR^{-\frac{\alpha p}{p-m}+Q+\rho}. \tag{4.6}$$

The constraint $1<p<m+\frac{\alpha}{Q}$ allows us to obtain the contradiction

$$\int_{\mathcal{Q}} |u|^p\,d\eta\,dt=\lim_{R\to\infty}\int_{\mathcal{Q}} |u|^p\varphi^\sigma\,d\eta\,dt=0 \;\Rightarrow\; u\equiv 0.$$

□

Remark 4.3. For $m=1$, as in the Euclidian case, we can obtain analogous estimates for the semi-group generated by the linear part of (1.2) via estimates of the semi-group generated by the one of linear part with $\alpha=2$. Whereupon, the existence of a local mild solution to (1.2) can be obtained by simple application of the Banach fixed point theorem (see [17] for the case $\alpha=2$). This suggests that the limiting exponent for (1.2) is in fact a Fujita's exponent.

5. Wave equation with linear damping (1.3)

Now, we consider the wave equation with a linear damping.

Definition 5.1. A locally integrable function $u\in L^{\max\{p,m\}}_{\rm loc}(\mathbb{R}^{2N+1}\times(0,T))$ is called a local weak solution of the differential equation (1.3) in $\mathbb{R}^{2N+1}\times(0,T)$ subject to the initial data $u(\eta,0)=u_0(\eta)$, $\frac{\partial u}{\partial t}(\eta,0)=u_1(\eta)$, $u_0,u_1\in L^1_{\rm loc}(\mathbb{R}^{2N+1})$ if the equality

$$\begin{aligned}&\int_{\mathcal{Q}_T}|u|^p\varphi\,d\eta\,dt+\int_{\mathbb{R}^{2N+1}}\Big(u_0(\eta)\varphi(\eta,0)+u_0(\eta)\frac{\partial\varphi}{\partial t}(\eta,0)+u_1(\eta)\varphi(\eta,0)\Big)\,d\eta\\ &=\int_{\mathcal{Q}_T}\Big(u\frac{\partial^2\varphi}{\partial t^2}-u\frac{\partial\varphi}{\partial t}+|u|^m(-\Delta_{\mathbb{H}})^{\alpha/2}\varphi\Big)\,d\eta\,dt\end{aligned} \tag{5.1}$$

is satisfied for any regular function

$$\varphi\in C^2((0,T];H^\alpha(\mathbb{R}^{2N+1}))\cap C^1([0,\,T];H^\alpha(\mathbb{R}^{2N+1})),$$

with $\varphi(.,T)=0,\varphi\ge 0$. The solution is called global if $T=+\infty$.

Theorem 5.2. *Let $1<p<m+\frac{\alpha}{Q}$, and $\int_{\mathbb{R}^{2N+1}}(u_1(\eta)+u_0(\eta))\,d\eta\ge 0$. Then (5.1) does not admit a nontrivial weak solution.*

Proof. Choosing

$$\varphi(\eta,t)=\Phi\Big(\frac{\tau^2+|x|^4+|y|^4}{R^4}\Big)\Phi\big(\frac{t^2}{R^{2\rho}}\big),\quad \rho=\frac{\alpha(p-1)}{p-m}$$

($\frac{\partial\varphi}{\partial t}(\eta,0)=0$), and using Lemma 2.2 together with the estimates, as before, we obtain

$$\int_{\mathcal{Q}}|u|^p\varphi^\sigma\,d\eta\,dt\le C\big(\mathcal{A}(\varphi)+\mathcal{B}(\varphi)+\mathcal{C}(\varphi)\big), \tag{5.2}$$

where

$$\mathcal{C}(\varphi):=\int_{\mathcal{Q}}\big|\frac{\partial\varphi}{\partial t}\big|^{\frac{p}{p-1}}\varphi^{\sigma-\frac{p}{p-1}}\,d\eta\,dt<\infty.$$

After using the scaled the variables

$$\tilde{\tau}=R^{-2}\tau,\quad \tilde{x}=R^{-1}x,\quad \tilde{y}=R^{-1}y,\quad \tilde{t}=R^{-\rho}t,$$

we obtain the estimate

$$\int_{\mathcal{Q}}|u|^p\varphi^\sigma\,d\eta\,dt+\int_{\mathbb{R}^{2N+1}}u_1(\eta)\varphi(\eta,0)\,d\eta+\int_{\mathbb{R}^{2N+1}}u_0(\eta)\varphi(\eta,0)\,d\eta\le CR^{-\frac{\alpha p}{p-m}+Q+\rho}.$$

The requirement $1<p<m+\frac{\alpha}{Q}$ allows us to conclude as in the previous cases. □

Remark 5.3. *We conjecture that the limiting exponent here is the critical exponent. However to show that in fact it is, one needs to obtain similar results as in* [21].

6. Life span of solutions

Our first result concerns the wave equation. We assume that the data $u_1(\eta)$ satisfies the condition

$$u_1(\eta)\ge|\eta|_{\mathbb{H}}^{-k},\quad x\in\mathbb{R}^{2N+1},\ k>Q. \tag{6.1}$$

Theorem 6.1. *Suppose that* (6.1) *is satisfied, $1<m<p<p_{m,\alpha}$, and let u be the solution of* (1.1) *with the initial data $u(\eta,0)=\mu u_0(\eta)$, where $\mu>0$. Denote by $[0,T_\mu)$ the life span of u. Then there exists a constant $C>0$ such that*

$$T_\mu\le C\mu^{1/\kappa}, \tag{6.2}$$

where $\kappa:=k-\frac{\alpha(p+1)}{2(p-m)}<0$.

Proof. We take, for $T_\mu>0$, $\varphi(\eta,t):=\varphi_1(\eta)\varphi_2(t)$, where

$$\varphi_1(\eta):=\Phi\Big(\frac{\tau^2+|x|^4+|y|^4}{T_\mu^4}\Big),\quad \varphi_2(t):=\big(\frac{t^2}{T_\mu^{2\rho}}\big).$$

We clearly note that: $\partial_t\varphi(\eta,0)=0$. Now, as for the estimate (3.3), we have, with $Q:=[0,T_\mu^2]\times\mathbb{R}^{2N+1}$,

$$\int_Q|u|^p\varphi^\sigma(\eta,t)\,d\eta\,dt+\mu\int_{\mathbb{R}^{2N+1}}u_1(\eta)\varphi(\eta,0)\,d\eta\le C\mathcal{A}+C\mathcal{B}. \tag{6.3}$$

Using the positivity of the first term in the left-hand side of (6.3), we have

$$\mu\int_{\mathbb{R}^{2N+1}}u_1(\eta)\varphi(\eta,0)\,d\eta\le C\mathcal{A}+C\mathcal{B}.$$

Next, by the assumption on the data $u_1(\eta)$, we get

$$\mu\int_{\mathbb{R}^{2N+1}}|\eta|^{-k}\varphi(\eta,0)\,d\eta\le\mu\int_{\mathbb{R}^{2N+1}}u_1(\eta)\varphi(\eta,0)\,d\eta.$$

Thus

$$\mu \int_{\mathbb{R}^{2N+1}} |\eta|^{-k} \varphi(\eta,0)\, d\eta \le C\mathcal{A} + C\mathcal{B}.$$

We pass to the new variables $\tilde{t} = T_\mu^{-\rho}$ and $\tilde{\eta} = (\tilde{\tau}, \tilde{x}, \tilde{y})$ such that $\tilde{\tau} = T_\mu^{-2}\tau$, $\tilde{x} = T_\mu^{-1}x$, $\tilde{y} = T_\mu^{-1}y$, to obtain

$$\mu T_\mu^{Q-k} \int_{\mathbb{R}^{2N+1}} |\tilde{\eta}| \Phi(\nu)^\sigma \, d\tilde{\eta} \le C T_\mu^{\theta}.$$

Whereupon

$$\mu \le C T^{k - \frac{\alpha(p+1)}{2(p-m)}} \Longrightarrow T_\mu \le C\mu^{1/\kappa},$$

where $\kappa := k - \frac{\alpha(p+1)}{2(p-m)} < 0$. This completes the proof of the theorem. □

The results concerning the heat equation and the damped wave equation are:

- For heat equation, we have the estimate $T_\mu \le C\mu^{1/\kappa}$, where $\kappa := k - \frac{\alpha}{p-m} < 0$, with $u_0(\eta) \ge |\eta|_{\mathbb{H}}^{-k}$;
- For wave equation with linear damping, we have the estimate $T_\mu \le C\mu^{1/\kappa}$, where $\kappa := k - \frac{\alpha}{p-m} < 0$, with $u_0(\eta) + u_1(\eta) \ge |\eta|_{\mathbb{H}}^{-k}$,

whose proofs are similar to the previous one and hence are omitted.

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