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ARBITRARY NUMBER OF LIMIT CYCLES FOR PLANAR DISCONTINUOUS PIECEWISE LINEAR DIFFERENTIAL SYSTEMS WITH TWO ZONES

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Abstract. For any given positive integer n we show the existence of a class of discontinuous piecewise linear differential systems with two zones in the plane having exactly n hyperbolic limit cycles. Moreover, all the points on the separation boundary between the two zones are of sewing type, except the origin which is the only equilibrium point.

1. Introduction and statement of main results

One of the most challenging problems in the qualitative theory of planar ordinary differential equations is the second part of the classical 16th Hilbert problem: the determination of an upper bound for the number of limit cycles (and their relative positions) for the class of polynomial vector fields of degree n. This problem remains unsolved if $n \geq 2$. The case $n = 1$, that is for the class of planar linear vector fields the problem has a trivial answer. However, this problem presents a surprising richness when adapted to the class of the planar piecewise linear systems.

Planar piecewise linear differential systems are widely studied nowadays because of their applicability in several branches of science. A landmark of such study was the work of Andronov et al. [1]. There is an expectation that these systems can present all the dynamical behaviors of the classical nonlinear differential systems.

In this article, we study the existence, number, stability and distribution of limit cycles for a class of piecewise linear differential systems in the plane. These issues must be studied taking into account the following aspects: the number and stability of equilibrium points as well as their locations with respect to the separation boundary $\mathcal{L}$ (which defines the number of zones) and the behavior of the linear vector fields on $\mathcal{L}$. Usually, the points of discontinuity on the separation boundary $\mathcal{L}$ are classified as sewing, sliding or tangency points. Here the term sliding is used in a broad sense meaning sliding or escaping points. See [9] and the references therein for more details.

Without being exhaustive we present below some results relative to the study of limit cycles in piecewise linear differential systems in the plane. For the simplest

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case, that is for planar piecewise linear differential systems with two zones separated by a straight line $\mathcal{L}$ and with only one equilibrium point $p \in \mathcal{L}$, we have the following results:

(i) With the additional hypothesis of the continuity on $\mathcal{L}\backslash\{p\}$, there is at most one limit cycle [4];
(ii) With the additional hypothesis that the points on $\mathcal{L}\backslash\{p\}$ are of sewing type, there is at most one limit cycle [10];
(iii) With the additional hypothesis of the existence of a sliding segment on $\mathcal{L}\backslash\{p\}$, there are examples with two limit cycles surrounding the sliding segment [5].

The answer to the existence and the number of limit cycles in piecewise linear systems with more than two zones in the plane was given in [6]. In that article the authors provided an example of a planar piecewise linear Liénard system with an arbitrary number of limit cycles all of them hyperbolic. More precisely, given a positive integer n the authors showed, using the averaging theory, that the system under consideration can have at least n limit cycles in the strip $|x| \leq 2n+2$ for a parameter $\epsilon \in \mathbb{R}$ sufficiently small. Llibre, Ponce and Zhang [8] proved a conjecture of [6] in which the upper bound for the number of limit cycles can be achieved for a fixed n.

Recent studies suggest that three is the maximum number of limit cycles for planar discontinuous piecewise linear differential systems with two zones separated by a straight line $\mathcal{L}$ and only one equilibrium point $p \notin \mathcal{L}$. Numerical examples which support this statement can be found in [2] and [7].

The separation boundary $\mathcal{L}$ between the two zones plays an important role in planar discontinuous piecewise linear differential systems with only one equilibrium point $p \notin \mathcal{L}$. The article [3] exhibits an example of such a system with seven limit cycles having $\mathcal{L}$ as a polygonal curve.

Here we show that a planar discontinuous piecewise linear differential system with two zones can have an arbitrary number of limit cycles, that is, the main idea in [8] still remains for only two zones. So, we study the following class of discontinuous piecewise linear differential with two zones in the plane

$$X' = \begin{cases} G^- X, & \mathcal{H}(X) < 0, \\ G^+ X, & \mathcal{H}(X) \geq 0, \end{cases} \tag{1.1}$$

where the prime denotes derivative with respect to the independent variable t, called here the time, $X = (x, y) \in \mathbb{R}^2$ and

$$G^\pm = \begin{pmatrix} g_{11}^\pm & g_{12}^\pm \\ g_{21}^\pm & g_{22}^\pm \end{pmatrix}, \tag{1.2}$$

are matrices with real entries satisfying the following assumptions:

(A1) $g_{12}^\pm < 0$;
(A2) G^- has complex eigenvalues with negative real parts, $\lambda_{1,2}^- = \gamma^- \pm i\omega^-$, while G^+ has complex eigenvalues with positive real parts, $\lambda_{1,2}^+ = \gamma^+ \pm i\omega^+$, where $\gamma^\pm, \omega^\pm \in \mathbb{R}$ and $\omega^\pm > 0$.

In addition,

(A3) the function $\mathcal{H}$ is at least continuous and the set $\mathcal{H}^{-1}(0)$ divides the plane in two unbounded components, that is the function $\mathcal{H}$ implicitly defines a simple planar curve homeomorphic to the real line and whose trace is unbounded.

A member of the class (1.1) will be denoted by $(G^-, G^+, \mathcal{H})$. Note that the hypothesis (A3) ensures the existence of only two zones whose separation (boundary) set is defined by $\mathcal{L}_{\mathcal{H}} = \{X \in \mathbb{R}^2 : \mathcal{H}(X) = 0\}$.

Our goal is to build a suitable function Ψ satisfying the assumptions on $\mathcal{H}$ and to choose matrices G^- and G^+ in the Jordan normal forms $J^\pm$ such that, for any positive integer n, the system (J^-, J^+, Ψ) has exactly n hyperbolic limit cycles. Furthermore, $0 = (0, 0) \in \mathcal{H}^{-1}(0)$ and $\mathcal{L}_{\mathcal{H}} \backslash \{0\}$ is a sewing set. As far as we know, it is the first example of such systems. We prove the following main theorem.

Theorem 1.1. *Given any positive integer n there is a planar piecewise linear differential system with two zones (J^-, J^+, Ψ) having n hyperbolic limit cycles.*

Novaes and Ponce [11] obtained examples of planar piecewise linear differential systems with two zones having n limit cycles, for every positive integer n. However, the limit cycles obtained can be non-hyperbolic.

This article is structured as follows. In Section 2 we present the proof of Theorem 1.1 and we give an example of such a system with ten limit cycles. In the last section we make some concluding remarks.

2. Proof of main results

We consider the case in which the matrices G^+ and G^- are in the Jordan normal forms; that is,

$$J^\pm = \begin{pmatrix} \gamma^\pm & -\omega^\pm \\ \omega^\pm & \gamma^\pm \end{pmatrix}. \tag{2.1}$$

The solutions of $X' = J^- X$ will be denoted by

$$(t, X_0) \mapsto X^-(t, X_0) = (x^-(t, X_0), y^-(t, X_0)),$$

while the solutions of $X' = J^+ X$ will be denoted by

$$(t, X_0) \mapsto X^+(t, X_0) = (x^+(t, X_0), y^+(t, X_0)),$$

where

$$\begin{aligned} x^\pm(t, X_0) &= e^{\gamma^\pm t} \left(\cos(\omega^\pm t) x_0 - \sin(\omega^\pm t) y_0\right), \\ y^\pm(t, X_0) &= e^{\gamma^\pm t} \left(\sin(\omega^\pm t) x_0 + \cos(\omega^\pm t) y_0\right). \end{aligned} \tag{2.2}$$

In Lemma 2.1 we present a result associated with the system (J^-, J^+, Φ), where

$$X \mapsto \Phi(X) = x - \phi(y)$$

and

$$y \mapsto \phi(y) = \rho v(y), \tag{2.3}$$

with $\rho > 0$, $s \in \mathbb{R} \mapsto v(s) = s\, u(s)$ and

$$s \in \mathbb{R} \mapsto u(s) = \begin{cases} 0, & s < 0, \\ 1, & s \geq 0 \end{cases}$$

is the unit step function.

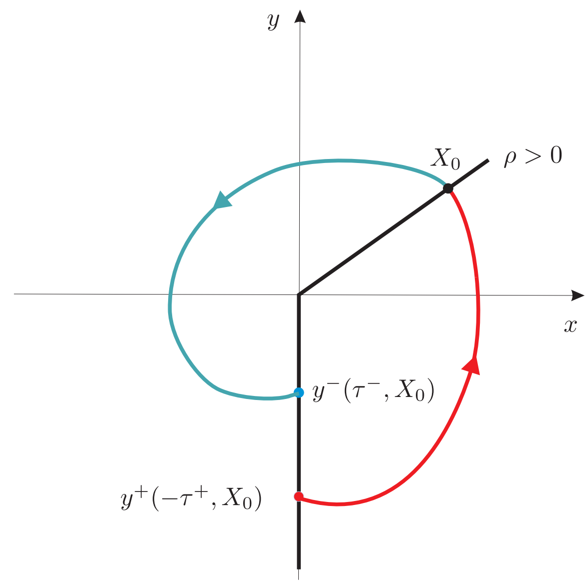

FIGURE 1. Displacement function $(y_0, \rho) \mapsto \delta_\phi(y_0, \rho)$.

Lemma 2.1. *Let $\eta^- < 0$ and $\eta^+ > 0$ be numbers satisfying $-\eta^- < \eta^+ < -3\eta^-$, where $\eta^\pm = \gamma^\pm/\omega^\pm$, and define*

$$\rho_c = \tan\Big(\pi\frac{\eta^+ + \eta^-}{\eta^+ - \eta^-}\Big).$$

Then, the origin of the system (J^-, J^+, Φ) is:

(a) *An unstable focus, if $0 < \rho < \rho_c$,*
(b) *A stable focus, if $\rho > \rho_c$,*
(c) *A center, if $\rho = \rho_c$.*

Proof. Let $X_0 = (x_0, y_0) = (\phi(y_0), y_0) \in \mathcal{L}_\Phi$ be any initial condition with $y_0 > 0$. The displacement function is defined by

$$(y_0, \rho) \mapsto \delta_\phi(y_0, \rho) = y^+(-\tau^+, X_0) - y^-(\tau^-, X_0), \tag{2.4}$$

where $\tau^- > 0$ is the smallest time such that $X^-(\tau^-, X_0) \in \mathcal{L}_\Phi$, and $\tau^+ > 0$ is the smallest time such that $X^+(-\tau^+, X_0) \in \mathcal{L}_\Phi$. See Figure 1.

From (2.2) and (2.3), the time τ^- is the solution of the equation $x^-(\tau^-, X_0) = 0$ and is given by

$$\tau^- = \tau^-(y_0, \rho) = \frac{1}{\omega^-}\Big(\arctan\big(\frac{x_0}{y_0}\big) + \pi\Big), \tag{2.5}$$

and τ^+ is the solution of $x^+(-\tau^+, X_0) = 0$, and is given by

$$\tau^+ = \tau^+(y_0, \rho) = -\frac{1}{\omega^+}\Big(\arctan\big(\frac{x_0}{y_0}\big) - \pi\Big). \tag{2.6}$$

Thus,

$$\begin{aligned} y^-(\tau^-, X_0) &= -e^{\gamma^-\tau^-}\sqrt{x_0^2 + y_0^2}, \\ y^+(-\tau^+, X_0) &= -e^{-\gamma^+\tau^+}\sqrt{x_0^2 + y_0^2}, \end{aligned} \tag{2.7}$$

and, therefore,

$$(y_0, \rho) \mapsto \delta_\phi(y_0, \rho) = \Delta_\phi(y_0, \rho)\sqrt{\phi(y_0)^2 + y_0^2},$$

where

$$(y_0, \rho) \mapsto \Delta_\phi(y_0, \rho) = \Big(e^{\gamma^-\tau^-} - e^{-\gamma^+\tau^+}\Big) = e^{-\gamma^+\tau^+}\Big(e^{\gamma^-\tau^- + \gamma^+\tau^+} - 1\Big). \tag{2.8}$$

From (2.5) and (2.6), we have

$$\rho \mapsto a(\rho) = \gamma^-\tau^- + \gamma^+\tau^+ = \pi(\eta^+ + \eta^-) - (\eta^+ - \eta^-)\arctan(\rho) \tag{2.9}$$

and $a(\rho) = 0$ implies

$$\rho = \rho_c = \tan\Big(\pi\frac{\eta^+ + \eta^-}{\eta^+ - \eta^-}\Big).$$

If $-\eta^- < \eta^+ < -3\eta^-$, then $\rho_c > 0$. So, for any $y_0 > 0$ such that

$$x_0 = \phi(y_0) = \rho_c y_0 u(y_0) = \rho_c y_0,$$

$\delta_\phi(y_0, \rho_c) \equiv 0$. This proves item (c). Items (a) and (b) follow from $x_0 = \rho v(y_0)$ and (2.9) since $s \mapsto \arctan(s)$ is a monotonically increasing function. $\square$

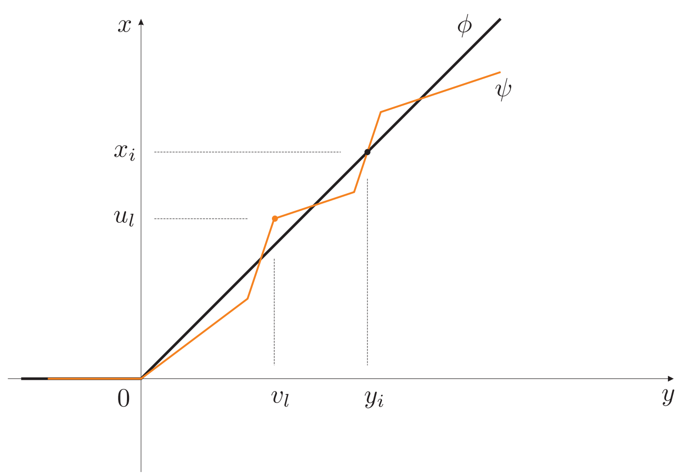

FIGURE 2. The graphs of the functions $y \mapsto \phi(y)$ and $y \mapsto \psi(y)$ are illustrated by continuous black and orange lines, respectively.

Given an integer number $n \geq 1$, consider the finite sequences $\{u_l\}_{l\in N}$ and $\{v_l\}_{l\in N}$, where

$$\begin{aligned} u_l &= (2l-3)ru(l-2), \\ v_l &= \frac{1}{\rho_c}\left(u_l + (-1)^l \varepsilon u(l-2)\right), \end{aligned} \tag{2.10}$$

for $l \in N = \{1, 2, \dots, n+2\}$, $r, \varepsilon \in \mathbb{R}$ satisfying $0 < \varepsilon < r$. So $v_1 < v_2 < \dots < v_{n+2}$. Consider

$$X \mapsto \Psi(X) = x - \psi(y),$$

where

$$y \mapsto \psi(y) = u_1 + \sum_{k=1}^{n+1} \alpha_k\left(v(y - v_k) - \beta_k v(y - v_{k+1})\right), \tag{2.11}$$

and the real numbers α_k and β_k are given by

$$\alpha_k = \frac{u_{k+1} - u_k}{v_{k+1} - v_k}, \quad k = 1, 2, \dots, n+1,$$
$$\beta_k = \begin{cases} 1, & k = 1, \dots, n, \\ 0, & k = n+1. \end{cases}$$

The sets $\mathcal{L}_\Phi$ and $\mathcal{L}_\Psi$ have intersections at the points $p_i = (x_i, y_i) \in \mathbb{R}^2$, with

$$\begin{gathered} x_i = 2ri, \\ y_i = \frac{1}{\rho_c} x_i = \frac{2r}{\rho_c} i, \end{gathered} \tag{2.12}$$

for $i = 1, \dots, n$. In fact,

$$\begin{aligned} y \mapsto g(y) &= \psi(y) - \phi(y) \\ &= u_1 + \sum_{k=1}^{n+1} \alpha_k \left(v(y - v_k) - \beta_k v(y - v_{k+1}) \right) - \rho_c v(y) \end{aligned}$$

is a continuous function on the closed interval $[v_j, v_{j+1}]$ satisfying $g(v_1) = 0$ and

$$g(v_j) = \sum_{k=1}^{n+1} F(j,k) = \sum_{k=1}^{j-1} F(j,k) + F(j,j) + \sum_{k=j+1}^{n+1} F(j,k) - \rho_c v(v_j),$$

for $j = 2, \dots, n+1$, where

$$F(j,k) = \alpha_k (v(v_j - v_k) - \beta_k v(v_j - v_{k+1})).$$

Since $\{v_l\}_{l \in N}$ is a monotone increasing finite sequence, it follows that

$$\begin{aligned} g(v_j) &= \sum_{k=1}^{j-1} F(j,k) - \rho v(v_j) \\ &= \sum_{k=1}^{j-1} \alpha_k (v_{k+1} - v_k) - \rho_c v_j \\ &= \sum_{k=1}^{j-1} (u_{k+1} - u_k) - \rho_c v_j = (1 + 2(j-2))r - \rho_c v_j \\ &= (2j-3)r - \big((2j-3)r + (-1)^j \varepsilon \big) = (-1)^{j+1} \varepsilon, \end{aligned}$$

for $j = 2, \dots, n+1$.

Thus $g(v_{i+1}) g(v_{i+2}) = -\varepsilon^2 < 0$ for $i = 1, \dots, n$ and by Bolzano's Theorem the function $y \mapsto g(y)$ has at least n zeros. Since the function $y \mapsto \psi(y)$ is piecewise linear and the union of straight lines of the form

$$y \in [v_l, v_{l+1}] \mapsto L_l(y) = \alpha_l (y - v_l) + u_l, \tag{2.13}$$

for $l = 1, \dots, n+1$, there exist exactly n zeros, that is there exists a unique $y_i \in [v_{i+1}, v_{i+2}]$ such that $g(y_i) = 0$. Moreover, it is easy to see that

$$y_i = \frac{v_{i+1} + v_{i+2}}{2} = \frac{1}{\rho_c} \frac{u_{i+1} + u_{i+2}}{2},$$

is such as in (2.12) for $i = 1, \dots, n$.

The n points $p_i = (x_i, y_i) \in \mathbb{R}^2$ given by (2.12) are zeros of the displacement function $y_0 \mapsto \delta_\psi(y_0)$ associated with the system (J^-, J^+, Ψ).

Here we studied only the case in which η^- and η^+ are such as in Lemma 2.1 and in addition $\eta^+ < -1/\eta^-$ although other choices are possible. The slope of the straight line $y \mapsto L_1(y) = \alpha_1 y$ is

$$\alpha_1 = \frac{1}{\rho_c}\left(1 + \frac{\varepsilon}{r}\right).$$

Thus, we take ρ_c such that

$$\frac{v_2 - v_1}{u_2 - u_1} = \frac{1}{\rho_c}\left(1 + \frac{\varepsilon}{r}\right) < \frac{2}{\rho_c} < -\frac{1}{\eta^-}$$

and

$$\frac{v_3 - v_1}{u_3 - u_1} = \frac{1}{\rho_c}\left(1 - \frac{\varepsilon}{3r}\right) > \frac{2}{3\rho_c} > \eta^+$$

for $0 < \varepsilon < r$. Therefore,

$$\begin{gathered} -\frac{1}{3\eta^+} < \eta^- < 0, \\ -2\eta^- < \tan\left(\pi\frac{\eta^+ + \eta^-}{\eta^+ - \eta^-}\right) < \frac{2}{3\eta^+}. \end{gathered} \tag{2.14}$$

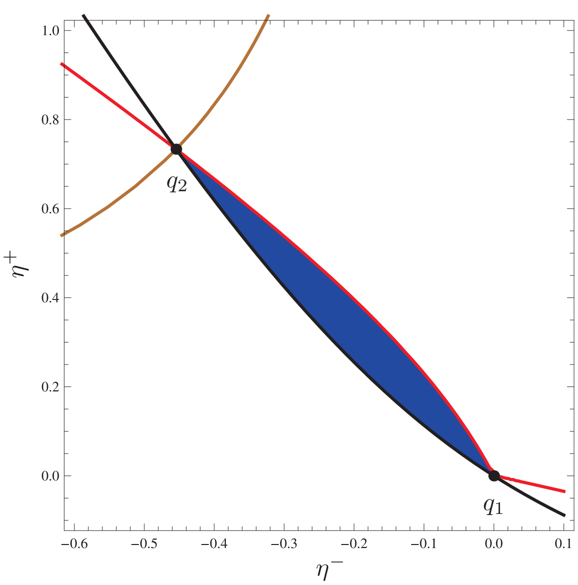

FIGURE 3. The set $\mathcal{R}$ is illustrated by the blue region. The continuous black, red and brown lines represent the graphs of the sets $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$, respectively.

From the inequalities

$$\begin{gathered} -\frac{1}{3\eta^+} < \eta^- < 0, \quad \eta^- < \eta^+, \\ -\eta^- < \eta^+ < -3\eta^-, \quad \eta^+ < -1/\eta^-, \\ -2\eta^- < \tan\left(\pi\frac{\eta^+ + \eta^-}{\eta^+ - \eta^-}\right) < \frac{2}{3\eta^+}, \end{gathered}$$

we obtain the following main functions

$$s \mapsto h_1(s) = -\frac{\pi - \arctan(2s)}{\pi + \arctan(2s)} s,$$
$$s \mapsto h_2(s) = -\frac{\pi - \arctan\left(\frac{2}{3s}\right)}{\pi + \arctan\left(\frac{2}{3s}\right)} s,$$
$$s \mapsto h_3(s) = -\frac{1}{3s}$$

and the set

$$\mathcal{R} = \big\{(\eta^-, \eta^+) \in \mathbb{R}^2 : \eta^+ > h_1(\eta^-),\ \eta^- < h_2(\eta^+)\big\}.$$

The set $\mathcal{R}$ is illustrated in Figure 3. The points q_1 and q_2 (displayed only with six decimals) are given by

$$q_1 = (\eta_1^-, \eta_1^+) = (0, 0) \in \mathcal{C}_1 \cap \mathcal{C}_2,$$
$$q_2 = (\eta_2^-, \eta_2^+) = (-0.454479, 0.733439) \in \mathcal{C}_1 \cap \mathcal{C}_2 \cap \mathcal{C}_3,$$

where

$$\mathcal{C}_1 = \{(\eta^-, \eta^+) \in \mathbb{R}^2 : \eta^+ = h_1(\eta^-)\},$$
$$\mathcal{C}_2 = \{(\eta^-, \eta^+) \in \mathbb{R}^2 : \eta^- = h_2(\eta^+)\},$$
$$\mathcal{C}_3 = \{(\eta^-, \eta^+) \in \mathbb{R}^2 : \eta^+ = h_3(\eta^-)\}.$$

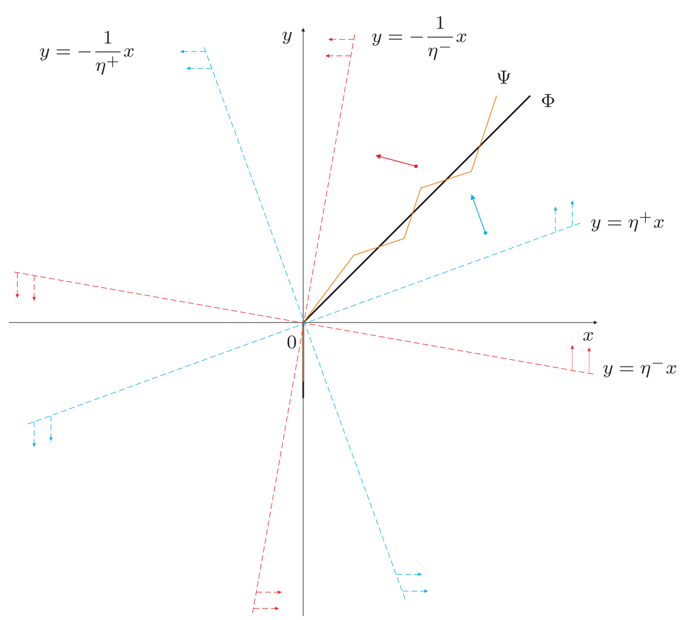

Figure 4. The dashed red lines are the nullclines associated with $X' = J^- X$ and the dashed blue ones are the nullclines associated with $X' = J^+ X$. The orange and black continuous lines are the graph of the sets $\Psi^{-1}(0)$ and $\Phi^{-1}(0)$, respectively.

It follows that the vector fields J^-X and J^+X are both transversal to the set $\mathcal{L}_\Psi$ if $(\eta^-, \eta^+) \in \mathcal{R}$. This means that the set $\mathcal{L}_\Psi \backslash \{0\}$ is a sewing set according to Figure 4. Therefore if $X_0 = (x_0, y_0) = (\psi(y_0), y_0) \in \mathcal{L}_\Psi$ and employing the same previous notation, there exists the smallest time $\tau^- > 0$ such that $X^-(\tau^-, X_0) \in \mathcal{L}_\Psi$ or more precisely $x^-(\tau^-, X_0) = 0$. In the same way there exists the smallest time $\tau^+ > 0$ such that $X^+(-\tau^+, X_0) \in \mathcal{L}_\Psi$ or $x^+(-\tau^+, X_0) = 0$. Moreover, the times τ^- and τ^+ are the same as given in (2.5) and (2.6); that is,

$$\tau^- = \tau^-(y_0) = \frac{1}{\omega^-}\Big(\arctan\big(\frac{x_0}{y_0}\big) + \pi\Big), \tag{2.15}$$

$$\tau^+ = \tau^+(y_0) = -\frac{1}{\omega^+}\Big(\arctan\big(\frac{x_0}{y_0}\big) - \pi\Big). \tag{2.16}$$

Thus, the following function is well defined

$$y_0 \mapsto \delta_\psi(y_0) = y^+(-\tau^+, X_0) - y^-(\tau^-, X_0), \tag{2.17}$$

where $y^-(\tau^-, X_0)$ and $y^-(\tau^-, X_0)$ are such as in (2.7).

The function $y_0 \mapsto \delta_\psi(y_0)$ can be rewritten as

$$y_0 \mapsto \delta_\psi(y_0) = \Delta_\psi(y_0)\sqrt{\psi(y_0)^2 + y_0^2},$$

where

$$y_0 \mapsto \Delta_\psi(y_0) = \Big(e^{\gamma^-\tau^-} - e^{-\gamma^+\tau^+}\Big) = e^{-\gamma^+\tau^+}\Big(e^{\gamma^-\tau^- + \gamma^+\tau^+} - 1\Big). \tag{2.18}$$

From (2.12), (2.18) and Lemma 2.1, it follows that the system (J^-, J^+, Ψ) has n limit cycles since $\delta_\psi(y_i) = 0$, for $i = 1, \dots, n$.

To show that the n limit cycles are all hyperbolic we take an open interval $I_i \subset [v_{i+1}, v_{i+2}]$ such that $y_i \in I_i$ for $i = 1, \dots, n$, since the function $y \mapsto \psi(y)$ is not differentiable at all the points in its domain. Thus, for $y_0 \in I_i$ the derivative of $y_0 \mapsto \delta_\psi(y_0)$ with respect to y_0 is

$$\frac{d}{dy_0}\delta_\psi(y_0) = \frac{d}{dy_0}\Delta_\psi(y_0)\sqrt{\psi(y_0)^2 + y_0^2} + \Delta_\psi(y_0)\frac{d}{dy_0}\sqrt{\psi(y_0)^2 + y_0^2}, \tag{2.19}$$

where

$$\begin{aligned} y_0 \mapsto \frac{d}{dy_0}\Delta_\psi(y_0) &= \frac{d}{dy_0}\Big(e^{\gamma^-\tau^-} - e^{-\gamma^+\tau^+}\Big) \\ &= \gamma^- e^{\gamma^-\tau^-}\frac{d}{dy_0}\tau^-(y_0) + \gamma^+ e^{-\gamma^+\tau^+}\frac{d}{dy_0}\tau^+(y_0). \end{aligned}$$

From (2.13) it results that

$$x_0 = L_{i+1}(y_0) = \alpha_{i+1}(y_0 - v_{i+1}) + u_{i+1}$$

and the derivatives of (2.15) and (2.16) with respect y_0 are

$$\begin{aligned} \frac{d}{dy_0}\tau^-(y_0) &= \frac{\alpha_{i+1}v_{i+1} - u_{i+1}}{\omega^-(L_{i+1}(y_0)^2 + y_0^2)}, \\ \frac{d}{dy_0}\tau^+(y_0) &= -\frac{\alpha_{i+1}v_{i+1} - u_{i+1}}{\omega^+(L_{i+1}(y_0)^2 + y_0^2)}. \end{aligned}$$

Therefore, for $y_0 = y_i \in I_i$ and from (2.10) and (2.12),

$$\frac{d}{dy_0}\tau^-(y_i) = -\frac{1}{\omega^-}(-1)^i\varepsilon S(\rho_c, r, \varepsilon, i),$$

$$\frac{d}{dy_0}\tau^+(y_i) = \frac{1}{\omega^+}(-1)^i \varepsilon S(\rho_c, r, \varepsilon, i),$$

where

$$S(\rho_c, r, \varepsilon, i) = \frac{\rho_c^2}{2ri(r+(-1)^i\varepsilon)(1+\rho_c^2)} > 0 \tag{2.20}$$

for $i = 1, \dots, n$ and $0 < \varepsilon < r$. So

$$\begin{aligned}\frac{d}{dy_0}\Delta_\psi(y_i) &= \gamma^- e^{\gamma^-\tau^-}\frac{d}{dy_0}\tau^-(y_i) + \gamma^+ e^{-\gamma^+\tau^+}\frac{d}{dy_0}\tau^+(y_i)\\ &= (-1)^i\varepsilon\left(\eta^+ e^{-\gamma^+\tau^+} - \eta^- e^{\gamma^-\tau^-}\right) S(\rho_c, r, \varepsilon, i)\end{aligned}$$

and taking into account the result (2.20) and that $\eta^+ e^{-\gamma^+\tau^+} - \eta^- e^{\gamma^-\tau^-} > 0$, $0 < \varepsilon < r$ and $\Delta_\psi(y_i) = 0$ it follows from (2.19) that

$$\frac{d}{dy_0}\delta_\psi(y_i) = \frac{d}{dy_0}\Delta_\psi(y_i)\sqrt{x_i^2 + y_i^2} \tag{2.21}$$

is different from zero. This implies that the n limit cycles are all hyperbolic. Furthermore if $i \in \{1, \dots, n\}$ is odd (or even) then the associated limit cycle is stable (or unstable). Note that with the choice (2.10) the first limit cycle is always an attractor limit cycle. Also note that the period of each limit cycle is constant and given by $\tau^- + \tau^+$ (see (2.15) and (2.16)) and can be made equal to 2π through a different time rescaling in each zone which becomes $\omega^- = \omega^+ = 1$. Theorem 1.1 is proved.

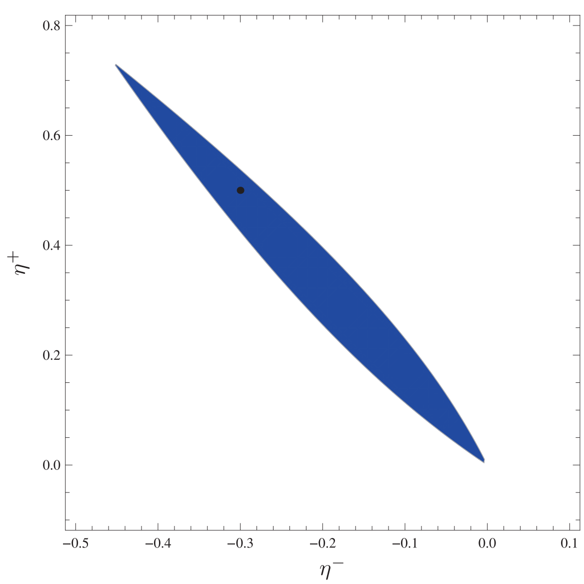

FIGURE 5. The set $\mathcal{R}$ is illustrated by the blue region. The black dot represents the pair $(\eta^-, \eta^+) = (-0.3, 0.5) \in \mathcal{R}$.

Now we present an example with ten limit cycles.

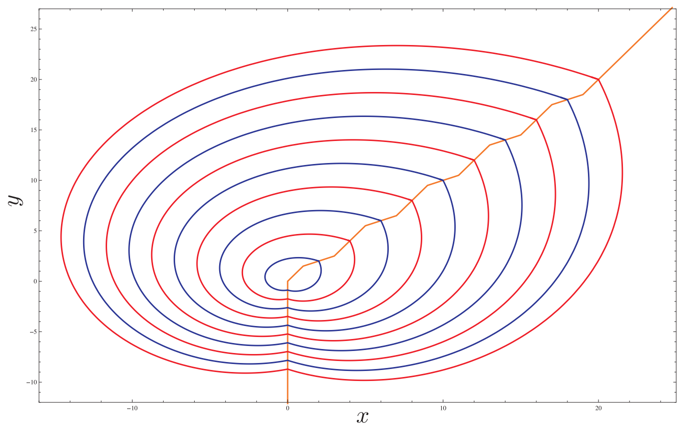

FIGURE 6. Phase portrait of the system (J^-, J^+, Ψ) with ten limit cycles for $(\eta^-, \eta^+) = (-0.3, 0.5) \in \mathcal{R}$, $r = 1$, $\varepsilon = 0.5$ and $n = 10$. The stable (unstable) limit cycles are illustrated by blue (red) lines.

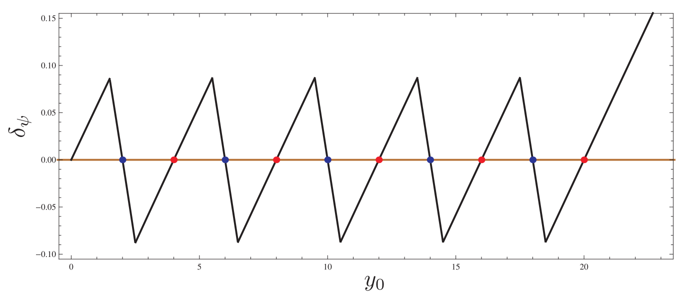

FIGURE 7. The continuous black line is the graph of the displacement function $y_0 \mapsto \delta_\psi(y_0)$ of (J^-, J^+, Ψ) for $(\eta^-, \eta^+) = (-0.3, 0.5) \in \mathcal{R}$, $r = 1$, $\varepsilon = 0.5$ and $n = 10$. The stable limit cycles are illustrated by blue dots and the unstable by red dots.

Example 2.2. In this example we consider the case $\eta^- = -0.3$ and $\eta^+ = 0.5$; that is, $\rho_c = 1$ and the pair $(\eta^-, \eta^+) \in \mathcal{R}$ (see Figure 5).

For $r = 1$, we choose $\varepsilon = 0.5$. According to Theorem 1.1, with these values and $n = 10$, there exists a system (J^-, J^+, Ψ) such that its phase portrait has exactly ten hyperbolic limit cycles. This result is summarized in Figures 6 and 7.

2.1. **Concluding remarks.** In this article, we study one of the main problems in the qualitative theory of planar differential equations: the number and distribution of limit cycles in piecewise linear differential systems with two zones in the plane.

We give a rigorous proof of the existence of an arbitrary number of limit cycles in piecewise linear differential systems with two zones in the plane. See Theorem 1.1. Based on our results we prove the conjecture about the existence of a piecewise linear differential system with two zones in the plane with exactly n limit cycles, for any $n \in \mathbb{N}$. See [3].

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