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POROSITY OF THE FREE BOUNDARY FOR SINGULAR p-PARABOLIC OBSTACLE PROBLEMS

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Abstract. In this article we establish the exact growth of the solution to the singular quasilinear p-parabolic obstacle problem near the free boundary from which follows its porosity.

1. Introduction

Let Ω be an open bounded domain of $\mathbb{R}^n$, $n \geq 2$, $T > 0$. We consider the problem: Find $u \in L^p(0,T;W^{1,p}(\Omega))$ such that:

(i) $u \geq 0$ in $\Omega_T = \Omega \times (0,T)$,
(ii) $L_p(u) = u_t - \Delta_p u = -f(x)$ in $\{u > 0\}$,
(iii) $u = g$ on $\partial_p \Omega_T = (\Omega \times \{0\}) \cup (\partial\Omega \times (0,T))$,

where $p > 1$, Δ_p is the p-Laplacian defined by $\Delta_p u = \operatorname{div}\big(|\nabla u|^{p-2}\nabla u\big)$, and f, g are functions defined in Ω_T and satisfying for two positive constants λ_0 and Λ_0

$$\lambda_0 \leq f \leq \Lambda_0 \quad \text{a.e. in } \Omega_T. \tag{1.1}$$

Moreover we assume that

$$f \text{ is non-increasing in } t. \tag{1.2}$$

$$g(x,0) = 0 \quad \text{a.e. in } \Omega. \tag{1.3}$$

$$g \text{ is non-decreasing in } t. \tag{1.4}$$

The variational formulation of the above problem is: Find

$$u \in K_g = \{v \in V^{1,p}(\Omega_T)/v = g \text{ on } \partial_p\Omega_T, \quad v \geq 0 \text{ a.e. in } \Omega_T\}$$

such that for all $h > 0$ and $t < T - h$:

$$\int_\Omega \partial_t u_h (v-u)dx + \int_\Omega \big(|\nabla u|^{p-2}\nabla u\big)_h.\nabla(v-u)dx + \int_\Omega f_h(v-u)dx \geq 0, \tag{1.5}$$

a.e. in $t \in (0,T)$, and for all $v \in K_g$, where

$$V^{1,p}(\Omega_T) = L^\infty(0,T;L^1(\Omega)) \cap L^p(0,T;W^{1,p}(\Omega)),$$

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and v_h is the Steklov average of a function v defined by

$$v_h(x,t) = \frac{1}{h}\int_t^{t+h} v(x,s)ds, \quad \text{if } t \in (0, T-h] \quad v_h(x,t) = 0, \quad \text{if } t > T-h\,.$$

Let us recall the following existence and uniqueness theorem of the solution of the problem (1.5) [8].

Theorem 1.1. *Assume that f and g satisfy* (1.1)–(1.4)*. Then there exists a unique solution u of the problem* (1.5) *which satisfies*

$$0 \le u \le M = \|g\|_{\infty,\Omega_T} \quad \text{in } \Omega_T, \tag{1.6}$$

$$u_t \ge 0 \quad \text{in } \Omega_T.$$

$$f\chi_{\{u>0\}} \le \Delta_p u - u_t \le f \quad \text{a.e. in } \Omega_T. \tag{1.7}$$

Remark 1.2. We deduce from (1.6)–(1.7) (see [4, Theorems 7 and 8]) that $u \in C^{0,\alpha}_{loc}(\Omega_T) \cap C^{1,\alpha}_{x,loc}(\Omega_T)$ for some $\alpha \in (0,1)$.

The main result of this article is as follows.

Theorem 1.3. *Assume that $1 < p < 2$ and that f and g satisfy* (1.1)–(1.4)*, and let u be the solution of* (1.5)*. Then for every compact set $K \subset \Omega_T$, the intersection $(\partial\{u > 0\}) \cap K \cap \{t = t_0\}$ is porous in $\mathbb{R}^n$ with porosity constant depending only on n, p, λ_0, Λ_0, M, and* $\operatorname{dist}(K, \partial_p\Omega_T)$*.*

We recall that a set $E \subset \mathbb{R}^n$ is called porous with porosity δ, if there is an $r_0 > 0$ such that for all $x \in E$ and all $r \in (0, r_0)$, there exists $y \in \mathbb{R}^n$ such that

$$B_{\delta r}(y) \subset B_r(x) \setminus E.$$

A porous set has Hausdorff dimension not exceeding $n - c\delta^n$, where $c = c(n) > 0$ is a constant depending only on n. In particular a porous set has Lebesgue measure zero.

Theorem 1.3 extends the result established in [8] in the quasilinear degenerate and linear cases $p \ge 2$. The proof is based on the exact growth of the solution of the problem (1.5) near the free boundary which is given by the next theorem.

Theorem 1.4. *Assume that $1 < p < 2$ and that f and g satisfy* (1.1)–(1.4)*, and let u be the solution of the problem* (1.5)*. Then there exists two positive constants $c_0 = c_0(n, p, \lambda_0)$ and $C_0 = C_0(n, p, \lambda_0, \Lambda_0, M)$ such that for every compact set $K \subset \Omega_T$, $(x_0, t_0) \in (\partial\{u > 0\}) \cap K$, the following estimates hold*

$$c_0 r^q \le \sup_{B_r(x_0)} u(.,t_0) \le C_0 r^q, \tag{1.8}$$

where $q = p/(p-1)$ is the conjugate of p.

Since the proof of Theorem 1.3 relies on the one of Theorem 1.4, it will be enough to prove the latter one. On the other hand we observe that the left hand side inequality in (1.8) was established in [8, Lemma 2.1] for any $p > 1$, while the right hand side inequality in (1.8) was established only for $p \ge 2$. In the next section, we shall establish the second inequality for a class of functions in the singular case i.e. for $1 < p < 2$. Then the right hand side inequality will follow exactly as in [8] and we refer the reader to that reference for the details. Hence the proof of Theorem 1.3 will follow.

For similar results in the quasilinear elliptic case, we refer to [5, 1, 2], respectively for the p-obstacle problem, the A-obstacle problem, and the $p(x)$-obstacle problem.

For the obstacle problem for a class of heterogeneous quasilinear elliptic operators with variable growth, we refer to [3].

2. A class of functions on the unit cylinder

In this section, we assume that $1 < p < 2$ and consider the family $\mathcal{F} = \mathcal{F}(p, n, M, \Lambda_0)$ of functions u defined on the unit cylinder $Q_1 = B_1 \times (-1, 1)$ by $u \in \mathcal{F}$ if it satisfies

$$u \in W^{1,p}(Q_1), \quad 0 \le -u_t + \Delta_p u \le \Lambda_0 \quad \text{in } Q_1, \tag{2.1}$$

$$0 \le u \le M \quad \text{in } Q_1, \tag{2.2}$$

$$u(0,0) = 0, \tag{2.3}$$

$$u_t \ge 0 \quad \text{in } Q_1. \tag{2.4}$$

The following theorem gives the growth of the elements of the family $\mathcal{F}$. This completes a result proved in [8] for the degenerate case $p \ge 2$.

Theorem 2.1. *There exists a positive constant $C = C(p, n, M, \Lambda_0)$ such that for every $u \in \mathcal{F}$, we have*

$$u(x,t) \le Cd(x,t) \quad \forall (x,t) \in Q_{1/2}$$

where $d(x,t) = \sup\{r : Q_r(x,t) \subset \{u > 0\}\}$ for $(x,t) \in \{u > 0\}$, and $d(x,t) = 0$ otherwise, and where $Q_r(x,t) = B_r(x) \times (s - r^q, s + r^q)$.

To prove Theorem 2.1, we need to introduce some notation inspired from [8]. For a nonnegative bounded function u, we define the quantities

$$Q_r^- = B_r \times (-r^q, 0), \quad S(r,u) = \sup_{(x,t)\in Q_r^-} u(x,t).$$

Also for $u \in \mathcal{F}$ define the set

$$\mathbb{M}(u) = \{j \in \mathbb{N} \cup \{0\} : AS(2^{-j-1}, u) \ge S(2^{-j}, u)\} \tag{2.5}$$

where $A = 2^q \max\big(1, 1/C_0\big)$ and C_0 is the constant in (1.8). As in [8], we first show a weaker version of the inequality.

Lemma 2.2. *There exists a constant $C_1 = C_1(p, n, M, \Lambda_0)$ such that*

$$S(2^{-j-1}, u) \le C_1 2^{-qj} \quad \forall u \in \mathcal{F},\ \forall j \in \mathbb{M}(u).$$

Proof. We argue by contradiction and assume that: for all $k \in \mathbb{N}$ there exist $u_k \in \mathcal{F}$ and $j_k \in \mathbb{M}(u_k)$ such that

$$S(2^{-j_k-1}, u_k) \ge k2^{-qj_k}. \tag{2.6}$$

Let $\alpha_k = 2^{-pj_k}(S(2^{-j_k-1}, u_k))^{2-p}$, and let

$$v_k(x,t) = \frac{u_k(2^{-j_k}x, \alpha_k t)}{S(2^{-j_k-1}, u_k)}$$

for $(x,t) \in Q_1$. First we observe that since $u(0,0) = 0$ and u is continuous, we have $\alpha_k \to 0$ as $k \to \infty$. Moreover, we have

$$\nabla v_k(x,t) = \frac{2^{-j_k}}{S(2^{-j_k-1}, u_k)} \nabla u_k(2^{-j_k}x, \alpha_k t)$$

$$v_{kt}(x,t) = \frac{\alpha_k}{S(2^{-j_k-1}, u_k)} u_{kt}(2^{-j_k}x, \alpha_k t) = \Big(\frac{2^{-qj_k}}{S(2^{-j_k-1}, u_k)}\Big)^{p-1} u_{kt}(2^{-j_k}x, \alpha_k t) \tag{2.7}$$

and

$$
\begin{aligned}
&\Delta_p v_k(x,t)\\
&= \operatorname{div}\big(|\nabla v_k|^{p-2}\nabla v_k\big)\\
&= \Big(\frac{2^{-j_k}}{S(2^{-j_k-1},u_k)}\Big)^{p-1}\operatorname{div}\big(|\nabla u_k(2^{-j_k}x,\alpha_k t)|^{p-2}\nabla u_k(2^{-j_k}x,\alpha_k t)\big)\\
&= 2^{-j_k}\Big(\frac{2^{-j_k}}{S(2^{-j_k-1},u_k)}\Big)^{p-1}\Delta_p u_k(2^{-j_k}x,\alpha_k t)\\
&= \Big(\frac{2^{-qj_k}}{S(2^{-j_k-1}},u_k)\Big)^{p-1}\Delta_p u_k(2^{-j_k}x,\alpha_k t).
\end{aligned}
\tag{2.8}
$$

We deduce from (2.7)–(2.8) that

$$
v_{kt}-\Delta_p v_k(x,t)=\Big(\frac{2^{-qj_k}}{S(2^{-j_k-1},u_k)}\Big)^{p-1}(u_{kt}-\Delta_p u_k)(2^{-j_k}x,\alpha_k t). \tag{2.9}
$$

Combining (2.1)–(2.6) and (2.9), we obtain

$$
0\le -v_{kt}+\Delta_p v_k\le \frac{\Lambda_0}{k^{p-1}} \quad \text{in } Q_1, \tag{2.10}
$$

$$
0\le v_k\le \frac{S(2^{-j_k},u_k)}{S(2^{-j_k-1},u_k)}\le A \quad \text{in } Q_1^-, \tag{2.11}
$$

$$
v_{kt}\ge 0 \quad \text{in } Q_1^-, \tag{2.12}
$$

$$
\sup_{Q_{1/2}^-} v_k = 1, \tag{2.13}
$$

$$
v_k(0,t)=0 \quad \forall t\in(-1,0). \tag{2.14}
$$

Taking into account (2.10)–(2.11), and using [6, Theorem 1.1] and [4, Theorem 1], we deduce that v_k is locally uniformly bounded in $L^\infty(Q_1)$ independently of k. Therefore we obtain from [4, Theorems 7 and 8], that v_k is uniformly bounded in $C^{0,\alpha}(\overline{Q_{3/4}})$ and in $C_x^{1,\alpha}(\overline{Q_{3/4}})$ independently of k, for a constant $\alpha=\alpha(n,p,A,\Lambda_0)\in(0,1)$. It follows then from Ascoli-Arzella's theorem that there exists a subsequence, still denoted by v_k, and a function $v\in C^{0,\alpha}(\overline{Q_{3/4}})\cap C_x^{1,\alpha}(\overline{Q_{3/4}})$ such that $v_k\to v$ and $\nabla v_k\to\nabla v$ uniformly in $\overline{Q_{3/4}}$. Moreover, using (2.10)–(2.14), we see that v satisfies

$$
\begin{gathered}
v_t-\Delta_p v=0 \quad \text{in } Q_{3/4}^-, \quad v, v_t\ge 0 \quad \text{in } Q_{3/4}^-,\\
\sup_{x\in Q_{1/2}^-} v(x,t)=1, \quad v(0,t)=0 \quad \forall t\in(-3/4,0).
\end{gathered}
$$

We discuss two cases:

Case 1: for all $(x,t)\in Q_{3/4}^-$ $v(x,t)=0$. In particular we have $v\equiv 0$ in $Q_{1/2}^-$ which contradicts the fact that $\sup_{x\in Q_{1/2}^-} v(x)=1$.

Case 2: There exists $(x_0,t_0)\in Q_{3/4}^-$ such that $v(x_0,t_0)>0$. Since $v(.,t_0)$ is not identically zero and $v(0,t_0/2)=0$, we get from the strong maximum principle (see [7]) that $v(x,t_0/2)=0$ for all $x\in B_{3/4}$. By the monotonicity of v with respect to t and the fact that v is nonnegative, we have necessarily $v(x,t)=0$ for all $(x,t)\in B_{3/4}\times(-3/4,t_0/2)$, which is in contradiction with the fact that $v(x_0,t_0)>0$. $\square$

Proof of Theorem 2.1. Using Lemma 2.2, the proof follows exactly as the one of [8, Theorem 2.2]. □

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