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MULTIPLE SOLUTIONS FOR PERTURBED p-LAPLACIAN PROBLEMS ON $\mathbb{R}^N$

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Abstract. We establish the existence of at least three solutions for a perturbed p-Laplacian problem on $\mathbb{R}^N$. Our approach is based on variational methods.

1. Introduction

In this work, we show the existence of at least three solutions for the nonlinear perturbed problem

$$\begin{gathered} -\Delta_p u + |u|^{p-2}u = \lambda\alpha(x)f(u) + \mu\beta(x)h(u) \quad x \in \mathbb{R}^N, \\ u \in W^{1,p}(\mathbb{R}^N), \end{gathered} \tag{1.1}$$

where $(\mathbb{R}^N, |\cdot|)$, $N > 1$, is the usual Euclidean space, $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ with $p > N$, stands for the p-Laplacian operator, $f, h : \mathbb{R} \to \mathbb{R}$ are continuous functions, $\alpha, \beta \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ are nonnegtive (not identically zero) radially symmetric maps, λ is a positive real parameter and μ is a non-negative parameter.

The main objective of this article is to investigate the existence and multiplicity solutions to the above elliptic equation defined on the whole space $\mathbb{R}^N$, by using variational methods. Many technical difficulties appear studying problems on unbounded domains (see [1, 2, 11, 14]). For instance, unlike bounded domains, no compact embedding is available for $W^{1,p}(\mathbb{R}^N)$; although the embedding $W^{1,p}(\mathbb{R}^N) \hookrightarrow L^\infty(\mathbb{R}^N)$ is continuous due to Morrey's theorem ($p > N$), it is far from being compact. However, the subspace of radially symmetric functions of $W^{1,p}(\mathbb{R}^N)$, denoted further by $W^{1,p}_r(\mathbb{R}^N)$, can be embedded compactly into $L^\infty(\mathbb{R}^N$) whenever $2 \le N < p < +\infty$ as proved in [12, Theorem 3.1] (see Lemma 2.4).

In this article, employing a three critical points theorem obtained in [3] which we recall in the next section (Theorem 2.1), we ensure the existence of at least three weak solutions for the problem (1.1). The aim of this work is to establish precise values of λ and μ for which the problem (1.1) admits at least three weak solutions. Our result is motivated by the recent work of Candito and Molica Bisci [9]. In that paper, problem (1.1) has infinitely many radial solutions when $\mu = 0$ and λ in a suitable interval.

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Theorem 2.1 has been used for establishing the existence of at least three solutions for eigenvalue problems in the papers [4, 5, 6]. Fora review on the subject, we refer the reader to [10].

2. Preliminaries

Our main tool is the following three critical points theorem.

Theorem 2.1 ([3, Theorem 2.6]). *Let X be a reflexive real Banach space, $\Phi : X \to \mathbb{R}$ be a coercive continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on X^*, and $\Psi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact, such that $\Phi(0) = \Psi(0) = 0$. Assume that there exist $r > 0$ and $\overline{x} \in X$, with $r < \Phi(\overline{x})$ such that*

(a1) $\frac{1}{r}\sup_{\Phi(x)\le r}\Psi(x) < \frac{\Psi(\overline{x})}{\Phi(\overline{x})}$,

(a2) *for each* $\lambda \in \Lambda_r :=]\frac{\Phi(\overline{x})}{\Psi(\overline{x})}, \frac{r}{\sup_{\Phi(x)\le r}\Psi(x)}[$ *the functional $\Phi - \lambda\Psi$ is coercive.*

Then, for each $\lambda \in \Lambda_r$ the functional $\Phi - \lambda\Psi$ has at least three distinct critical points in X.

The standard Sobolev space $W^{1,p}(\mathbb{R}^N)$ is equipped with the norm

$$\|u\| := \Big(\int_{\mathbb{R}^N} |\nabla u(x)|^p dx + \int_{\mathbb{R}^N} |u(x)|^p dx\Big)^{1/p}.$$

Since by hypotheses $p > N$, $W^{1,p}(\mathbb{R}^N)$ is continuously embedded in $L^\infty(\mathbb{R}^N)$ and we obtain the following lemma.

Lemma 2.2 ([12, Remark 2.2]). *Let $u \in W^{1,p}(\mathbb{R}^N)$. Then*

$$\|u\|_\infty \le \frac{2p}{p-N}\|u\| \tag{2.1}$$

for every $u \in W^{1,p}(\mathbb{R}^N)$.

We also note that, in the low-dimensional case, every function $u \in W^{1,p}(\mathbb{R}^N)$ admits a continuous representation (see [7, p. 166]). In the sequel we will replace u by this element. Let $O(N)$ stands for the orthogonal group of $\mathbb{R}^N$ and $B(0,s)$ denotes the open N-dimensional ball of center zero, radius $s > 0$, and standard Lebesgue measure , $\text{meas}(B(0,s))$. Finally, we set

$$\|\alpha\|_{B(0,s/2)} := \int_{B(0,s/2)} \alpha(x)dx.$$

We say that a function $u \in W^{1,p}(\mathbb{R}^N)$ is a weak solution of (1.1) if

$$\begin{aligned}
&\int_{\mathbb{R}^N} |\nabla u(x)|^{p-2}\nabla u(x)\cdot\nabla v(x)dx + \int_{\mathbb{R}^N} |u(x)|^{p-2}u(x)v(x)dx \\
&- \lambda\int_{\mathbb{R}^N} \alpha(x)f(u(x))v(x)dx - \mu\int_{\mathbb{R}^N} \beta(x)h(u(x))v(x)dx = 0
\end{aligned}$$

for every $v \in W^{1,p}(\mathbb{R}^N)$.

For completeness, we also recall here the principle of symmetric criticality that plays a central role in many problems from differential geometry and physics, and in partial differential equations.

The action of a topological group G on the Banach space $(X, \|\cdot\|_X)$ is a continuous map $\varsigma : G \times X \to X : (g, x) \to \varsigma(g, u) =: gu$, such that

$$1u = u, \quad (gm)u = g(mu), \quad u \mapsto gu \text{ is linear.}$$

The action is said to be isometric if $\|gu\|_X = \|u\|_X$, for every $g \in G$. Moreover, the space of G-invariant points is defined by

$$\mathrm{Fix}(G) := u \in X : gu = u, \forall g \in G,$$

and a map $m : X \to \mathbb{R}$ is said to be G-invariant if $m \circ g = m$ for every $g \in G$.

Theorem 2.3 (Palais (1979))**.** *Assume that the action of the topological group G on the Banach space X is isometric. If $J \in C^1(X;\mathbb{R})$ is G-invariant and if u is a critical point of J restricted to Fix(G), then u is a critical point of J.*

The action of the group $O(N)$ on $W^{1,p}(\mathbb{R}^N)$ can be defined by $(gu)(x) := u(g^{-1}x)$, for every $g \in W^{1,p}(\mathbb{R}^N)$ and $x \in \mathbb{R}^N$. It is clear that this group acts linearly and isometrically, which means $\|u\| = \|gu\|$, for every $g \in O(N)$ and $u \in W^{1,p}(\mathbb{R}^N)$. Defining the subspace of radially symmetric functions of $W^{1,p}(\mathbb{R}^N)$ by

$$X := W^{1,p}_r(\mathbb{R}^N) := \{u \in W^{1,p}(\mathbb{R}^N) : gu = u, \forall g \in O(N)\},$$

we can state the following crucial embedding result due to Kristály and principally based on a Strauss-type estimation (see [16]).

Lemma 2.4. *The embedding $W^{1,p}_r(\mathbb{R}^N) \hookrightarrow L^\infty(\mathbb{R}^N)$, is compact whenever $2 \le N < p < +\infty$.*

See [12, Theorem 3.1] for details. We also cite a recent monograph by Kristály, Rădulescu and Varga [13] and the classical book of Willem [17] as a reference for these topics.

For the sake of convenience, we define

$$F(t) = \int_0^t f(\xi)d\xi \quad \text{for all } t \in \mathbb{R}, \quad H(t) = \int_0^t h(\xi)d\xi \quad \text{for all } t \in \mathbb{R}.$$

3. Main results

Fix $\tau > 0$ such that

$$\kappa := \frac{\|\alpha\|_{B(0,\tau/2)}}{\omega_\tau\big(\frac{2p}{p-N}\big)^p\big\{\frac{\sigma(N,p)}{\tau^p} + l(p,N)\big\}\|\alpha\|_1} > 0 \tag{3.1}$$

where $\sigma(N,p) := 2^{p-N}(2^N - 1)$, as well as

$$l(p,N) := \frac{1 + 2^{N+p}NB_{(1/2,1)}(N,p+1)}{2^N}$$

in which $B_{(1/2,1)}(N, p+1)$ denotes the generalized incomplete beta function defined as follows:

$$B_{(1/2,1)}(N,p+1) := \int_{1/2}^1 t^{N-1}(1-t)^p dt.$$

We also note that $\omega_\tau := \mathrm{meas}(B(0,\tau)) = \tau^N \frac{\tau^{N/2}}{\Gamma(1+\frac{N}{2})}$, where Γ is the Gamma function defined by

$$\Gamma(t) := \int_0^{+\infty} z^{t-1}e^{-z}dz \quad (\forall t > 0).$$

To introduce our result, we fix three constants $c > 0$ and ζ such that
$$\frac{1}{\kappa F(\zeta)} > \frac{c^p}{\sup_{|t|\le c} F(t)}\,.$$
Taking
$$\lambda \in \Lambda := \Big]\frac{\zeta^p\omega_\tau}{\|\alpha\|_{B(0,\tau/2)}F(\zeta)}\Big\{\frac{\sigma(N,p)}{p\tau^p}+\frac{g(p,N)}{p}\Big\},\ \frac{c^p}{p(\frac{2p}{p-N})^p\|\alpha\|_1\max_{|t|\le c}F(t)}\Big[,$$
we set
$$\delta_{\lambda,h} := \min\Big\{\frac{c^p-\lambda\sup_{|t|\le c}F(t)}{H^c},\ \frac{1-\lambda F(\zeta)}{pH_\zeta}\Big\} \tag{3.2}$$
and
$$\overline{\delta}_{\lambda,h} := \min\Big\{\delta_{\lambda,g},\ \frac{1}{\max\{0,\|\beta\|_1\}\limsup_{|t|\to\infty}\frac{H(t)}{t^p}}\Big\}, \tag{3.3}$$
where we define $r/0 = +\infty$, so that, for instance, $\overline{\delta}_{\lambda,h} = +\infty$ when
$$\limsup_{|t|\to\infty}\frac{\|\beta\|_1 H(t)}{t^p} \le 0,$$
and $H_c = H^\zeta = 0$.

Now, we formulate our main result.

Theorem 3.1. *Assume that there exist constants $c > 0$ and $\zeta > 0$ with*
$$c^p < \Big(\frac{2p}{p-N}\Big)^p\zeta^p\omega_\tau\Big[\frac{\sigma(N,p)}{\tau^p}+g(p,N)\Big]$$
such that

- (A2) $\frac{\sup_{|t|\le c}F(t)}{c^p} < \kappa F(\zeta)$;
- (A3) $\limsup_{|t|\to+\infty}\frac{\|\alpha\|_1F(t)}{t^p} \le 0$.

Then, for each
$$\lambda \in \Lambda := \Big]\frac{\zeta^p\omega_\tau}{\|\alpha\|_{B(0,\tau/2)}F(\zeta)}\Big\{\frac{\sigma(N,p)}{p\tau^p}+\frac{g(p,N)}{p}\Big\},\ \frac{c^p}{p\big(\frac{2p}{p-N}\big)^p\|\alpha\|_1\max_{|t|\le c}F(t)}\Big[$$
and for every function $h:\mathbb{R}\to\mathbb{R}$ satisfying the condition
$$\limsup_{|t|\to\infty}\frac{\|\beta\|_1 H(t)}{t^p} < +\infty,$$
there exists $\overline{\delta}_{\lambda,h} > 0$ given by (3.3) such that, for each $\mu \in [0,\overline{\delta}_{\lambda,h}[$, problem (1.1) admits at least three distinct weak solutions in X.

Proof. To apply Theorem 2.1 to our problem, we introduce the functionals $\Phi,\Psi : X\to\mathbb{R}$ for each $u\in X$, as follows
$$\Phi(u) = \frac{1}{p}\|u\|_r^p,$$
$$\Psi(u) = \int_{\mathbb{R}^N}\Big[\alpha(x)F(u(x))+\frac{\mu}{\lambda}\beta(x)H(u(x))\Big]dx.$$
Now we show that the functionals Φ and Ψ satisfy the required conditions. It is well known that Ψ is a differentiable functional whose differential at the point $u\in X$ is
$$\Psi'(u)(v) = \int_{\mathbb{R}^N}[\alpha(x)f(u(x))+\frac{\mu}{\lambda}\beta(x)h(u(x))]v(x)dx,$$

for every $v \in X$, as well as, is sequentially weakly upper semicontinuous. Furthermore, $\Psi' : X \to X^*$ is a compact operator. Moreover, Φ is continuously differentiable and whose differential at the point $u \in X$ is

$$\Phi'(u)v = \int_{\mathbb{R}^N} |\nabla u(x)|^{p-2}\nabla u(x) \cdot \nabla v(x)dx + \int_{\mathbb{R}^N} |u(x)|^{p-2}u(x)v(x)dx,$$

for every $v \in X$, while by standard arguments, one has that Φ is Gâteaux differentiable and sequentially weakly lower semicontinuous, and its Gâteaux derivative Φ' admits a continuous inverse on X^*. Clearly, the weak solutions of the problem (1.1) are exactly the solutions of the equation $\Phi'(u) - \lambda\Psi'(u) = 0$. Put $r = \frac{c^p}{(\frac{2p}{p-N})^p p}$ and

$$w(x) = \begin{cases} 0, & x \in \mathbb{R}^N \setminus B(0,\tau), \\ \frac{2\zeta}{\tau}(\tau - |x|), & x \in B(0,\tau) \setminus B(0,\tau/2), \\ \zeta, & x \in B(0,\tau/2). \end{cases} \tag{3.4}$$

It is easy to see that $w \in X$ and

$$\|w\|_r^p = \zeta^p\omega_\tau\big[\frac{\sigma(N,p)}{\tau^p} + g(p,N)\big].$$

Indeed

$$\begin{aligned}\int_{\mathbb{R}^N} |\nabla w(x)|^p dx &= \int_{B(0,\tau)\setminus B(0,\tau/2)} \frac{2^p\zeta^p}{\tau^p} dx \\ &= \frac{2^p\zeta^p}{\tau^p}(\text{meas}(B(0,\tau))) - \text{meas}(B(0,\tau/2)) = \frac{2^{p-N}\zeta^p\omega_\tau}{\tau^p}(2^N - 1),\end{aligned}$$

and

$$\begin{gathered}\int_{\mathbb{R}^N} |w(x)|^p dx = \int_{B(0,\tau/2)} \zeta^p dx + \int_{B(0,\tau/2)} \frac{2^p\zeta^p}{\tau^p}(\tau - |x|)^p dx \\ \zeta^p\Big(\int_{B(0,\tau/2)} dx + \frac{2^p}{\tau^p}\int_{B(0,\tau)\setminus B(0,\tau/2)} (\tau - |x|)^p dx\Big) = \omega_\tau\zeta^p g(p,N).\end{gathered}$$

Note that the last equality holds owing to

$$I_p := \int_{B(0,\tau)\setminus B(0,\tau/2)} (\tau - |x|)^p dx = N\omega_\tau\tau^p B_{(1/2,1)}(N, p+1). \tag{3.5}$$

The easiest way to compute this integral is to go through a general polarcoordinates transformation. Let

$$\begin{gathered} x_1 = \rho\cos\theta_1, \\ x_j = \rho\cos\theta_1\cos\theta_2\cdots\cos\theta_{j_1}\sin\theta_j, \quad (j = 2,\cdots,N-1) \\ x_N = \rho\cos\theta_1\cos\theta_2\cdots\cos\theta_{N_1}, \quad \text{for } \rho \in [\bar{\mu}\tau,\tau],\ \theta_j \in (-\pi/2,\pi/2], \\ j = 1,\cdots,N-2 \text{ and } \theta_{N-1} \in (-\pi,\pi]. \end{gathered}$$

The Jacobian of this transformation is

$$dx_1\cdots dx_N = \rho^{N-1}\Big\{\prod_{j=1}^{N-1} |\cos\theta_j|^{N-j-1}\Big\}d\rho d\theta_1\cdots d\theta_{N-1}.$$

Hence, one has

$$I_p = \Big(\int_{\tau/2}^{\tau} (\tau - \rho)^p\rho^{N-1}d\rho\Big)\Big(\int_{-\pi}^{\pi} d\theta_{N-1}\Big)\prod_{j=1}^{N-2}\int_{-\pi/2}^{\pi/2} |\cos\theta_j|^{N-j-1}d\theta_j.$$

On the other hand, since

$$\prod_{j=1}^{N-2}\int_{-\pi/2}^{\pi/2}|\cos\theta_j|^{N-j-1}d\theta_j=\prod_{j=1}^{N-2}\Gamma\big(\frac{N-j}{2}\big)\Gamma\big(\frac{1}{2}\big)/\Gamma\big(\frac{N-j+1}{2}\big),$$

taking into account that

$$\prod_{j=1}^{N-2}\Gamma\big(\frac{N-j}{2}\big)\Gamma\big(\frac{1}{2}\big)/\Gamma\big(\frac{N-j+1}{2}\big)=\frac{N\pi^{N/2-1}}{2\Gamma(\frac{N}{2}+1)},$$

an elementary computation gives (3.5). Moreover, from the condition

$$c^p<\zeta^p\omega_\tau\Big[\frac{\sigma(N,p)}{\tau^p}+g(p,N)\Big]\Big(\frac{2p}{N-p}\Big)^p$$

one has $0<r<\Phi(w)$. Exploiting the embedding $X\hookrightarrow L^\infty(\mathbb{R}^N)$ (by relation (1.1)), one has $\max_{t\in\mathbb{R}^N}|v(t)|\leq c$ for all $v\in X$ such that $\|v\|_r^p<pr$ and it follows that

$$\begin{aligned}\sup_{u\in\Phi^{-1}(]-\infty,r])}\Psi(u)&=\sup_{u\in\Phi^{-1}(]-\infty,r])}\int_{\mathbb{R}^N}\big[\alpha(x)F(u(x))+\frac{\mu}{\lambda}\beta(x)H(u(x))\big]dx\\&\leq\int_{\mathbb{R}^N}\alpha(x)\sup_{|t|\leq c}F(t)dx+\frac{\mu}{\lambda}H^c.\end{aligned}$$

On the other hand, from the definition of Ψ, we infer

$$\begin{aligned}\Psi(w)&=\int_{\mathbb{R}^N}\alpha(x)F(w(x))dx+\frac{\mu}{\lambda}\int_{\mathbb{R}^N}\beta(x)H(w(x))dx\\&=\int_{\mathbb{R}^N}\alpha(x)F(w(x))dx+\frac{\mu}{\lambda}\int_{\mathbb{R}^N}\beta(x)H(w(x))dx\\&\geq\int_{\mathbb{R}^N}\alpha(x)F(w(x))dx+\|\beta\|_1\frac{\mu}{\lambda}\inf_{[0,\eta]}H\\&=\int_{\mathbb{R}^N}\alpha(x)F(w(x))dx+\|\beta\|_1\frac{\mu}{\lambda}H_\eta.\end{aligned}$$

Therefore, owing to Assumption (A2), we have

$$\begin{aligned}&\frac{\sup_{u\in\Phi^{-1}(]-\infty,r])}\Psi(u)}{r}\\&=\frac{\sup_{u\in\Phi^{-1}(]-\infty,r])}\int_{\mathbb{R}^N}[\alpha(x)F(u(x))+\frac{\mu}{\lambda}\beta(x)H(u(x))]dx}{r}\\&\leq\frac{\int_{\mathbb{R}^N}\sup_{|t|\leq c}\alpha(x)F(t)dx+\frac{\mu}{\lambda}\|\beta\|_1H^c}{\frac{c^p}{(\frac{2p}{N-p})^pp}}\end{aligned}\tag{3.6}$$

and

$$\begin{aligned}\frac{\Psi(w)}{\Phi(w)}&=\frac{\int_{\mathbb{R}^N}\alpha(x)F(w(x))dx+\frac{\mu}{\lambda}\int_{\mathbb{R}^N}\beta(x)H(w(x))dx}{\frac{\zeta^p\omega_\tau[\frac{\sigma(N,p)}{\tau^p}+g(p,N)]}{p}}\\&\geq\frac{\int_{\mathbb{R}^N}\alpha(x)F(w(x))dx+\|\beta\|_1\frac{\mu}{\lambda}H_\eta}{\frac{\zeta^p\omega_\tau[\frac{\sigma(N,p)}{\tau^p}+g(p,N)]}{p}}.\end{aligned}\tag{3.7}$$

Since $\mu < \delta_{\lambda,g}$, one has

$$\mu < \frac{\frac{c^p}{(\frac{2p}{N-p})^p p} - \lambda \int_{\mathbb{R}^N} \sup_{|t|\le c} \alpha(x)F(t)dx}{H^c},$$

which means

$$\frac{\int_{\mathbb{R}^N} \sup_{|t|\le c} \alpha(x)F(t)dx + \frac{\mu}{\lambda} H^c}{\frac{c^p}{(\frac{2p}{N-p})^p p}} < \frac{1}{\lambda}.$$

Furthermore,

$$\mu < \frac{\zeta^p \omega_\tau [\frac{\sigma(N,p)}{\tau^p} + g(p,N)] - p\lambda \int_{\mathbb{R}^N} \alpha(x)F(w(x))dx}{p\|\beta\|_1 H_\eta},$$

and this means

$$\frac{\int_{\mathbb{R}^N} \alpha(x)F(w(x))dx + \|\beta\|_1 \frac{\mu}{\lambda} H_\eta}{\frac{\zeta^p \omega_\tau \left[\frac{\sigma(N,p)}{\tau^p} + g(p,N)\right]}{p}} > \frac{1}{\lambda}.$$

Then

$$\frac{\int_{\mathbb{R}^N} \sup_{|t|\le c} \alpha(x)F(t)dx + \frac{\mu}{\lambda} H^c}{\frac{c^p}{(\frac{2p}{N-p})^p p}} < \frac{1}{\lambda} < \frac{\int_{\mathbb{R}^N} \alpha(x)F(w(x))dx + \|\beta\|_1 \frac{\mu}{\lambda} H_\eta}{\frac{\zeta^p \omega_\tau [\frac{\sigma(N,p)}{\tau^p} + g(p,N)]}{p}}. \quad (3.8)$$

Hence from (3.6)-(3.8), the condition (a1) of Theorem 2.1 is verified.

Finally, since $\mu < \overline{\delta}_{\lambda,g}$, we can fix $l > 0$ such that

$$\limsup_{|t|\to\infty} \frac{\|\beta\|_1 H(t)}{t^p} < l$$

and $\mu l < \frac{1}{p(\frac{2p}{N-p})^p \|\alpha\|_1}$. Therefore, there exists a function $q \in L^1(\mathbb{R}^N)$ such that

$$\|\beta\|_1 H(t) \le l t^p + q(x) \quad \text{for all } x \in \mathbb{R}^N \text{ and for all } t \in \mathbb{R}.$$

Now, fix

$$0 < \epsilon < \frac{1 - p\mu l \left(\frac{2p}{N-p}\right)^p \|\alpha\|_1}{p\lambda \left(\frac{2p}{N-p}\right)^p \|\alpha\|_1}.$$

From (A3) there is a function $q_\epsilon \in L^1(\mathbb{R}^N)$ such that

$$\|\alpha\|_1 F(t) \le \epsilon t^p + q_\epsilon(x) \quad \text{for all } x \in \mathbb{R}^N \text{and for all } t \in \mathbb{R}.$$

It follows that, for each $u \in X$,

$$\begin{aligned} &\Phi(u) - \lambda\Psi(u) \\ &= \frac{1}{p}\|u\|^p - \lambda \int_{\mathbb{R}^N} [\alpha(x)F(u(x)) + \frac{\mu}{\lambda}\beta(x)H(u(x))]dx \\ &\ge \Big(\frac{1}{p} - \lambda\varepsilon\big(\frac{2p}{N-p}\big)^p \|\alpha\|_1 - \nu l \big(\frac{2p}{N-p}\big)^p \|\alpha\|_1\Big)\|u\|^p - \lambda\|q_\epsilon\|_1 - \mu\|q\|_1, \end{aligned}$$

and thus

$$\lim_{\|u\|\to+\infty} (\Phi(u) - \lambda\Psi(u)) = +\infty,$$

There was a strange symbol between $\frac{1}{p}$ and λ; so am not sure that $-$ is the correct symbol

which means the functional $\Phi - \lambda\Psi$ is coercive, and the condition (a2) of Theorem 2.1 is satisfied. Since, from (3.6) and (3.8),

$$\lambda \in \Big]\frac{\Phi(w)}{\Psi(w)}, \frac{r}{\sup_{\Phi(x)\le r} \Psi(x)}\Big[,$$

Theorem 2.1, with $\overline{x} = w$, assures the existence of three critical points for the functional $\Phi - \lambda\Psi$, and the proof is complete. $\square$

Remark 3.2. The methods used here can be applied studying discrete boundary value problems as in [8], and also non-smooth variational problems as in [15].

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