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ANISOTROPIC SINGULARITY OF SOLUTIONS TO ELLIPTIC EQUATIONS IN A MEASURE FRAMEWORK

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ABSTRACT. In this article we study the weak solutions of elliptic equation

$$\begin{gathered}-\Delta u = 2\frac{\partial\delta_0}{\partial\nu} \quad \text{in } \Omega,\\ u = 0 \quad \text{on } \partial\Omega,\end{gathered}$$

where Ω is an open bounded C^2 domain of $\mathbb{R}^N$ with $N \geq 2$ containing the origin, ν is a unit vector and $\frac{\partial\delta_0}{\partial\nu}$ is defined in the distribution sense, i.e.

$$\langle\frac{\partial\delta_0}{\partial\nu}, \zeta\rangle = \frac{\partial\zeta(0)}{\partial\nu}, \quad \forall\zeta \in C_0^1(\Omega).$$

We prove that this problem admits a unique weak solution u in the sense that

$$\int_\Omega u(-\Delta)\xi dx = 2\frac{\partial\xi(0)}{\partial\nu}, \quad \forall\xi \in C_0^2(\Omega).$$

Moreover, u has an anisotropic singularity and can be approximated, as $t \to 0^+$, by the solutions of

$$\begin{gathered}-\Delta u = \frac{\delta_{t\nu} - \delta_{-t\nu}}{t} \quad \text{in } \Omega,\\ u = 0 \quad \text{on } \partial\Omega.\end{gathered}$$

1. Introduction

The simplest and the most important Laplacian equation

$$-\Delta u = \delta_0 \quad \text{in } \mathbb{R}^N \tag{1.1}$$

comes up in a wide variety of physical contexts. In a typical interpretation, δ_0 denotes the electrostatic particle and u does the electrostatic potential. The unique solution of (1.1) is called fundamental solution of

$$-\Delta u = 0 \quad \text{in } \mathbb{R}^N \setminus \{0\}. \tag{1.2}$$

It is well known that the fundamental solution is

$$\Gamma(x) = \begin{cases} c_0|x|^{2-N} & \text{for } N \geq 3,\\ -c_0\log(|x|) & \text{for } N = 2,\end{cases}$$

where $c_0 > 0$, has isotropic singularity, i.e. $\Gamma \to +\infty$ from any direction near the origin. This kind of particle is called isotropic source.

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In contrast with electrostatic particle, the magnetic particle (we do not focus on the generation) has totally different phenomena: given a magnetic particle in the origin, we have to put its polar direction ν, if we denote by u the magnetic potential, then it could be observed that u would tend to $+\infty$ at the origin from the direction ν, but to $-\infty$ at the origin from the direction $-\nu$. We call this phenomena as anisotropic singularity from the mathematical point and magnetic particle as anisotropic source. Our aim of this paper is to study the anisotropy singular phenomena in partial differential equations.

Let Ω be a bounded C^2 domain in $\mathbb{R}^N$ with $N \geq 2$ containing the origin, δ_0 be the Dirac mass concentrated at the origin. Our purpose in this article is to investigate the weak solution to semilinear elliptic problem

$$\begin{gathered} -\Delta u = 2\frac{\partial \delta_0}{\partial \nu} \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.3}$$

where ν is a unit vector in $\mathbb{R}^N$, Δ denotes the Laplacian operator and $\frac{\partial \delta_0}{\partial \nu}$ is defined in the distribution sense that

$$\langle \frac{\partial \delta_0}{\partial \nu}, \xi \rangle = \frac{\partial \xi(0)}{\partial \nu}, \quad \forall \xi \in C_0^1(\Omega).$$

It is worth mentioning that

$$2\frac{\partial \delta_0}{\partial \nu} = \frac{\partial \delta_0}{\partial \nu} + \big[-\frac{\partial \delta_0}{\partial(-\nu)}\big],$$

which shows that the anisotropic source $2\frac{\partial \delta_0}{\partial \nu}$ consists by two directions sources, so we may call it as dipole source.

Before starting our main results in this paper, we introduce the definition of the weak solution to (1.3).

Definition 1.1. A measurable function u is a weak solution of (1.3) if $u \in L^1(\Omega)$ and

$$\int_\Omega u(-\Delta)\xi dx = 2\frac{\partial \xi(0)}{\partial \nu}, \quad \forall \xi \in C_0^2(\Omega). \tag{1.4}$$

Now we are ready to state our main theorem on the existence, uniqueness and asymptotic behavior of weak solutions for (1.3).

Theorem 1.2. *Assume that Ω is a bounded C^2 domain in $\mathbb{R}^N$ with $N \geq 2$ containing the origin, δ_0 denotes the Dirac mass concentrated at the origin, ν is a unit vector in $\mathbb{R}^N$.*

Then (1.3) *admits a unique weak solution u, which has following asymptotic behavior at the origin*

$$\lim_{t\to 0^+} \frac{u(te)}{P_\nu(te)} = 1 \quad \text{for } e \in \partial B_1(0),\ e \cdot \nu \neq 0, \tag{1.5}$$

where

$$P_\nu(x) = c_N \frac{x \cdot \nu}{|x|^N}, \quad \forall x \in \mathbb{R}^N \setminus \{0\} \tag{1.6}$$

with

$$c_N = \frac{2}{|\partial B_1(0)|} > 0.$$

We notice that the weak solution u of (1.3) with $\Omega = B_1(0)$ has to change signs. Indeed, letting ξ be the solution of

$$\begin{gathered} -\Delta u = 1 \quad \text{in } B_1(0), \\ u = 0 \quad \text{on } \partial B_1(0), \end{gathered}$$

we observe that $\frac{\partial \xi(0)}{\partial \nu} = 0$, if u keeps nonnegative, a contradiction is obtained from (1.4). Furthermore, thanks to (1.5), the solution u inherits the anisotropic singularity of P_ν and we prove that P_ν is a weak solution of

$$-\Delta u = 2\frac{\partial \delta_0}{\partial \nu} \quad \text{in } \mathbb{R}^N.$$

For more on anisotropic singularities results, we refer to [6, 14]. The proof of Theorem 1.2 is addressed in Section 2.

In Section 3, we approximate the weak solution u by weak solutions of

$$\begin{gathered} -\Delta u = \mu_t \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.7}$$

where $\mu_t = \frac{\delta_{t\nu} - \delta_{-t\nu}}{t}$. The existence and uniqueness of weak solution of (1.7) could see the references [1, 3, 13]. We remark that the source μ_t consists of isotropic source. But the limit of $\{\mu_t\}$ as $t \to 0^+$ is $2\frac{\partial \delta_0}{\partial \nu}$, which is an anisotropic source.

In Section 4 we consider the weak solution of elliptic equations with multipole source, which consists of a multipole sources by addressing in one point, i.e.

$$\partial_n \delta_0 = \sum_{i=0}^{n-1} 2\frac{\partial \delta_0}{\partial \nu_i},$$

where $n \in \mathbb{N}$ and ν_i is unit vector in $\mathbb{R}^N$ with $i = 0, 1, \dots, n-1$. Here $\partial_n \delta_0$ could be called a multipole source. In particular case that $N = 2$, we are interested in the period of the corresponding weak solution to elliptic equation with multipole source. Precisely, we may obtain a $\frac{2\pi}{n}$-period singularities of solution to (1.3), if n is odd and the dipole source is replaced by a proper multipole source

$$\nu_i = \Big(\cos(\frac{2i\pi}{n}), \sin(\frac{2i\pi}{n})\Big).$$

2. Proof of Theorem 1.2

To prove Theorem 1.2, we analyzing the function P_ν, and for be convenience we let $\nu = e_N := (0, \dots, 0, 1)$ in this section. Also for convenience, we abbreviate P_{e_N} by P_N, and $\frac{\partial \delta_0}{\partial e_N}$ by $\frac{\partial \delta_0}{\partial x_N}$.

Proposition 2.1. *Let*

$$P_N(x) = c_N \frac{x_N}{|x|^N}, \quad \forall x \in \mathbb{R}^N \setminus \{0\},$$

where $c_N = \frac{2}{|\partial B_1(0)|}$. Then the function P_N is the unique weak solution of

$$\begin{gathered} -\Delta u = 2\frac{\partial \delta_0}{\partial x_N} \quad \textit{in } \mathbb{R}^N, \\ u(x) \to 0 \quad \textit{as } \ |x| \to \infty; \end{gathered} \tag{2.1}$$

that is,
$$\int_{\mathbb{R}^N} P_N(-\Delta)\xi dx = 2\frac{\partial \xi(0)}{\partial x_N}, \quad \forall \xi \in C_0^2(\mathbb{R}^N).$$

Proof. (Existence) By direct computation, we derive that P_N is a classical solution of
$$-\Delta u = 0 \quad \text{in } \mathbb{R}^N \setminus \{0\}.$$
Thus, for $\epsilon > 0$ and $\xi \in C_0^2(\mathbb{R}^N)$,
$$\begin{aligned}
0 &= \int_{\mathbb{R}^N\setminus B_\epsilon(0)} (-\Delta)P_N \xi dx \\
&= \int_{\mathbb{R}^N\setminus B_\epsilon(0)} \nabla P_N \cdot \nabla \xi dx - \int_{\partial B_\epsilon(0)} \frac{\partial P_N}{\partial \vec{n}} \xi dS(x) \\
&= \int_{\mathbb{R}^N\setminus B_\epsilon(0)} P_N(-\Delta)\xi dx + \int_{\partial B_\epsilon(0)} \frac{\partial \xi}{\partial \vec{n}} P_N dS(x) - \int_{\partial B_\epsilon(0)} \frac{\partial P_N}{\partial \vec{n}} \xi dS(x),
\end{aligned} \tag{2.2}$$
where $\vec{n}$ is a unit normal vector pointing outward of $\mathbb{R}^N \setminus B_\epsilon(0)$. We claim that
$$\lim_{\epsilon\to 0} \int_{\partial B_\epsilon(0)} \frac{\partial P_N}{\partial \vec{n}} \xi dS(x) = \frac{2(N-1)}{N} \frac{\partial \xi(0)}{\partial x_N} \tag{2.3}$$
and
$$\lim_{\epsilon\to 0} \int_{\partial B_\epsilon(0)} \frac{\partial \xi}{\partial \vec{n}} P_N dS(x) = -\frac{2}{N} \frac{\partial \xi(0)}{\partial x_N}. \tag{2.4}$$
Indeed, since $\xi \in C_0^2(\mathbb{R}^N)$, then for $|x|$ small,
$$\begin{gathered}
\xi(x) = \xi(0) + \nabla\xi(0)\cdot x + O(|x|^2), \\
\nabla\xi(x) = \nabla\xi(0) + O(|x|).
\end{gathered}$$
Moreover, for $x \in \partial B_\epsilon(0)$, there holds that
$$\vec{n}_x = -\frac{x}{|x|}, \quad \nabla P_N(x)\cdot \vec{n}_x = c_N(N-1)\frac{x_N}{|x|^{N+1}},$$
then we have
$$\begin{aligned}
&\int_{\partial B_\epsilon(0)} \frac{\partial P_N}{\partial \vec{n}} \xi dS(x) \\
&= c_N(1-N)\epsilon^{-N-1} \int_{\partial B_\epsilon(0)} x_N[\xi(0) + \nabla\xi(0)\cdot x + O(|x|^2)] dS(x) \\
&= c_N(N-1)\frac{\partial \xi(0)}{\partial x_N} \epsilon^{-N-1} \Big[\int_{\partial B_\epsilon(0)} x_N^2 dS(x) + O(1) \int_{\partial B_\epsilon(0)} |x|^3 dS(x) \Big] \\
&= c_N(N-1)\frac{\partial \xi(0)}{\partial x_N} \Big[\int_{\partial B_1(0)} x_N^2 dS(x) + O(1)\epsilon \Big] \\
&= c_N(N-1)\frac{\partial \xi(0)}{\partial x_N} \Big[\frac{|\partial B_1(0)|}{N} + O(1)\epsilon \Big] \\
&\to \frac{2(N-1)}{N} \frac{\partial \xi(0)}{\partial x_N} \quad \text{as } \epsilon \to 0^+
\end{aligned}$$
and
$$\int_{\partial B_\epsilon(0)} \frac{\partial \xi}{\partial \vec{n}} P_N dS(x) = c_N \epsilon^{-N} \int_{\partial B_\epsilon(0)} x_N[-\nabla\xi(0)\cdot \frac{x}{|x|} + O(|x|)] dS(x)$$

$$
\begin{aligned}
&= -c_N \frac{\partial \xi(0)}{\partial x_N} \epsilon^{-N-1} \Big[\int_{\partial B_\epsilon(0)} x_N^2 dS(x) + O(1)\epsilon \Big] \\
&\to -\frac{2}{N} \frac{\partial \xi(0)}{\partial x_N} \quad \text{as } \epsilon \to 0^+,
\end{aligned}
$$

which imply (2.3) and (2.4). Passing to the limit in (2.2) as $\epsilon \to 0^+$, we obtain that P_N is a weak solution of (2.1).

(Uniqueness) Let P be a weak solution of (2.1) and then $w := P - P_N$ is a weak solution to

$$
\begin{gathered}
-\Delta u = 0 \quad \text{in } \mathbb{R}^N, \\
u(x) \to 0 \quad \text{as } |x| \to \infty
\end{gathered}
$$

Let $\{\eta_n\} \subset C_0^\infty(\mathbb{R}^N)$ be a sequence of radially decreasing and symmetric mollifiers such that $\operatorname{supp}(\eta_n) \subset B_{\varepsilon_n}(0)$ with $\varepsilon_n \le \frac{1}{n}$ and $w_n = w * \eta_n$. We observe that

$$
w_n \to w \quad \text{ a.e. in } \mathbb{R}^N \text{ and in } L^1_{\rm loc}(\mathbb{R}^N) \text{ as } n \to \infty. \tag{2.5}
$$

By the Fourier transformation, we have

$$
\eta_n * (-\Delta)\xi = (-\Delta)(\xi * \eta_n);
$$

then

$$
\int_{\mathbb{R}^N} w(-\Delta)(\xi * \eta_n) dx = \int_{\mathbb{R}^N} w * \eta_n (-\Delta)\xi dx.
$$

It follows that w_n is a classical solution of

$$
\begin{gathered}
-\Delta u = 0 \quad \text{in } \mathbb{R}^N, \\
u(x) \to 0 \quad \text{as } |x| \to \infty.
\end{gathered} \tag{2.6}
$$

By Maximum Principle, (2.6) has only zero as a classical solution. Therefore, we have $w_n \equiv 0$ in $\mathbb{R}^N$. Thanks to (2.5), we have $w = 0$ a.e. in $\mathbb{R}^N$. This completes the proof $\square$

Remark 2.2 ([8])**.** Let $\mathbb{R}^N_+ = \mathbb{R}^{N-1} \times \mathbb{R}_+$ and $\mathbb{R}^N_- = \mathbb{R}^{N-1} \times \mathbb{R}_-$. Then $P_+ := P_N$ in $\bar{\mathbb{R}}^N_+$ is a weak solution of

$$
\begin{gathered}
-\Delta u = 0 \quad \text{in } \mathbb{R}^N_+, \\
u = \delta_0 \quad \text{on } \mathbb{R}^{N-1} \times \{0\}
\end{gathered} \tag{2.7}
$$

and $P_- := P_N$ in $\bar{\mathbb{R}}^N_-$ is a weak solution of

$$
\begin{gathered}
-\Delta u = 0 \quad \text{in } \mathbb{R}^N_-, \\
u = -\delta_0 \quad \text{on } \mathbb{R}^{N-1} \times \{0\}.
\end{gathered} \tag{2.8}
$$

Here the definitions of weak solution are give as

$$
\int_{\mathbb{R}^N_\pm} P_\pm (-\Delta)\zeta dx = \frac{\partial \zeta(0)}{\partial x_N}, \quad \forall \zeta \in C_0^2(\mathbb{R}^N_\pm).
$$

This indicates that the weak solution of (2.1) could be joint to the weak solutions of (2.7) and (2.8).

We are ready to prove Theorem 1.2 by using the function P_N.

Proof of Theorem 1.2. (Existence) Without loss of generality, we prove only the case $\nu = e_N$. Let $\eta : \mathbb{R}^N \to \mathbb{R}$ be a C^∞ nonnegative function such that

$$\eta = \begin{cases} 1 & \text{in } B_{\sigma_0}(0), \\ 0 & \text{in } \mathbb{R}^N \setminus B_{2\sigma_0}(0), \end{cases}$$

where $\sigma_0 > 0$, in the throughout of this paper, is a positive number such that $B_{3\sigma_0}(0) \subset \Omega$. Denote

$$W = \eta P_N \quad \text{in } \mathbb{R}^N.$$

We notice that $-\Delta W = 0$ in $B_{\sigma_0}(0) \setminus \{0\}$ and $\Omega \setminus B_{2\sigma_0}(0)$; thus, denoting $f = \Delta W$ in $\mathbb{R}^N \setminus \{0\}$ and $f(0) = 0$, one has that $f \in C_0^1(\Omega)$. It is well-known that there exists a unique solution $w \in C^2(\Omega) \cap C(\bar{\Omega})$ to problem

$$\begin{gathered} -\Delta u = f \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega. \end{gathered}$$

Next we prove that $u = w + W$ is a weak solution of (1.3). It is obvious that

$$-\Delta u = 0 \quad \text{in } \Omega \setminus \{0\},$$

which implies that for $\xi \in C_0^2(\Omega)$ and $\epsilon \in (0, \sigma_0)$,

$$\begin{aligned} 0 &= \int_{\Omega \setminus B_\epsilon(0)} (-\Delta) u \, \xi dx \\ &= \int_{\Omega \setminus B_\epsilon(0)} u(-\Delta)\xi dx + \int_{\partial B_\epsilon(0)} \frac{\partial \xi}{\partial \vec{n}} u dS(x) - \int_{\partial B_\epsilon(0)} \frac{\partial u}{\partial \vec{n}} \xi dS(x), \\ &= \int_{\Omega \setminus B_\epsilon(0)} u(-\Delta)\xi dx + \int_{\partial B_\epsilon(0)} \frac{\partial \xi}{\partial \vec{n}} P_N dS(x) - \int_{\partial B_\epsilon(0)} \frac{\partial P_N}{\partial \vec{n}} \xi dS(x) \\ &\quad + \int_{\partial B_\epsilon(0)} \frac{\partial \xi}{\partial \vec{n}} w dS(x) - \int_{\partial B_\epsilon(0)} \frac{\partial w}{\partial \vec{n}} \xi dS(x). \end{aligned} \tag{2.9}$$

Since $w \in C_0^2(\Omega)$, it follows that

$$\lim_{\epsilon \to 0^+} \Big[\int_{\partial B_\epsilon(0)} \frac{\partial \xi}{\partial \vec{n}} w dS(x) - \int_{\partial B_\epsilon(0)} \frac{\partial w}{\partial \vec{n}} \xi dS(x) \Big] = 0$$

and by (2.3) and (2.4), one has that

$$\lim_{\epsilon \to 0^+} \Big[\int_{\partial B_\epsilon(0)} \frac{\partial \xi}{\partial \vec{n}} P_N dS(x) - \int_{\partial B_\epsilon(0)} \frac{\partial P_N}{\partial \vec{n}} \xi dS(x) \Big] = -2 \frac{\partial \xi(0)}{\partial x_N}.$$

Thus, passing to the limit in (2.9) as $\epsilon \to 0$, we obtain

$$\int_\Omega u(-\Delta)\xi dx = 2 \frac{\partial \xi(0)}{\partial x_N}, \quad \forall \xi \in C_0^2(\Omega).$$

(Uniqueness) Let v be a weak solution of (2.1). Then $\varphi := u - v$ is a weak solution of

$$\begin{gathered} -\Delta \varphi = 0 \quad \text{in } \Omega, \\ \varphi = 0 \quad \text{on } \partial\Omega. \end{gathered}$$

By Kato's inequality [13, Theorem 2.4] (see also [7]), we have that $\varphi = 0$ a.e. in Ω.

Now we prove (1.5). Since w is $C_0^2(\Omega)$ and

$$\lim_{t \to 0^+} W(te) \operatorname{sign}(e \cdot \nu) = +\infty \quad \text{for } e \in \partial B_1(0),\ e \cdot \nu \neq 0,$$

this implies (1.5) by the fact $u = w + W$. The proof is complete $\square$

3. Approximations

In this section, we prove that the anisotropic source could be approximated by isotropy sources. Denote

$$\mu_t = \frac{\delta_{te_N} - \delta_{-te_N}}{t}, \tag{3.1}$$

where $t \in (0, \sigma_0/2)$ and $e_N = (0, \dots, 0, 1)$. From [13, Theorem 3.7] there exists a unique solution u_t to problem

$$\begin{gathered} -\Delta u = \mu_t \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{3.2}$$

In fact, u_t could be expressed by Green's function G_Ω as follows

$$u_t(x) = \int_\Omega G_\Omega(x, y) d\mu_t(y) = \frac{G_\Omega(x, te_N) - G_\Omega(x, -te_N)}{t}. \tag{3.3}$$

Proposition 3.1. *Assume that Ω is a bounded C^2 domain in $\mathbb{R}^N$ containing the origin, μ_t given in (3.1) with $t \in (0, \sigma_0/2)$, u_t is the unique weak solution of (3.2) and u is the unique weak solution of (1.3), where $\sigma_0 > 0$ such that $B_{3\sigma_0}(0) \subset \Omega$. Then*

$$u_t \to u \quad \text{a.e. in } \Omega \text{ and in } L^p(\Omega) \text{ as } t \to 0^+,$$

where $p \in [1, \frac{N}{N-1})$. Moreover,

$$u(x) = 2\frac{\partial G_\Omega(x, 0)}{\partial x_N}, \quad \forall x \in \Omega \setminus \{0\}.$$

In the proof of this proposition, we use (3.3) to get the converge $u_t \to u$ almost every where in Ω and the Marcinkiewicz estimates for the converge in $L^p(\Omega)$ with $p \in [1, \frac{N}{N-1})$. To this end, we introduce following lemmas.

Lemma 3.2. *Let Ω be a bounded C^2 domain in $\mathbb{R}^N$ containing the origin and μ_t given in (3.1) with $t \in (0, \sigma_0/2)$, then*

$$\mu_t \rightharpoonup 2\frac{\partial \delta_0}{\partial x_N} \quad \text{as } t \to 0^+$$

in the sense that

$$\lim_{t \to 0^+} \langle \mu_t, \xi \rangle = 2\frac{\partial \xi(0)}{\partial x_N}, \quad \forall \xi \in C_0^1(\Omega).$$

Proof. For $\xi \in C_0^2(\Omega)$, we have

$$\langle \mu_t, \xi \rangle = \frac{\langle \delta_{te_N}, \xi \rangle - \langle \delta_{-te_N}, \xi \rangle}{t} = \frac{\xi(te_N) - \xi(-te_N)}{t}$$

and

$$\begin{aligned} \lim_{t \to 0^+} \frac{\xi(te_N) - \xi(-te_N)}{t} &= \lim_{t \to 0^+} \frac{\xi(te_N) - \xi(0)}{t} + \lim_{t \to 0^+} \frac{\xi(0) - \xi(-te_N)}{t} \\ &= 2\frac{\partial \xi(0)}{\partial x_N}, \end{aligned}$$

which completes the proof. $\square$

Lemma 3.3. *Let Ω be a bounded C^2 domain in $\mathbb{R}^N$ containing the origin, μ_t be given by* (3.1) *with $t \in (0, \sigma_0/2)$, and u_t be the unique weak solution of* (3.2)*. Then*

$$\lim_{t\to 0^+} u_t(x) = 2\frac{\partial G_\Omega(x,0)}{\partial x_N}, \quad \forall x \in \Omega \setminus \{0\}.$$

Proof. For $x \in \Omega \setminus \{0\}$, $G_\Omega(x,\cdot)$ is C^2 in $\{te_N, t \in (0, |x|/2)\}$, and

$$\lim_{t\to 0^+} \frac{G_\Omega(x, te_N) - G_\Omega(x, -te_N)}{t} = 2\frac{\partial G_\Omega(x,0)}{\partial y_N} = 2\frac{\partial G_\Omega(x,0)}{\partial x_N}.$$

Along with (3.3), we obtain

$$\lim_{t\to 0^+} u_t(x) = 2\frac{\partial G_\Omega(x,0)}{\partial x_N}, \quad \forall x \in \Omega \setminus \{0\}.$$

This completes the proof. □

Before starting the Marcinkiewicz estimate, we recall some definitions and properties of Marcinkiewicz spaces.

Definition 3.4. Let $\Theta \subset \mathbb{R}^N$ be a domain and μ be a positive Borel measure in Θ. For $\kappa > 1$, $\kappa' = \kappa/(\kappa - 1)$ and $u \in L^1_{\rm loc}(\Theta, d\mu)$, we set

$$\|u\|_{M^\kappa(\Theta,d\mu)} = \inf\Big\{c \in [0,\infty] : \int_E |u| d\mu \le c\Big(\int_E d\mu\Big)^{1/\kappa'}, \ \forall E \subset \Theta, \ E \text{ Borel}\Big\} \tag{3.4}$$

$$M^\kappa(\Theta, d\mu) = \{u \in L^1_{\rm loc}(\Theta, d\mu) : \|u\|_{M^\kappa(\Theta,d\mu)} < \infty\}. \tag{3.5}$$

Here $M^\kappa(\Theta, d\mu)$ is called the Marcinkiewicz space of exponent κ, or weak L^κ-space and $\|\cdot\|_{M^\kappa(\Theta,d\mu)}$ is a quasi-norm.

Proposition 3.5 ([2, 5])**.** *Assume that $1 \le q < \kappa < \infty$ and $u \in L^1_{\rm loc}(\Theta, d\mu)$. Then there exists $c_3 > 0$ dependent of q, κ such that*

$$\int_E |u|^q d\mu \le c_3 \|u\|_{M^\kappa(\Theta,d\mu)} \Big(\int_E d\mu\Big)^{1-q/\kappa},$$

for any Borel subset E of Θ.

The next estimate plays an important role in $u_t \to u$ in $L^p(\Omega)$ with $p \in [1, \frac{N}{N-1})$.

Lemma 3.6. *Assume that $\Omega \subset \mathbb{R}^N$ $(N \ge 2)$ is a bounded C^2 domain containing the origin and*

$$\mathbb{G}_\Omega[\mu_t](x) = \int_\Omega G_\Omega(x,y) d\mu_t(y).$$

(i) For $N \ge 3$ there exists $c_1 > 0$ such that

$$\|\mathbb{G}_\Omega[\mu_t]\|_{M^{\frac{N}{N-1}}(\Omega,dx)} \le c_1;$$

(ii) for $N = 2$, for any $\sigma \in (0, \frac12)$, there exists $c_\sigma > 0$ such that

$$\|\mathbb{G}_\Omega[\mu_t]\|_{M^{\frac{2}{1+\sigma}}(\Omega,dx)} \le c_\sigma.$$

Proof. We observe that for $x, y \in \Omega$ with $x \ne y$,

$$G_\Omega(x,y) = \begin{cases} c_0|x-y|^{2-N} + \Gamma_\Omega(x,y) & \text{if } N \ge 3, \\ -c_0 \log|x-y| + \Gamma_\Omega(x,y) & \text{if } N = 2, \end{cases}$$

where Γ_Ω is a C^2 and harmonic function.

For any $t \in (0, \sigma_0/2)$, we divide the domain Ω into

$$O_t := \{x \in \Omega : |x| < t/2\} \quad \text{and} \quad Q_t := \{x \in \Omega : |x| \geq t/2\},$$

then for $N \geq 3$ and $x \in O_t \setminus \{0\}$,

$$\begin{aligned}
|\mathbb{G}_\Omega[\mu_t](x)| &= |\frac{G_\Omega(x, te_N) - G_\Omega(x, -te_N)}{t}| \\
&\leq c_2\Big[|\frac{|x - te_N|^{2-N} - |x + te_N|^{2-N}}{t}| + 1\Big] \\
&\leq c_3\Big[|\frac{\partial |x|^{2-N}}{\partial x_N}| + 1\Big] \\
&\leq 2c_3(N-2)|x|^{1-N}
\end{aligned}$$

and for $x \in Q_t$,

$$\begin{aligned}
|\mathbb{G}_\Omega[\mu_t](x)| &= |\frac{G_\Omega(x, te_N) - G_\Omega(x, -te_N)}{t}| \\
&\leq c_4 \frac{|x - te_N|^{2-N} + |x + te_N|^{2-N}}{t} \\
&\leq 2c_4 \frac{|x - te_N|^{2-N} + |x + te_N|^{2-N}}{|x|},
\end{aligned}$$

where $c_2, c_3, c_4 > 0$. Therefore, for some $c_5 > 0$,

$$|\mathbb{G}_\Omega[\mu_t](x)| \leq c_5\Big[|x|^{1-N} + \frac{|x - te_N|^{2-N} + |x + te_N|^{2-N}}{|x|}\Big], \quad \forall x \in \Omega \setminus \{0\}. \quad (3.6)$$

For $N = 2$, we obtain that for some $c_6 > 0$,

$$|\mathbb{G}_\Omega[\mu_t](x)| \leq c_6\Big[|x|^{1-N} + \frac{|\log(|x - te_N|)| + |\log(x + te_N)|}{|x|}\Big], \quad \forall x \in \Omega \setminus \{0\}. \quad (3.7)$$

Let E be a Borel subset of Ω, then there exists $r_E > 0$ such that $|E| = |B_{r_E}(0)|$. Therefore, for $N \geq 3$, we deduce that

$$\begin{aligned}
\int_E |\mathbb{G}_\Omega[\mu_t](x)|\,dx &\leq c_5 \int_E \Big(|x|^{1-N} + \frac{|x - te_N|^{2-N} + |x + te_N|^{2-N}}{|x|}\Big)\,dx \\
&\leq c_5 \int_{B_{r_E}(0)} |x|^{1-N} dx + c_5 r_E^{-1} \int_{B_{r_E}(te_N)} |x - te_N|^{2-N}\,dx \\
&\quad + c_5 r_E^{-1} \int_{B_{r_E}(-te_N)} |x + te_N|^{2-N}\,dx + 2c_5 r_E^{2-N} \int_{B_{r_E}(0)} |x|^{-1}\,dx \\
&\leq c_7 r_E = c_8 |B_{r_E}(0)|^{\frac{1}{N}} \\
&= c_8 |E|^{\frac{1}{N}},
\end{aligned}$$

where $c_7, c_8 > 0$. This implies that

$$\|\mathbb{G}_\Omega[\mu_t]\|_{M^{\frac{N}{N-1}}(\Omega, dx)} \leq c_8.$$

For $N = 2$, we assume that $r_E \in (0, 1/2)$,

$$\begin{aligned}
&\int_E |\mathbb{G}_\Omega[\mu_t](x)|\,dx \\
&\leq c_6 \int_E \Big(|x|^{-1} + \frac{|\log(|x - te_N|)| + |\log(|x + te_N|)|}{|x|}\Big)\,dx
\end{aligned}$$

$$
\begin{aligned}
&\le c_6 \int_{B_{r_E}(0)} |x|^{-1} dx + c_6 r_E^{-1} \int_{B_{r_E}(te_N)} |\log(|x-te_N|)|\, dx \\
&\quad + c_6 r_E^{-1} \int_{B_{r_E}(-te_N)} |\log(|x-te_N|)|\, dx + 2c_6 |\log r_E| \int_{B_{r_E}(0)} |x|^{-1}\, dx \\
&\le c_9 r_E [-\log(r_E)] \\
&\le c_{10} |B_{r_E}(0)|^{\frac{1}{2}} [-\log(|B_{r_E}(0)|)]
\end{aligned}
$$

where $c_9, c_{10} > 0$. Then for any $\sigma \in (0, \frac{1}{2})$, there exists $c_\sigma > 0$ such that

$$
\int_E |u_t(x)| dx \le c_\sigma |B_{r_E}(0)|^{\frac{1-\sigma}{2}} = c_\sigma |E|^{\frac{1-\sigma}{2}},
$$

which implies

$$
\|\mathbb{G}_\Omega[\mu_t]\|_{M^{\frac{2}{1+\sigma}}(\Omega, dx)} \le c_\sigma.
$$

This ends the proof. □

Proof of Proposition 3.1. We observe that u_t is the unique weak solution of (3.2); that is,

$$
\int_\Omega u_t(-\Delta)\xi dx = \frac{\xi(te_N) - \xi(-te_N)}{t}, \quad \forall \xi \in C_0^2(\Omega). \tag{3.8}
$$

On the one hand, by Lemma 3.2, we have

$$
\lim_{t\to 0^+} \frac{\xi(te_N) - \xi(-te_N)}{t} = 2\frac{\partial \xi(0)}{\partial x_N}.
$$

On the other hand, by Lemma 3.3, we have

$$
u_t \to 2\frac{\partial G_\Omega(\cdot, 0)}{\partial x_N} \quad \text{a.e. in } \Omega
$$

and combining Proposition 3.5 and Lemma 3.6, $\{u_t\}$ is relatively compact in $L^p(\Omega)$ for any $p \in [1, \frac{N}{N-1})$. Therefore, up to some subsequence, passing to the limit of $t \to 0^+$ in the identity (3.8), it implies that $\frac{\partial G_\Omega(\cdot,0)}{\partial x_N}$ is a weak solution of (1.3) and then Proposition 3.1 follows by uniqueness of weak solution to (1.3). □

4. Multipole singularities

In this section we discuss the weak solution of elliptic equation with multiple-polar source. We construct multiple-polar source by

$$
\partial_n \delta_0 = \sum_{i=0}^{n-1} 2\frac{\partial \delta_0}{\partial \nu_i}, \tag{4.1}
$$

where $n \in \mathbb{N}$ and ν_i is unit vector in $\mathbb{R}^N$ with $i = 0, \dots, n-1$.

Proposition 4.1. *Assume that $N = 2$, $\partial_n \delta_0$ is defined in (4.1) with n odd number and*

$$
\nu_i = \Big(\cos(\frac{2i\pi}{n}), \sin(\frac{2i\pi}{n})\Big).
$$

Then the problem

$$
\begin{gathered}
-\Delta u = 2\partial_n \delta_0 \quad \text{in } \Omega, \\
u = 0 \quad \text{on } \partial\Omega
\end{gathered} \tag{4.2}
$$

admits a unique weak solution v_n such that the function $\lim_{r\to 0} v_n(r,\theta) r^{N-1}$ has $2\pi/n$-period.

Proof. Since the Laplacian operator is linear, $v_n = \sum_{i=0}^{n-1} u_i$, where u_i is the unique solution of (1.3) replaced ν by ν_i. The uniqueness of v_n follows the proof of Theorem 1.2.

Next we prove that v_n has $\frac{2\pi}{n}$-period singularity. It follows by Proposition 2.1 that

$$\begin{gathered}
-\Delta u = 2\partial_n \delta_0 \quad \text{in } \mathbb{R}^N, \\
u(x) \to 0 \quad \text{as } |x| \to \infty,
\end{gathered} \tag{4.3}$$

has a unique weak solution w_n satisfying

$$\begin{aligned}
w_n(x) &= \sum_{i=0}^{n-1} P_{\nu_i}(x) = c_N |x|^{-1} \sum_{i=0}^{n-1} \frac{x}{|x|} \cdot \nu_i \\
&= c_N r^{-1} \sum_{i=0}^{n-1} \big[\cos\theta \cos(\frac{2i\pi}{n}) + \sin\theta \sin(\frac{2i\pi}{n}) \big] \\
&= c_N r^{-1} \sum_{i=0}^{n-1} \cos(\theta - \frac{2i\pi}{n})
\end{aligned}$$

where (r, θ) is the polar coordinates of x. We observe that if n is even, letting $n = 2j$, then $\cos(\theta - \frac{2i\pi}{n}) = -\cos(\theta - \frac{2(i+j)\pi}{n})$, which implies that $w_n = 0$ in $\mathbb{R}^N \setminus \{0\}$.

When n is odd, the function $\theta \mapsto \sum_{i=0}^{n-1} \cos(\theta - \frac{2i\pi}{n})$ is nontrivial and has $\frac{2\pi}{n}$-period. Similar to the proof of Theorem 1.2, we can prove that

$$\lim_{r \to 0^+} \frac{v_n(te)}{w_n(te)} = 1 \quad \text{for } e \in \partial B_1(0),\ e \cdot \nu_i \neq 0,\ i = 0, \dots, n-1.$$

This completes the proof. □

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