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BOUNDARY-VALUE PROBLEMS FOR RIEMANN-LIOUVILLE FRACTIONAL DIFFERENTIAL INCLUSIONS IN BANACH SPACES

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Abstract. In this article, we sudy the existence of solutions of boundary-value problems for Riemann-Liouville fractional differential inclusions of order $r \in (2, 3]$ in a Banach space.

1. Introduction

This article concerns the existence of solutions for boundary-value problems (BVP for short), for fractional order differential inclusions. We consider the boundary-value problem

$$D^r y(t) \in F(t, y), \quad \text{for a.e. } t \in J = [0, T], \tag{1.1}$$

$$y(0) = 0, \quad y'(0) = 0, \quad y''(T) = 0, \tag{1.2}$$

where $2 < r \leq 3$, D^r is the Riemann-Liouville fractional derivative, $F : J \times E \to \mathcal{P}(E)$ is a multivalued map, $\mathcal{P}(E)$ is the family of all nonempty subsets of E, and $(E, |\cdot|)$ denotes a Banach space.

Differential equations of fractional order have recently proved to be valuable tools in the modeling of many phenomena in various fields of science and engineering. Indeed, there are numerous applications in viscoelasticity, electrochemistry, control, porous media, electromagnetism, and so on. There has been a significant development in fractional differential equations in recent years; see the monographs of Hilfer [22], Kilbas et al. [26, 27], Delbosco et al. [18], Miller et al. [29], Heymans et al. [21], Podlubny[32, 33], Kaufman et al. [25], Karakostas et al. [24], Momani and Hadid [30], and the papers by Agarwal et al. [1, 2, 3], Bai et al. [8, 9], Benchohra et al. [11, 12, 13], and Hamani et al. [19].

In this article, we present existence results for the problem (1.1)-(1.2), when the right hand side is convex valued. This result relies on the set-valued analog of Mönch's fixed point theorem combined with the technique of measure of noncompactness. Recently, this has proved to be a valued tool in solving fractional differential equation and inclusions in Banach spaces; for details, see the papers of Lasota et al. [28], Agarwal et al. [4] and Benchohra et al. [14, 15, 16]. This

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result extends to the multivalued case some previous results in the literature, and constitutes an interesting contribution to this emerging field.

2. Preliminaries

In this section, we introduce definitions, and preliminary facts that will be used in the remainder of this paper. Let $C(J,E)$ be the Banach space of all continuous functions from J into E with the norm

$$\|y\| = \sup\{|y(t)| : 0 \leq t \leq T\},$$

and we let $L^1(J,E)$ denote the Banach space of functions $y : J \to E$ which are Bochner integrable with norm

$$\|y\|_{L^1} = \int_0^T |y(t)|dt.$$

$AC^1(J,E)$ is the space of functions $y : J \to E$, which are absolutely continuous whose first derivative, y', is absolutely continuous.

Let $(E, |\cdot|)$ be a Banach space. Let $P_{cl}(E) = \{A \in \mathcal{P}(E) : A \text{ closed}\}$, $P_c(E) = \{A \in \mathcal{P}(E) : A \text{ convex}\}$, $P_{cp,c}(E) = \{A \in \mathcal{P}(E) : A \text{ compact and convex}\}$. A multivalued map $G : E \to \mathcal{P}(E)$ has a fixed point if there is $x \in E$ such that $x \in G(E)$. The fixed point set of the multivalued operator G will be denoted by $FixG$. A multivalued map $G : J \to P_{cl}(\mathbb{R})$ is said to be measurable if for every $y \in \mathbb{R}$, the function $t \mapsto d(y, G(t)) = \inf\{|y - z| : z \in G(t)\}$ is measurable.

Let X, Y be two sets, and $N : X \to \mathcal{P}(Y)$ be a set-valued map. We define the graph of N, as

$$\operatorname{graph}(N) = \{(x,y) : x \in X,\ y \in N(X)\}.$$

For more details on multivalued maps see the books of Deimling [17], Aubin et al. [6, 7] and Hu and Papageorgiou [23].

Let $R > 0$, and

$$B = \{x \in E : |x| \leq R\}, \quad U = \{x \in C(J,E) : \|x\| \leq R\},$$

Clearly U is a closed subset of $C(J,B)$.

Definition 2.1 ([27, 32])**.** The fractional (arbitrary) order integral of the function $h \in L^1([a,b], \mathbb{R}_+)$ of order $r \in \mathbb{R}_+$ is defined by

$$I_a^r h(t) = \int_a^t \frac{(t-s)^{r-1}}{\Gamma(r)} h(s)\, ds,$$

where Γ is the gamma function. When $a = 0$, we write $I^r h(t) = h(t) * \varphi_r(t)$, where $\varphi_r(t) = \frac{t^{r-1}}{\Gamma(r)}$ for $t > 0$, and $\varphi_r(t) = 0$ for $t \leq 0$, and $\varphi_r \to \delta(t)$ as $r \to 0$, where δ is the delta function.

Definition 2.2 ([27, 32])**.** For a function h given on the interval $[a,b]$, the r Riemann-Liouville fractional-order derivative of h, is defined by

$$(D_{a+}^r h)(t) = \frac{1}{\Gamma(n-r)} \Big(\frac{d}{dt}\Big)^n \int_a^t (t-s)^{n-r-1} h(s)\, ds.$$

Here $n = [r] + 1$ and $[r]$ denotes the integer part of r.

For convenience, we recall the definition of the Kuratowski measure of noncompactness.

Definition 2.3 ([5, 10])**.** Let E be a Banach space and let Ω_E be the family of bounded subsets of E. The Kuratowski measure of noncompactness is the map $\alpha : \Omega_E \to [0,\infty)$ defined by

$$\alpha(M) = \inf\{\epsilon > 0 : M \subset \cup_{j=1}^{m} M_j, \operatorname{diam}(M_j) \le \epsilon\},$$

where $M \in \Omega_E$.

Properties:

(1) $\alpha(M) = 0 \Leftrightarrow \overline{M}$ is compact (M is relatively compact).
(2) $\alpha(M) = \alpha(\overline{M})$.
(3) $M_1 \subset M_2 \Rightarrow \alpha(M_1) \le \alpha(M_2)$.
(4) $\alpha(M_1 + M_2) \le \alpha(M_1) + \alpha(M_2)$.
(5) $\alpha(cM) = c\alpha(M)$, $c \in \mathbb{R}$.
(6) $\alpha(\operatorname{conv} M) = \alpha(M)$.

More properties of α can be found in [5, 10].

Definition 2.4. A multivalued map $F : J \times E \to \mathcal{P}(E)$ is said to be Carathéodory if

(1) $t \to F(t,u)$ is measurable for each $u \in E$.
(2) $u \to F(t,u)$ is upper semicontinuous for almost all $t \in J$.

For each $y \in C(J,E)$, define the set of selections of F by

$$S_{F,y} = \{v \in L^1(J,E) : v(t) \in F(t,y(t)) \text{ a.e. } t \in J\}.$$

Theorem 2.5 ([20])**.** *Let E be a Banach space and $C \subset L^1(J,E)$ be countable with $|u(t)| \le h(t)$ for a.e. $t \in J$, and every $u \in C$, where $h \in L^1(J,\mathbb{R}_+)$. Then the function $\phi(t) = \alpha(C(t))$ belongs to $L^1(J,\mathbb{R}_+)$ and satisfies*

$$\alpha\Big(\Big\{\int_0^T u(s)\,ds : u \in C\Big\}\Big) \le 2\int_0^T \alpha(C(s))\,ds.$$

Let us now recall the set-valued analog of Mönch's fixed point theorem.

Theorem 2.6 ([31])**.** *Let K be a closed, convex subset of a Banach space E, U a relatively open subset of K, and $N : \overline{U} \to \mathcal{P}_c(K)$. Assume* graph$(N)$ *is closed, N maps compact sets into relatively compact sets, and that, for some $x_0 \in U$, the following two conditions are satisfied:*

$$M \subset \overline{U},\ M \subset \operatorname{conv}(x_0 \cup N(M)) \text{ and } \overline{M} = \overline{U} \text{ with } C \subset M \text{ countable imply that } \overline{M} \text{ is compact}, \tag{2.1}$$

$$x \in (1-\lambda)x_0 + \lambda N(x) \quad \text{for all } x \in \overline{U}\backslash U,\ \lambda \in (0,1). \tag{2.2}$$

Then there exists $x \in \overline{U}$ with $x \in N(x)$.

Lemma 2.7 ([28])**.** *Let J be a compact real interval. Let F be a Carathéodory multivalued map and let Θ be a linear continuous map from $L^1(J,E) \to C(J,E)$. Then the operator*

$$\Theta \circ S_{F,y} : C(J,E) \to \mathcal{P}_{cp,c}(C(J,E)), \quad y \mapsto (\Theta \circ S_{F,y})(y) = \Theta(S_{F,y})$$

is a closed graph operator in $C(J,E) \times C(J,E)$.

3. Main results

Let us start by defining what we mean by a solution of the problem (1.1)–(1.2).

Definition 3.1. A function $y \in AC^2([0,T],E)$ is said to be a solution of (1.1)–(1.2) if there exist a function $v \in L^1(J,E)$ with $v(t) \in F(t,y(t))$, for a.e. $t \in J$, such that $D^r y(t) = v(t)$ on J, and the condition $y(0)=0$, $y'(0)=0$, $y''(T)=0$ are satisfied.

For the existence of solutions for the problem (1.1)–(1.2), we need the following auxiliary lemma.

Lemma 3.2 ([9]). *Let $r > 0$, and $h \in C(0,T) \cap L^1(0,T)$. Then*

$$I^r D^r h(t) = h(t) + c_1 t^{r-1} + c_2 t^{r-2} + \dots + c_n t^{r-n}$$

for some $c_i \in \mathbb{R}$, $i = 1,\dots,n$, where n is the smallest integer greater than or equal to r.

Lemma 3.3. *Let $2 < r \le 3$ and let $h : [0,T] \to E$ be continuous. A function y is a solution of the fractional integral equation*

$$\begin{aligned} y(t) &= \frac{1}{\Gamma(r)} \int_0^t (t-s)^{r-1} h(s)\,ds \\ &\quad - \frac{t^{r-1}}{(r-1)(r-2)\Gamma(r-2)} \int_0^T (T-s)^{r-3} h(s)\,ds. \end{aligned} \tag{3.1}$$

if and only if y is a solution of the fractional BVP

$$D^r y(t) = h(t), \quad t \in [0,T], \tag{3.2}$$

$$y(0) = 0, \quad y'(0) = 0, \quad y''(T) = 0. \tag{3.3}$$

Proof. Assume y satisfies (3.1). Then Lemma 3.2 implies that

$$y(t) = c_1 t^{r-1} + c_2 t^{r-2} + c_3 t^{r-3} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{r-1} h(s)\,ds.$$

From (3.3), a simple calculation gives

$$c_1 = -\frac{1}{(r-1)(r-2)\Gamma(r-3)} \int_0^T (T-s)^{r-3} h(s)\,ds,$$
$$c_2 = 0, quad c_3 = 0\,.$$

Hence we get equation (3.1). Conversely, it is clear that if y satisfies equation (3.1), then equations (3.2)-(3.3) hold. □

Theorem 3.4. *Assume the following hypotheses hold:*

- (H1) *$F : J \times \mathbb{R} \to \mathcal{P}_{cp,c}(\mathbb{R})$ is a Carathéodory multivalued map.*
- (H2) *For each $R > 0$, there exists a function $p \in L^1(J,E)$ and such that*

$$\|F(t,u)\|_{\mathcal{P}} = \sup\{|v|,\ v(t) \in F(t,y)\} \le p(t)$$

for each $(t,y) \in J \times E$ with $|y| \le R$, and

$$\lim_{R\to+\infty} \inf \frac{\int_0^T p(t)dt}{R} = \delta < \infty.$$

(H3) *There exists a Carathéodory function* $\psi : J \times [0, 2R] \to \mathbb{R}_+$ *such that*
$$\alpha(F(t, M)) \le \psi(t, \alpha(M)), \quad \text{a.e. } t \in J \text{ and each } M \subset B,$$
and $\phi \equiv 0$ *is the unique solution in* $C(J, [0, 2R])$ *of the inequality*
$$\begin{aligned}\phi(t) \le 2[&\frac{1}{\Gamma(r)} \int_0^t (t-s)^{r-1} \varphi(s, \phi(s))\, ds \\ &- \frac{t^{r-1}}{(r-1)(r-2)\Gamma(r-2)} \int_0^T (T-s)^{r-3} \varphi(s, \phi(s))\, ds],\end{aligned} \tag{3.4}$$
for $t \in J$.

Then the BVP (1.1)–(1.2) *has at least one solution on* $C(J, B)$*, provided that*
$$\delta < \Big[\frac{T}{\Gamma(r+1)} + \frac{T^2}{(r-1)(r-2)\Gamma(r-1)}\Big]. \tag{3.5}$$

Proof. First we transform problem (1.1)–(1.2) into a fixed point problem. Consider the multivalued operator
$$\begin{aligned}N(y) = \Big\{h \in C(J, E) : (Ny)(t) = &\frac{1}{\Gamma(r)} \int_0^t (t-s)^{r-1} v(s)\, ds \\ &- \frac{t^{r-1}}{(r-1)(r-2)\Gamma(r-2)} \int_0^T (T-s)^{r-3} v(s)\, ds,\ v \in S_{F,y}\Big\}.\end{aligned}$$

Clearly, from Lemma 3.3, the fixed points of N are solutions to (1.1)–(1.2). We shall show that N satisfies the assumptions of the set-valued analog of Mönch's fixed point theorem. The proof will be given in several steps.

Step 1: $N(y)$ is convex for each $y \in C(J, E)$. Indeed, if h_1, h_2 belong to $N(y)$, then there exist $v_1, v_2 \in S_{F,y}$ such that for each $t \in J$ we have
$$\begin{aligned}h_i(t) = &\frac{1}{\Gamma(r)} \int_0^t (t-s)^{r-1} v_i(s)\, ds \\ &- \frac{t^{r-1}}{(r-1)(r-2)\Gamma(r-2)} \int_0^T (T-s)^{r-3} v_i(s)\, ds, \quad i = 1, 2.\end{aligned}$$
Let $0 \le d \le 1$. Then, for each $t \in J$, we have
$$\begin{aligned}&(dh_1 + (1-d)h_2)(t) \\ &= \frac{1}{\Gamma(r)} \int_0^t (t-s)^{r-1} [dv_1(s) + (1-d)v_2(s)]\, ds \\ &\quad + \frac{t^{r-1}}{(r-1)(r-2)\Gamma(r-2)} \int_0^T (T-s)^{r-3} [dv_1(s) + (1-d)v_2(s)]\, ds.\end{aligned}$$
Since $S_{F,y}$ is convex (because F has convex values), we have $dh_1 + (1-d)h_2 \in N(y)$.

Step 2: $N(M)$ is relatively compact for each compact $M \subset \overline{U}$. Let $M \subset \overline{U}$ be a compact set and let (h_n) by any sequence of elements of $N(M)$. We show that (h_n) has a convergent subsequence by using the Arzéla-Ascoli criterion of compactness in $C(J, E)$. Since $h_n \in N(M)$ there exist $y_n \in M$ and $v_n \in S_{F,y_n}$ such that
$$h_n(t) = \frac{1}{\Gamma(r)} \int_0^t (t-s)^{r-1} v_n(s)\, ds$$

$$
- \frac{t^{r-1}}{(r-1)(r-2)\Gamma(r-2)} \int_0^T (T-s)^{r-3} v_n(s)\, ds.
$$

Using Theorem 2.5 and the properties of the measure of noncompactness of Kuratowski, we have

$$
\begin{aligned}
\alpha(\{h_n(t)\}) \leq 2[&\frac{1}{\Gamma(r)} \int_0^t \alpha(\{(t-s)^{r-1} v_n(s)\})\, ds \\
&- \frac{t^{\alpha-1}}{(r-1)(r-2)\Gamma(r-2)} \int_0^T \alpha(\{(T-s)^{r-3} v_n(s)\})\, ds].
\end{aligned} \tag{3.6}
$$

On the other hand, since $M(s)$ is compact in E, the set $\{v_n(s) : n \geq 1\}$ is compact. Consequently, $\alpha(\{v_n(s) : n \geq 1\}) = 0$ for a.e. $s \in J$. Furthermore

$$
\begin{gathered}
\alpha(\{(t-s)^{r-1} v_n(s)\}) = (t-s)^{r-1} \alpha(\{v_n(s) : n \geq 1\}) = 0, \\
\alpha(\{(T-s)^{r-1} v_n(s)\}) = (T-s)^{r-1} \alpha(\{v_n(s) : n \geq 1\}) = 0,
\end{gathered}
$$

for a.e. $t, s \in J$. Now (3.6) implies that $\{h_n(t) : n \geq 1\}$ is relatively compact in E, for each $t \in J$. In addition, for each t_1 and t_2 from J, $t_1 < t_2$, we have

$$
\begin{aligned}
&|h_n(t_2) - h_n(t_1)| \\
&= \Big| \frac{1}{\Gamma(r)} \int_0^{t_1} [(t_2-s)^{r-1} - (t_1-s)^{r-1}] v_n(s)\, ds + \frac{1}{\Gamma(r)} \int_{t_1}^{t_2} (t_2-s)^{r-1} v_n(s)\, ds \Big| \\
&\quad + \frac{(t_2-t_1)^{r-1}}{(r-1)(r-2)\Gamma(r-2)} \int_0^T (T-s)^{r-3} |v_n(s)|\, ds \\
&\leq \frac{p(t)}{\Gamma(r)} \int_0^{t_1} [(t_1-s)^{r-1} - (t_2-s)^{r-1}]\, ds + \frac{p(t)}{\Gamma(r)} \int_{t_1}^{t_2} (t_2-s)^{r-1}\, ds \\
&\quad + \frac{p(t)(t_2-t_1)^{r-1}}{(r-1)(r-2)\Gamma(r-2)} \int_0^T (T-s)^{r-3}\, ds \\
&\leq \frac{p(t)}{\Gamma(r+1)} [(t_2-t_1)^r + t_1^r - t_2^r] + \frac{p(t)}{\Gamma(r+1)} (t_2-t_1)^r + \frac{p(t)(t_2-t_1)^{r-1}}{\Gamma(r-1)} \\
&\leq \frac{p(t)}{\Gamma(r+1)} (t_2-t_1)^r + \frac{p(t)}{\Gamma(r+1)} (t_1^r - t_2^r) + \frac{T^r p(t)(t_2-t_1)^{r-1}}{\Gamma(r-1)}.
\end{aligned} \tag{3.7}
$$

As $t_1 \to t_2$, the right-hand side of the above inequality tends to zero. This shows that $\{h_n : n \geq 1\}$ is equicontinuous. Consequently, $\{h_n : n \geq 1\}$ is relatively compact in $C(J, E)$.

Step 3: The graph of N is closed. Let $(y_n, h_n) \in \operatorname{graph}(N)$, $n \geq 1$, with $\|y_n - y\|$, $\|h_n - h\| \to 0$,as $n \to \infty$. We must show that $(y, h) \in \operatorname{graph}(N)$.

$(y_n, h_n) \in \operatorname{graph}(N)$ means that $h_n \in N(y_n)$, which means that there exists $v_n \in S_{F,y_n}$, such that for each $t \in J$,

$$
\begin{aligned}
h_n(t) = &\frac{1}{\Gamma(r)} \int_0^t (t-s)^{r-1} v_n(s)\, ds \\
&- \frac{t^{r-1}}{(r-1)(r-2)\Gamma(r-2)} \int_0^T (T-s)^{r-3} v_n(s)\, ds.
\end{aligned}
$$

Consider the continuous linear operator $\Theta : L^1(J,E) \to C(J,E)$,

$$\begin{aligned}\Theta(v)(t) \mapsto h_n(t) = \frac{1}{\Gamma(r)}\int_0^t (t-s)^{r-1} v_n(s)\,ds \\ - \frac{t^{r-1}}{(r-1)(r-2)\Gamma(r-2)}\int_0^T (T-s)^{r-3} v_n(s)\,ds.\end{aligned}$$

Clearly, $\|h_n(t)-h(t)\| \to 0$ as as $n\to\infty$. From Lemma 2.7 it follows that $\Theta\circ S_F$ is a closed graph operator. Moreover, $h_n(t)\in\Theta(S_{F,y_n})$. Since $y_n\to y$, Lemma 2.7 implies

$$h(t) = \frac{1}{\Gamma(r)}\int_0^t (t-s)^{r-1} v(s)\,ds - \frac{t^{r-1}}{(r-1)(r-2)\Gamma(r-2)}\int_0^T (T-s)^{r-3} v(s)\,ds,$$

for some $v\in S_{F,y}$.

Step 4. Suppose $M\subset\overline{U}$, $M\subset \operatorname{conv}(\{0\}\cup N(M))$, and $\overline{M}=\overline{C}$ for some countable set $C\subset M$. Using an estimation of type (3.7), we see that $N(M)$ is equicontinuous. Then from $M\subset\operatorname{conv}(\{0\}\cup N(M))$, we deduce that M is equicontinuous, too. To apply the Arzéla-Ascoli theorem, it remains to show that $M(t)$ is relatively compact in E for each $t\in J$. Since $C\subset M\subset\operatorname{conv}(\{0\}\cup N(M))$ and C is countable, we can find a countable set $H=\{h_n : n\ge 1\}\subset N(M)$ with $C\subset\operatorname{conv}(\{0\}\cup H)$. Then, there exist $y_n\in M$ and $v_n\in S_{F,y_n}$ such that

$$\begin{aligned}h_n(t) = \frac{1}{\Gamma(r)}\int_0^t (t-s)^{r-1} v_n(s)\,ds \\ - \frac{t^{r-1}}{(r-1)(r-2)\Gamma(r-2)}\int_0^T (T-s)^{r-3} v_n(s)\,ds.\end{aligned}$$

From $M\subset\overline{C}\subset\overline{\operatorname{conv}}(\{0\}\cup H))$, and according to Theorem 2.5, we have

$$\alpha(M(t)) \le (\alpha(\overline{C}(t)) \le \alpha(H(t)) = \alpha(\{h_n((t) : n\ge 1\}).$$

Using(3.6), we obtain

$$\begin{aligned}\alpha(M(t)) \le 2[\frac{1}{\Gamma(r)}\int_0^t \alpha(\{(t-s)^{r-1} v_n(s)\})\,ds \\ - \frac{t^{\alpha-1}}{(r-1)(r-2)\Gamma(r-2)}\int_0^T \alpha(\{(T-s)^{r-3} v_n(s)\})\,ds].\end{aligned}$$

Now, since $v_n(s)\in M(s)$, we have

$$\begin{aligned}\alpha(M(t)) \le 2[\frac{1}{\Gamma(r)}\int_0^t \alpha(\{(t-s)^{r-1} v_n(s) : n\ge 1\})\,ds \\ - \frac{t^{\alpha-1}}{(r-1)(r-2)\Gamma(r-2)}\int_0^T \alpha(\{(T-s)^{r-3} v_n(s) : n\ge 1\})\,ds].\end{aligned}$$

Also, since $v_n(s)\in M(s)$, we have

$$\begin{gathered}\alpha(\{(t-s)^{r-1} v_n(s);\, n\ge 1\}) = (t-s)^{r-1}\alpha(M(s)),\\ \alpha(\{(T-s)^{r-1} v_n(s);\, n\ge 1\}) = (T-s)^{r-1}\alpha(M(s)).\end{gathered}$$

It follows that

$$\alpha(M(t)) \le 2[\frac{1}{\Gamma(r)}\int_0^t (t-s)^{r-1}\alpha(M(s))\,ds$$

$$
\begin{aligned}
&- \frac{t^{\alpha-1}}{(r-1)(r-2)\Gamma(r-2)}\int_0^T (T-s)^{r-3}\alpha(M(s))\,ds]\\
&\le 2[\frac{1}{\Gamma(r)}\int_0^t (t-s)^{r-1}\psi(s,\alpha(M(s)))\,ds\\
&- \frac{t^{\alpha-1}}{(r-1)(r-2)\Gamma(r-2)}\int_0^T (T-s)^{r-3}\psi(s,\alpha(M(s)))\,ds].
\end{aligned}
$$

Also, the function ϕ given by $\phi(t)=\alpha(M(t))$ belongs to $C(J,[0,2R])$. Consequently by (H3), $\phi\equiv 0$; that is, $\alpha(M(t))=0$ for all $t\in J$. Now, by the Arzéla-Ascoli theorem, M is relatively compact in $C(J,E)$.

Step 5. Let $h\in N(y)$ with $y\in\overline{U}$. Since $|y(s)|\le R$ and by (H2), we have $N(\overline{U})\subseteq\overline{U}$, because if it were not true, then there exists a function $y\in\overline{U}$, but $\|N(y)\|_{\mathcal{P}}>R$ and

$$
\begin{aligned}
h(t) &= \frac{1}{\Gamma(r)}\int_0^t (t-s)^{r-1}v(s)\,ds\\
&\quad - \frac{t^{r-1}}{(r-1)(r-2)\Gamma(r-2)}\int_0^T (T-s)^{r-3}v(s)\,ds,
\end{aligned}
$$

for some $v\in S_{F,y}$. On the other hand,

$$
\begin{aligned}
R &\le \|N(y)\|_{\mathcal{P}}\\
&\le \frac{1}{\Gamma(r)}\int_0^t (t-s)^{r-1}|v(s)|\,ds\\
&\quad - \frac{t^{r-1}}{(r-1)(r-2)\Gamma(r-2)}\int_0^T (T-s)^{r-3}|v(s)|\,ds\\
&\le \frac{T}{\Gamma(r+1)}\int_0^t p(s)\,ds\\
&\quad - \frac{T^2}{(r-1)(r-2)\Gamma(r-1)}\int_0^T p(s)\,ds\\
&\le \big[\frac{T}{\Gamma(r+1)}+\frac{T^2}{(r-1)(r-2)\Gamma(r-1)}\big]\int_0^T p(s)\,ds.
\end{aligned}
$$

Dividing both sides by R and taking the lower limits as $R\to\infty$, we conclude that

$$
\big[\frac{T}{\Gamma(r+1)}+\frac{T^2}{(r-1)(r-2)\Gamma(r-1)}\big]\delta\ge 1
$$

which contradicts (3.5). Hence $N(\overline{U})\subseteq\overline{U}$.

As a consequence of Steps 1-5 and Theorem 2.6, we conclude that N has a fixed point $y\in C(J,B)$ which is a solution of problem (1.1)-(1.2). □

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