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# RIEMANN PROBLEM FOR A TWO-DIMENSIONAL QUASILINEAR HYPERBOLIC SYSTEM

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Abstract. This article concerns the study of the Riemann problem for a two-dimensional non-strictly hyperbolic system of conservation laws. The initial data are three constant states separated by three lines and are chosen so that one of the three interfaces of the initial data projects a planar delta shock wave. Based on the generalized characteristic analysis, the global solutions are constructed completely. The solutions reveal a variety of geometric structures for the interactions of delta shock waves with rarefaction waves, shock waves and contact discontinuities.

## 1. Introduction

In this article, we study the Riemann problem of the two-dimensional system

$$\begin{aligned} &u_t + (u^2)_x + (uv)_y = 0, \\ &v_t + (uv)_x + (v^2)_y = 0, \end{aligned} \tag{1.1}$$

with initial data

$$(u,v)\big|_{t=0} = \begin{cases} (u_1,v_1), & y>0, \\ (u_2,v_2), & x<0,\ y<0, \\ (u_3,v_3), & x>0, y<0, \end{cases} \tag{1.2}$$

where $(u_i, v_i)$, $i = 1, 2, 3$ are constant states. It was shown in [4] that it is most suitable for the choice of initial data as constants in each of the three sectors such as in the form (1.2), because it keeps the essential components of the two-dimensional Riemann problem for a system of conservation laws and while the number of cases is less than that in other choices. Thus, the choice of initial data in the form (1.2) is able to reveal the formation and development of singularity of solution to the system (1.1). Furthermore, the technique developed for the two-dimensional Riemann problem with three constant initial data as in the form (1.2) can be easily generalized to other choices of initial data. In addition, with the choice of initial data in the form (1.2), the restriction of wave pattern is sufficient for deriving the expression of exact numerical fluxes such as the positive scheme [22] and the Godunov scheme [5].

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System (1.1) can be considered as a simplification of two-dimensional Euler equations for it can be derived directly from the two-dimensional isentropic Euler equations by letting the pressure and density be constants in the last two momentum equations [21]. In fact, the simplified system (1.1) is also able to explain some interesting phenomena in gas dynamics such as diffractions along wedges [21]. The system (1.1) also belongs to the system of type

$$\begin{aligned} u_t + (uf(u,v))_x + (ug(u,v))_y = 0, \\ v_t + (vf(u,v))_x + (vg(u,v))_y = 0, \end{aligned} \tag{1.3}$$

where $f(u,v) = u$ and $g(u,v) = v$. Equations like (1.3) occur in a variety of applications, including oil recovery, elastic theory and magneto-hydrodynamics [2]. The Riemann problem for (1.3) is much more complicated than the scalar case, but it is simpler than the problem for general hyperbolic system. This is due to the fact that the domain of mixed type does not appear in the study of self-similar solutions for (1.3). Thus the study of the Riemann problem for (1.3) can be regarded as a necessary step to more complicated and practical cases such as the conjectures on the two-dimensional Riemann solutions for the Euler equations [25]. When $f(u,v) = g(u,v) = u$ and $v = \rho$, the Riemann problem for the system (1.3) was investigated in [17, 19].

There have been many studies on system (1.1) from various aspects. Tan and Zhang [21] firstly studied the four quadrant Riemann problem of (1.1), namely the initial data are four constant states in each quadrant of $(x,y)$ plane, and they discovered that a kind of new nonlinear wave called delta shock wave was there. About the delta shock wave solution in the multi-dimensional hyperbolic conservation laws, we can also see [6, 9, 10, 11, 13, 14, 16, 18, 20, 26] and the reference therein. Yang and Zhang [24] verified the analytic solutions in [21] numerically using the MmB preserving scheme. Lopes-Filho and Nussenzveig Lopes [12] have investigated the singularity formation about the evolution along a characteristic of the compression rate of nearby characteristics for (1.1). Wang [23] proposed an example to show that the solution for (1.1) is not unique. The three constant Riemann problem for (1.1), that is the initial data take three constant states in three angular domains in the $(x,y)$ plane, was studied in [2, 3, 13, 14, 15]. Huang and Yang [7] constructed the solutions for the two constant Riemann problem of (1.1). The non-selfsimilar Riemann problem on (1.1) was also considered by Chen, Wang and Yang [1] and they discovered the triple-shock pattern there.

Our goal in this article is to construct explicitly the global solutions for (1.1) and (1.2). The initial data as (1.2) simplify the complexity of the structures of the four quadrant Riemann solutions, while the essential ingredients of two-dimensional Riemann problems can hold. Following [21], we assume that the initial data (1.2) are chosen so that only one planar elementary wave appears at each interface of the initial data. The justification of this choice lies in that the majority of physical observations only involve the study of a single propagation wave type [8]. The most attractive feature of the system (1.1) is in that the plane delta shock wave appears in the solutions of Rieamnn problem (1.1) and (1.2) for some certain initial data. Thus, we draw our attention on the cases that one of the three planar elementary waves is a planar delta shock wave, which are different from the previous results. Using the method of generalized characteristic analysis, we solve the Riemann problem (1.1) and (1.2) analytically and nine exact entropy solutions with different geometric structures are constructed globally. The solutions reveal various interactions of delta shock waves with the classical waves involving contact discontinuities, shock waves and rarefaction waves. The evolution of the planar delta shock wave is presented in detail. The results of the present note provide a preparation of theoretical analysis for the numerical simulation for (1.1).

The rest of this article is organized as follows. In section 2, we provide some basic properties of system (1.1) for completeness, including the characteristics, bounded discontinuities and delta shock waves. In section 3, we classify the Riemann problem according to the combinations of the exterior waves. Then the global solutions are constructed by the method of generalized characteristic analysis.

## 2. Preliminaries

In this section, we briefly review some basic properties of system (1.1) for readers' convenience, and the detailed study can be found in [21].

Since both (1.1) and (1.2) are invariant under the self-similar transformation $(t,x,y) \to (\alpha t, \alpha x, \alpha y)$ with $\alpha > 0$, we seek the self-similar solution of the form $(u,v)(t,x,y) = (u,v)(\xi,\eta)$ where $(\xi,\eta) = (x/t, y/t)$. The system for this form of solution is

$$\begin{aligned} -\xi u_\xi - \eta u_\eta + (u^2)_\xi + (uv)_\eta = 0, \\ -\xi v_\xi - \eta v_\eta + (uv)_\xi + (v^2)_\eta = 0, \end{aligned} \tag{2.1}$$

and the initial data (1.2) become boundary values at infinity

$$\lim_{\xi^2+\eta^2\to\infty} (u,v) = \begin{cases} (u_1,v_1), & \eta > 0, \\ (u_2,v_2), & \xi < 0, \eta < 0, \\ (u_3,v_3), & \xi > 0, \eta < 0. \end{cases} \tag{2.2}$$

System (2.1) has two eigenvalues

$$\lambda_1 = \frac{v-\eta}{u-\xi}, \quad \lambda_2 = \frac{2v-\eta}{2u-\xi}, \tag{2.3}$$

which are called pseudo-characteristics of (1.1) for a given solution $(u,\rho)(\xi,\eta)$. The $\lambda_1$ pseudo-characteristic field is linearly degenerate and the $\lambda_2$ pseudo-characteristic field is genuinely nonlinear if $u\eta - v\xi \neq 0$.

Define the characteristic curves $\Gamma_i$ $(i = 1,2)$ in the $(\xi,\eta)$ plane by

$$\Gamma_i : \frac{d\eta}{d\xi} = \lambda_i. \tag{2.4}$$

The singularity point for $\Gamma_i$, denoted by $P_i$, is $P_i = (iu, iv)$, $i = 1,2$. We call the curve $\eta/\xi = v/u$ the base curve denoted by $B$, which consists of singularity points for $\Gamma_i$ and the degenerate hyperbolic points $\lambda_1 = \lambda_2$. We stipulate the direction of characteristic curves $\Gamma_i$ from infinity to the singularity point $P_i$ $(i = 1,2)$, which is motivated by virtue of the increase of time [11].

(i) Smooth solution. If $v/u =$ constant in some domain, we call it a simple wave of the second kind, which satisfies $u_\xi + \lambda_2 u_\eta = 0$. The simple wave is called a rarefaction wave (abbr. $R$), if all the $\lambda_2-$ characteristic curves and their extensions in the positive directions do not intersect until they reach the corresponding base curve.

(ii) Bounded discontinuity solution. Let $\eta = \eta(\xi)$ be a smooth discontinuity of a bounded discontinuous solution in the $(\xi, \eta)$ plane. Solving the Rankine-Hugoniot condition, we obtain the following two kinds of discontinuities.

A contact discontinuity (abbr. $J$) satisfies

$$\frac{d\eta}{d\xi} = \sigma_1 = \frac{\eta - v_+}{\xi - u_+} = \frac{\eta - v_-}{\xi - u_-}, \tag{2.5}$$

which is the $\lambda_1-$ characteristic line for both sides. Hereafter, $(u_\pm, v_\pm)$ represent the limit states on two sides of the discontinuity $\eta = \eta(\xi)$.

A shock wave (abbr. $S$) satisfies

$$\frac{d\eta}{d\xi} = \sigma_2 = \frac{\eta - (v_+ + v_-)}{\xi - (u_+ + u_-)}, \quad \frac{v_+}{u_+} = \frac{v_-}{u_-}, \tag{2.6}$$

and the entropy condition which can be defined as "three incoming, one outgoing", that is, at any point of the discontinuity, three of the characteristic lines, two $\Gamma_2$s and one $\Gamma_1$, come into the point and the remaining one, $\Gamma_1$, goes out. Similarly to characteristic curves, we orient the integral curve of $d\eta/d\xi = \sigma_i$ to point towards the singularity point $(\xi, \eta) = (u_+ + (i-1)u_-, v_+ + (i-1)v_-)$, $i = 1, 2$.

(iii) Delta shock wave. A discontinuity in $(u, v)(\xi, \eta)$ at $\xi = \xi(\eta)$ is called a delta shock wave (abbr. $\delta$) if it satisfies

$$\frac{d\xi}{d\eta} = \frac{\xi - (u_+ + u_-)}{\eta - (v_+ + v_-)}, \tag{2.7}$$

and the entropy condition which can be defined as "none outgoing", that means that all of the characteristic lines on both sides of the discontinuity curve do not go out at every point of the discontinuity. Similarly, the direction of a delta shock wave is towards its singular point $(\xi, \eta) = (u_+ + u_-, v_+ + v_-)$.

## 3. Construction of Solutions involving one $\delta$

We consider now the Riemann problem (1.1) and (1.2). It is obvious that, outside a sufficiently large circle in the $(\xi, \eta)$ plane, the solution must be constant states connected by three one-dimensional planar waves $(u, v)(\xi)$ or $(u, v)(\eta)$, which are called planar elementary waves or exterior waves. In this note, we deal with the cases in which exactly one of the three one-dimensional waves from infinity is a delta shock wave. We assume first that the exterior wave connecting states $(u_2, v_2)$ and $(u_3, v_3)$ is a delta shock wave $\delta_{23}$, so that $u_2 > 0 > u_3$ should be satisfied. According to the remaining two exterior waves, we find that there exist five different combinations which lead to topologically distinct solutions. The combinations are as follows: 1. $R_{12}\delta_{23}R_{31}$, 2. $R_{12}\delta_{23}S_{31}$, 3. $R_{12}\delta_{23}J_{31}$, 4. $S_{12}\delta_{23}J_{31}$, 5. $J_{12}\delta_{23}J_{31}$.

What we need to do in the following is to extend the exterior solutions inwards to construct our global Riemann solutions. We will deal with this problem case by case according to the above classification. Here and below, $\delta_{ij}$ denotes the delta shock wave with $(u_i, v_i)$ and $(u_j, v_j)$ on its two sides, also for $R_{ij}$, $S_{ij}$, $J_{ij}$.

**Case 1.** $R_{12}\delta_{23}R_{31}$ The occurrence of this case depends on the condition: $v_2 < v_1$, $v_3 < v_1$ and $u_1/v_1 = u_2/v_2 = u_3/v_3$, where the value of $u_1/v_1$ has two possibilities: $u_1/v_1 > 0$ or $u_1/v_1 < 0$. We only need to construct the solution for $u_1/v_1 > 0$ since the other case can be treated in the same way.

By the theory of Cauchy problems, we know that the determination domain of the constant state $(u_3, v_3)$ is $\Omega_1 = \{(\xi,\eta)\big|\xi > u_2 + u_3, \eta < 2v_3\}$, namely, $(u, v)(\xi, \eta) = (u_3, v_3)$ when $(\xi, \eta) \in \Omega_1$. So $\delta_{23}$ will stay straight until it meets the point $(\xi_0, \eta_0) = (u_2 + u_3, 2v_3)$.

Let $R_{30}$ (resp. $R_{10}$) denote the part of $R_{31}$ where the $v$-component of the solution satisfies $v_3 \le v < 0$ (resp. $0 \le v \le v_1$). Then $\delta_{23}$ will penetrate $R_{30}$ to form a new delta shock wave $\delta_{2R} : \xi = \xi(\eta)$ which satisfies

$$\begin{gathered} \frac{d\xi}{d\eta} = \frac{\xi - (u + u_2)}{\eta - (v + v_2)}, \\ \eta = 2v, \\ \frac{u}{v} = \frac{u_2}{v_2}, \quad v_3 \le v < 0, \\ \xi_0 = u_2 + u_3, \quad \eta_0 = 2v_3. \end{gathered} \tag{3.1}$$

From this equation, we find that the tangent line of this discontinuity always points to the singularity points $(\xi, \eta) = (u + u_2, v + v_2)$. Therefore the integral curve of (3.1) is convex. Substituting $v = \eta/2$, $u = u_2\eta/2v_2$ into the first equation in (3.1) yields

$$\frac{d\xi}{d\eta} = \frac{2\xi - 2u_2 - u_2\eta/v_2}{\eta - 2v_2}. \tag{3.2}$$

With the initial condition $(\xi_0, \eta_0) = (u_2 + u_3, 2v_3)$ in mind, an easy calculation leads to

$$\xi - 2u_2 = \frac{u_2}{v_2}(\eta - 2v_2) - \frac{u_2}{4v_2(v_3 - v_2)}(\eta - 2v_2)^2. \tag{3.3}$$

The delta shock wave $\delta_{2R}$ cannot cancel the whole rarefaction wave $R_{31}$ and it ends at the point $(\xi_1, \eta_1) = (u_2v_2/(v_2 - v_3), 0)$, where a shock wave $S_{2R}$ develops by the "three incoming, one outgoing" entropy condition. The shock wave penetrates part of the rarefaction wave $R_{10}$ and it has the same expression as (3.3), namely the curve of $S_{2R}$ is the continuation of that of $\delta_{2R}$. It can be found from (3.3) that $d\xi/d\eta \to u_2/v_2$ as $\eta \to 2v_2$ which means that $S_{2R}$ vanishes tangentially to the point $(2u_2, 2v_2)$.

We illustrate the global structure of the solution in Figure 1. For convenience, we use some notations in the following figures. $(i)$, $(\bar{i})$, $(i + j)$, represent points $(\xi, \eta) = (u_i, v_i)$, $(\xi, \eta) = (2u_i, 2v_i)$, $(\xi, \eta) = (u_i + u_j, v_i + v_j)$, respectively. And ⓘ stands for the state $(u_i, v_i)$.

**Case 2.** $R_{12}\delta_{23}S_{31}$ This case happens if and only if $v_2 < 0 < v_1 < v_3$, $u_3 < u_1 < 0 < u_2$ and $u_1/v_1 = u_2/v_2 = u_3/v_3$.

The construction of solution for this case is analogous to that in Case 1. The difference lies in that the shock wave $S_{R3}$ penetrates the whole rarefaction wave $R_{10}$ and ends at the point $(\xi_0, 2v_1)$ with the slope

$$\frac{d\eta}{d\xi} = \frac{\eta - (v_1 + v_3)}{\xi - (u_1 + u_3)},$$

where $\xi_0$ can be obtained by substituting $\eta = 2v_1$ into

$$\xi - 2u_3 = \frac{u_3}{v_3}(\eta - 2v_3) - \frac{u_3}{4v_3(v_2 - v_3)}(\eta - 2v_3)^2.$$

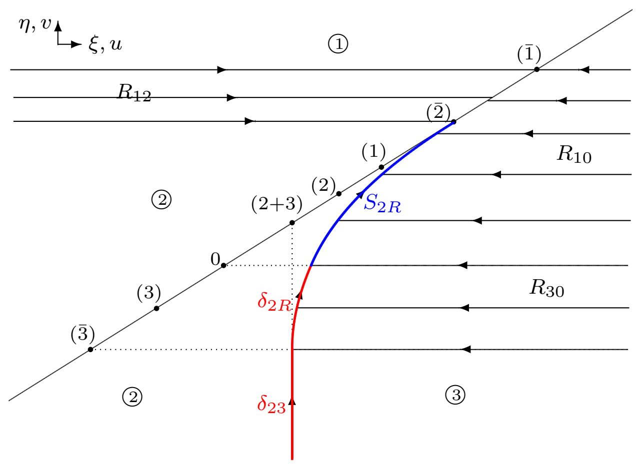


FIGURE 1. Solution for Case 1 when $u_1/v_1 > 0$.

Thereafter the shock wave stays straight with $(u_1, v_1)$ and $(u_3, v_3)$ as the limit states on two sides until it matches with $S_{31}$ at the singularity point $(u_1 + u_3, v_1 + v_3)$. See Figure 2.

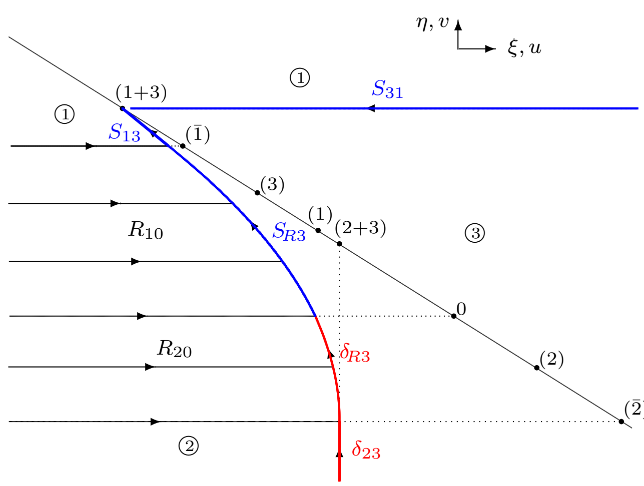


FIGURE 2. Solution for Case 2.

**Case 3.** $R_{12}\delta_{23}J_{31}$ The appearance of this case depends on the conditions $v_2 < v_1$, $v_1 = v_3$ and $u_1/v_1 = u_2/v_2$. The discussion for this case can be further divided into three subcases according to the values of $u_1/v_1$ and $v_1$: a. $u_1/v_1 > 0$; b. $u_1/v_1 < 0$ and $v_1 < 0$; c. $u_1/v_1 < 0$ and $v_1 > 0$.

**Subcase 3a.** $u_1/v_1 > 0$. Without loss of generality, we assume that $v_1 < 2v_2$. Since the determination domain of constant state $(u_3, v_3)$ is $\Omega_2 = \{(\xi, \eta) \big| \xi > u_2 + u_3, \eta <$

$v_3\}$, it follows that $(u,v)(\xi,\eta)=(u_3,v_3)$ when $(\xi,\eta)\in\Omega_2$. By the fact that $J_{31}$ intersects the base curve of constant $(u_1,v_1)$ only at the point $(\xi,\eta)=(u_1,v_1)$, $J_{31}$ will stay straight until it meets the point $(\xi,\eta)=(u_2+u_3,v_3)$. So we have

$$\lim_{\eta\to J_{31}+0}(u(\xi,\eta),v(\xi,\eta))=(u_1,v_1).$$

Solving the boundary value problem at the point $(u_2+u_3,v_3)$ with the boundary conditions

$$\lim_{\xi\to\delta_{23}-0}(u(\xi,\eta),v(\xi,\eta))=(u_2,v_2),\quad \lim_{\eta\to J_{31}+0}(u(\xi,\eta),v(\xi,\eta))=(u_1,v_1),$$

we find that a shock wave, denoted by $S_{21}$, is the solution. Here and in what follows, $\xi\to\delta_{ij}-0$ (resp. $\delta_{ij}+0$) means that for any point $(\xi_0,\eta_0)\in\delta_{ij}$, $(\xi,\eta)\to(\xi_0,\eta_0)$ with $\xi<\xi_0$ (resp. $\xi>\xi_0$). The similar notation is $\eta\to\delta_{ij}\pm 0$.

The shock wave $S_{21}:\eta-v_3=v_2(\xi-u_2-u_3)/(u_1-u_3)$ cannot keep straight after it meets the rarefaction wave $R_{12}$. Then the shock wave $S_{R1}$ begins to cancel $R_{12}$ and stops tangentially at the point $(2u_1,2v_1)$. See Figure 3.

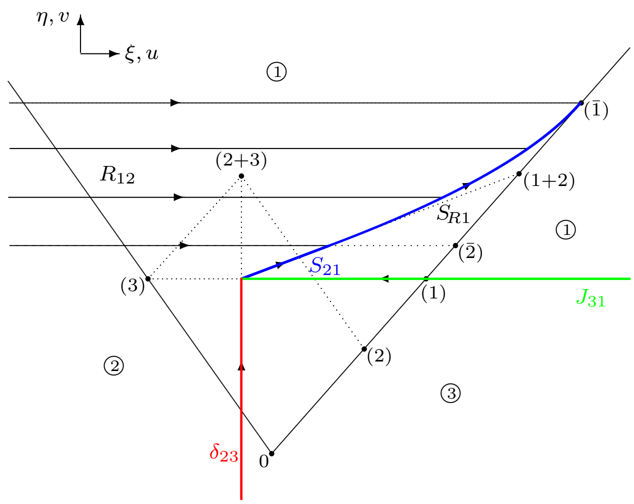


FIGURE 3. Solution for Subcase 3a.

**Subcase 3b.** $u_1/v_1<0$ and $v_1<0$. It is obvious that $(u,v)(\xi,\eta)=(u_2,v_2)$ when $\xi<u_2+u_3$ and $\eta<2v_2$. The delta shock wave $\delta_{23}$ cannot arrive at its singularity point $(u_2+u_3,v_2+v_3)$ for the reason that it will interact with $R_{12}$. The interaction gives rise to a new delta shock wave $\delta_{R3}:\xi=\xi(\eta)$ which will penetrate $R_{12}$ and has a varying speed expressed as

$$\begin{gathered}\frac{d\xi}{d\eta}=\frac{\xi-(u+u_3)}{\eta-(v+v_3)},\\ \eta=2v,\\ \frac{u}{v}=\frac{u_2}{v_2},\quad v_2\le v\le v_1\\ \xi_0=u_2+u_3,\quad \eta_0=2v_2.\end{gathered}\tag{3.4}$$

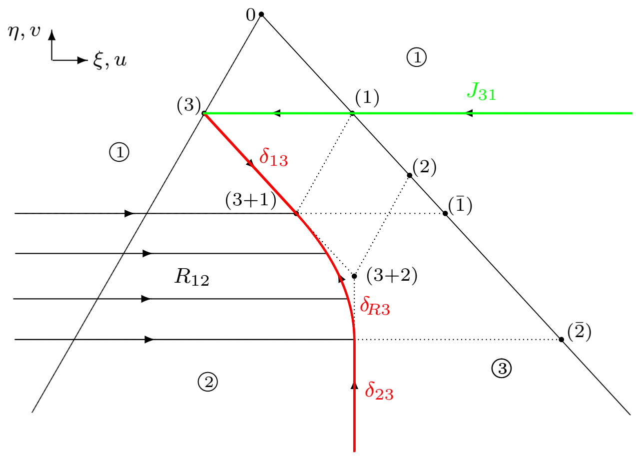


FIGURE 4. Solution for Subcase 3b.

A similar calculation as in Case 1 leads to

$$\xi - v_3(\frac{u_3}{v_3} + \frac{u_2}{v_2}) = \frac{u_2}{v_2}(\eta - 2v_3) - \frac{u_2}{4v_2(v_2 - v_3)}(\eta - 2v_3)^2. \tag{3.5}$$

It can be found that the curve $\xi = \xi(\eta)$ lies below the line $\xi = u_3 + u_2(\eta - v_3)/v_2$ which consists of singularity points for the curve. In fact, in view of $u_2/v_2 < 0$ and $v_2 < v_3$, it can be derived from (3.5) that

$$\xi - u_3(\frac{u_3}{v_3} + \frac{u_2}{v_2}) < \frac{u_2}{v_2}(\eta - 2v_3),$$

which gives $\xi < u_3 + u_2(\eta - v_3)/v_2$. Therefore $\delta_{R3}$ is able to cancel the whole $R_{12}$ completely and disappears tangentially at the point $(u_1 + u_3, v_1 + v_3)$.

The contact discontinuity $J_{31}$ can go straight until it reaches its singularity point $(u_3, v_3)$ where a delta shock wave $\delta_{13}$ should be constructed to separate the states $(u_3, v_3)$ and $(u_1, v_1)$. Finally $\delta_{13}$ matches with $\delta_{R3}$ at its singularity point $(u_1 + u_3, v_1 + v_3)$. See Figure 4.

**Subcase 3c.** $u_1/v_1 < 0$ and $v_1 > 0$. The discussion for the interaction of $\delta_{23}$ and $R_{20}$ is the same as that in Case 3b. At the point $(u_3 - v_3u_2/(v_2 - v_3), 0)$, the delta shock wave is decomposed into a contact discontinuity $J_{43}$ and a shock wave $S_{4R}$ with the intermediate state $(u_4, v_4)$ between them. Here $(u_4, v_4)$ denotes the crossing point of the base curve of constant $(u_1, v_1)$ and $\delta_{R3}$'s tangent line at the point $(u_3 - v_3u_2/(v_2 - v_3), 0)$ which passes through the point $(u_3, v_3)$.

The contact discontinuity $J_{43}$ connecting states $(u_4, v_4)$ and $(u_3, v_3)$ matches with $J_{31}$ at their common singularity point $(u_3, v_3)$ where we find a centered rarefaction wave $R_{41}$ is the solution by solving the boundary value problem with the boundary conditions

$$\lim_{\xi \to J_{43} - 0}(u(\xi, \eta), v(\xi, \eta)) = (u_4, v_4), \quad \lim_{\eta \to J_{31} + 0}(u(\xi, \eta), v(\xi, \eta)) = (u_1, v_1).$$

Then the shock wave $S_{4R}$ which connects the states on $R_{10}$ and $(u_4, v_4)$ must interact with $R_{41}$, penetrate it and finally terminate at the point $(2u_1, 2v_1)$. The shock wave and rarefaction wave are the second kind of waves, so their interaction can be obtained similarly to the situation of the scalar conservation law. See Figure 5.

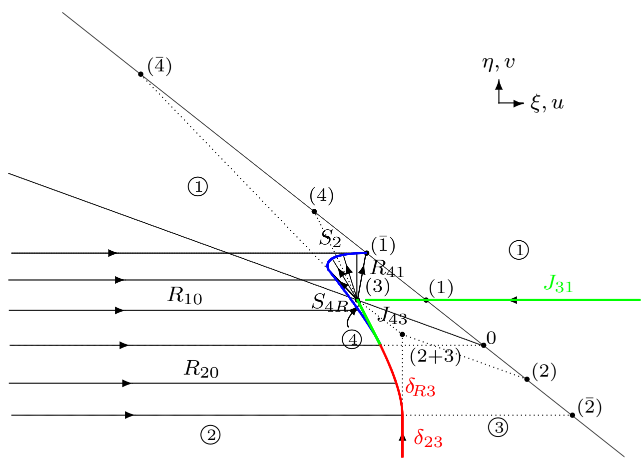


Figure 5. Solution for Subcase 3c.

Case 4 $S_{12}\delta_{23}J_{31}$. This case occurs when $v_2 > v_1$, $v_2.v_1 > 0$, $v_1 = v_3$ and $u_1/v_1 = u_2/v_2$ are satisfied. We also proceed our discussion through two subcases according to the value of $u_1/v_1$: a. $u_1/v_1 > 0$; b. $u_1/v_1 < 0$.

**Subcase 4a.** $u_1/v_1 > 0$. Similarly to Case 3a, the collision of $\delta_{23}$ and $J_{31}$ happens at the point $(u_2+u_3, v_3)$ where a rarefaction wave $R_{21}$ if the point $(u_2+u_3, v_3)$ lies above the line $\eta = v_1/u_1\xi$ or a shock wave $S_{21}$ otherwise should be constructed. When $R_{21}$ appears, the exterior wave $S_{12}$ will penetrate it and finally end at the point $(2u_1, 2v_1)$. See Figure 6. If $S_{21}$ forms, the exterior wave $S_{12}$ can go straight until it arrives at its singularity point $(u_1+u_2, v_1+v_2)$ which is also the ending point of $S_{21}$.

**Subcase 4b.** $u_1/v_1 < 0$. It is clear to see that $(u,v)(\xi,\eta) = (u_2, v_2)$ when $\xi < u_2 + u_3$ and $\eta < v_2 + v_3$. The exterior waves $\delta_{23}$ and $J_{31}$ can arrive at their singularity points respectively, while the shock wave $S_{12}$ cannot and it stops at the point $(u_2 + u_3, v_2 + v_3)$. Then a delta shock wave $\delta_{31} : \eta = v_3 + u_1/v_1(\xi - u_3)$ should be constructed to separate the states $(u_1, v_1)$ and $(u_3, v_3)$ lying between $J_{31}$ and $S_{12}$, the ending point of which is also $(u_2 + u_3, v_2 + v_3)$. See Figure 7.

**Case 5.** $J_{12}\delta_{23}J_{31}$. In this case, the initial data satisfy $v_1 = v_2 = v_3$. There are two subcases corresponding to topologically distinct solutions in view of the sign of $u_1/v_1$.

**Subcase 5a.** $u_1/v_1 > 0$. The three exterior waves collide at the point $(u_2+u_3, v_3)$, where the solution in the region $\{(\xi,\eta)\big|\eta > v_3\}$ is a shock wave $S_{\infty 1}$ penetrates a centered rarefaction wave $R_{1\infty}$. Here the rarefaction wave $R_{1\infty}$ connects the states $(u_1, v_1)$ and infinity, also for $S_{\infty 1}$. See Figure 8

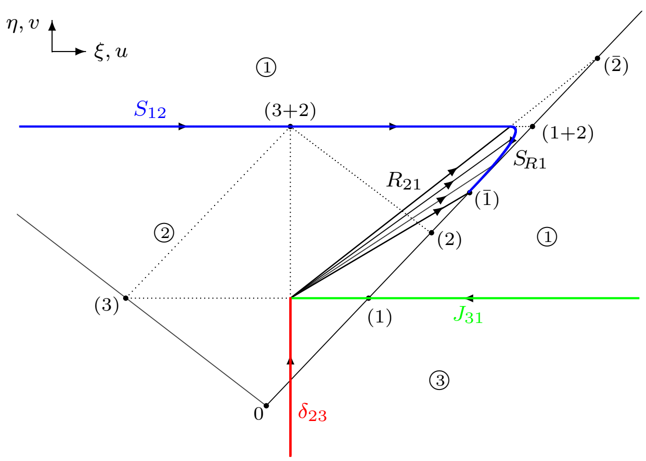


FIGURE 6. Solution involving $R_{21}$ for Subcase 4a.

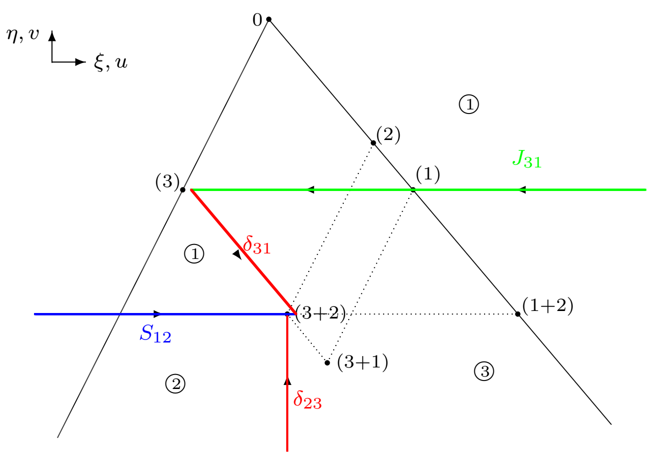


FIGURE 7. Solution for Subcase 4b.

**Subcase 5b.** $u_1/v_1 < 0$. Different from the above subcase, only two exterior waves $J_{12}$ and $J_{31}$ interact for this subcase, while the exterior wave $\delta_{23}$ can keep straight and arrive at its singularity point $(u_2+u_3, v_2+v_3)$. The interaction of the two contact discontinuities results in a delta shock wave $\delta_{32}$ connecting the states $(u_2, v_2)$ and $(u_3, v_3)$. Such a delta shock wave $\delta_{32}$ is not unique, the expression of which may be any line starting from any point $(\xi, v_3)$ where $u_3 < \xi < u_2$ and ending at the point $(u_2+u_3, v_2+v_3)$. So the solution for this subcase is not unique. See Figure 9.

So far, we have finished the construction of solutions to the Riemann problem (1.1) and (1.2) when the exterior wave connecting states $(u_2, v_2)$ and $(u_3, v_3)$ is a

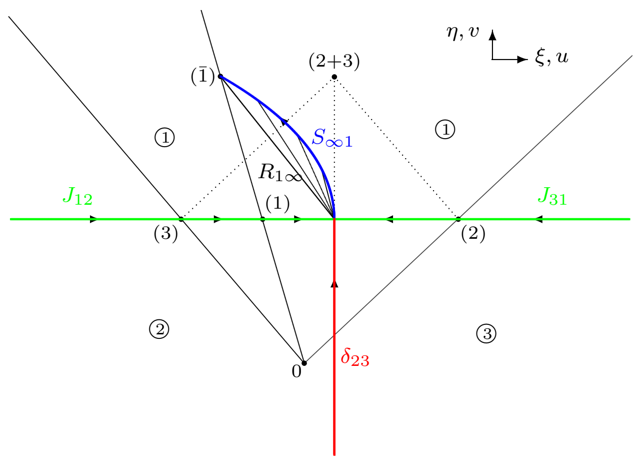


FIGURE 8. Solution for Subcase 5a.

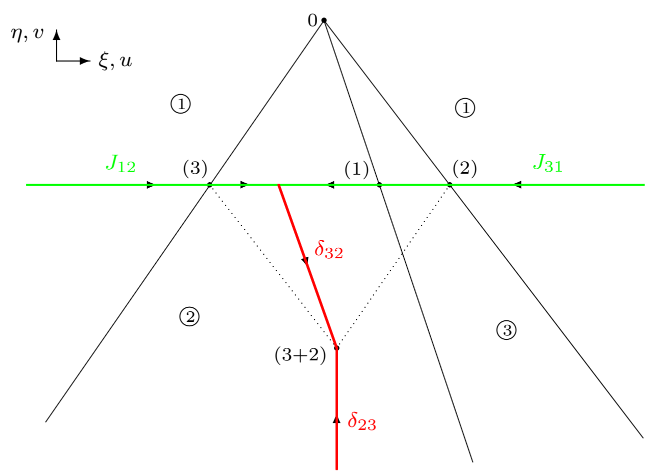


FIGURE 9. Solution for Subcase 5b.

delta shock wave $\delta_{23}$, and the other two exterior waves are classical waves as shock waves, rarefaction waves and contact discontinuities. The formation and evolution of singularities in the solutions of Riemann problem (1.1) and (1.2) are analyzed in details, which is a major difficulty in solving hyperbolic systems of conservation laws. For the other cases when the exterior waves involves only one delta shock wave propagating along in the $x-$ direction, two delta shock waves, or three delta shock waves, the discussion is complicated and we will study them in the future.

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