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BOUNDARY CONTROLLABILITY FOR A NONLINEAR BEAM EQUATION

XIAO-MIN CAO

Abstract. This article concerns a nonlinear system modeling the bending vibrations of a nonlinear beam of length $L > 0$. First, we derive the existence of long time solutions near an equilibrium. Then we prove that the nonlinear beam is locally exact controllable around the equilibrium in $H^4(0, L)$ and with control functions in $H^2(0, T)$. The approach we used are open mapping theorem, local controllability established by linearization, and the induction.

1. Introduction and statement of main results

We consider a controllability problem for a system modeling the bending vibrations of a nonlinear beam of length $L > 0$. Let ϕ denote the deflection of the beam, the left end of the beam is fixed, and an appropriate shear force u exerted on the right end of the beam, then the equations of motion describing beam bending are

$$\begin{gathered} \phi_{tt} + (a(x, \phi)\phi'')'' = b(x, \phi, \phi', \phi''), \quad x \in (0, L),\ t \in (0, T), \\ \phi(x, 0) = \phi_0(x), \quad \phi_t(x, 0) = \phi_1(x), \quad x \in (0, L), \\ \phi(0, t) = 0, \quad \phi'(0, t) = 0, \quad t \in (0, T), \\ \phi(L, t) = 0, \quad \phi'(L, t) = u(t), \quad t \in (0, T), \end{gathered} \tag{1.1}$$

where $a(x, y)$, $b(x, y_1, y_2, y_3)$ are smooth functions on $[0, L] \times \mathbb{R}$ and $[0, L] \times \mathbb{R}^3$, respectively, such that

$$a(x, y) > 0, \quad \forall (x, y) \in [0, L] \times \mathbb{R} \tag{1.2}$$

$$b(x, 0, 0, 0) = 0, \quad \forall x \in [0, L] \tag{1.3}$$

Let $\phi_0, \phi_1, \widehat{\phi}_0, \widehat{\phi}_1$ be given functions and $T > 0$ be given. If there is boundary function u on $(0, T)$ such that the solution of (1.1) satisfies $\phi(T) = \widehat{\phi}_0, \phi_t(T) = \widehat{\phi}_1$ on $[0, L]$, we say the system (1.1) is exactly controllable from (ϕ_0, ϕ_1) to $(\widehat{\phi}_0, \widehat{\phi}_1)$ at time T by controlled moment on the right end.

Boundary exact controllability on linear beam problems has been studied for many years, see [8, 9, 11, 12, 15, 17, 18], and many others.

To the best of our knowledge, there is little about the boundary control of nonlinear beam equation in the existing related papers. Recently, Yao and G. Weiss [27]

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consider a dynamical system with boundary input and output describing the bending vibrations of a quasi-linear beam, where the nonlinearity comes from Hooke's law. they show that the structure of the boundary input and output forces the system to admit global solutions at least when the initial data and the boundary input are small in a certain sense. And they prove that the norm of the state of the system decays exponentially if the input becomes zero after a finite time. Cindea and Tucsnak [3] study the exact controllability of a nonlinear plate equation by the means of a control which acts on an internal region of the plate. For rectangular domains, they obtain that the Berger system is locally exactly controllable in arbitrarily small time and for every open and nonempty control region.

Recently, using moment theory and Nash-Moser theorem, Beauchard [2] prove that the linear beam equation with clamped ends is locally controllable in a $H^{5+\varepsilon}\times H^{3+\varepsilon}((0,1),R)$ neighborhood of a particular trajectory of the free system, with $\varepsilon>0$ and with control functions in $H_0^1((0,T),R)$.

In the present work we will study boundary controllability for the nonlinear beam equation (1.1) by using some ideas of [21] for the nonlinear system of wave equations. First, we will derive the existence of long time solutions near an equilibrium. Then, the locally exact controllability of the system around the equilibrium will be established.

Let us choose some Sobolev spaces to formulate the problems. To get a smooth control, we assume initial data $\phi_0\in H^4(0,L),\phi_1\in H^2(0,L)$ to study the controllability of the system around the equilibrium in $H^4(0,L)$ at time T via a boundary control $u\in H^2(0,T)$. Obviously, control function we obtained is more smooth than the previous results [2, 9].

We say $\omega\in H^4(0,L)$ is an equilibrium of the system (1.1) if

$$(a(x,\omega)\omega'')''=b(x,\omega,\omega',\omega''). \tag{1.4}$$

Let $\phi_0\in H^4(0,L)$, $\phi_1\in H^2(0,L)$, $u\in H^2(0,T)$. We say these functions satisfy compatibility conditions if

$$\begin{gathered}\phi_0(0)=0,\phi_0'(0)=0,\phi_1(0)=0,\phi_1'(0)=0,\\ \phi_0(L)=0,\phi_0'(L)=u(0),\phi_1(L)=0,\phi_1'(L)=\dot u(0).\end{gathered}$$

Set

$$V(0,L)=\{\varphi|\varphi\in H^2(0,L),\varphi(0)=\varphi(L)=\varphi'(0)=0.\} \tag{1.5}$$

The next result shows that near one equilibrium, the system has solutions of long time.

Theorem 1.1. *Let $w\in H^4(0,L)\cap V(0,L)$ be an equilibrium of (1.1). Let $T>0$ be arbitrary given. Then there is $\varepsilon_T>0$, which depends on the time T, such that if $\phi_0\in H^4(0,L)\cap V(0,L),\phi_1\in V(0,L)$ satisfy*

$$\|\phi_0-\omega\|_4<\varepsilon_T,\|\phi_1\|_2<\varepsilon_T,$$

and $u\in H^2(0,T)$ satisfies the compatibility conditions with (ϕ_0,ϕ_1) and $\|u\|_2<\varepsilon_T$, then system (1.1) has a solution

$$\phi\in C([0,T],H^4(0,L))\cap C^1([0,T],H^2(0,L))\cap C^2([0,T],L^2(0,L)).$$

Near an equilibrium, we have the following locally exact controllability results.

Theorem 1.2. *Let $\omega \in H^4(0,L) \cap V(0,L)$ be an equilibrium of* (1.1). *Then, for $T > 0$ given, there is $\varepsilon_T > 0$ such that for any $(\phi_0^i, \phi_1^i) \in (H^4(0,L) \cap V(0,L)) \times V(0,L)(i = 1, 2)$ with*

$$\|\phi_0^i - \omega\|_4 < \varepsilon_T, \|\phi_1^i\|_2 < \varepsilon_T,$$

we can find $u \in H^2(0,T)$ which is compatible with (ϕ_0^1, ϕ_1^1) such that the solution of the system (1.1) *with the initial data (ϕ_0^1, ϕ_1^1) satisfies*

$$\phi(T) = \phi_0^2, \quad \phi_t(T) = \phi_1^2.$$

2. Existence of long time solutions near equilibria

The existence of short time solutions for the nonlinear beam equation can be proved using standard arguments such as the nonlinear semigroups theory or the Galerkin method and fixed point arguments [1, 4, 16, 20]. We only study some energy estimates of the short time solutions to have long time solutions when initial data are close to an equilibrium here.

We suppose that the equilibrium is the zero in this section. If an equilibrium $\omega \in H^4(0,L) \cap V(0,L)$ is not zero, we can make a transform by $\phi = \omega + \psi$, and consider the problem

$$\begin{gathered}
\psi_{tt} + (a(x, \omega + \psi)(\psi'' + \omega''))'' = b(x, \omega + \psi, \omega' + \psi', \omega'' + \psi''),\\
x \in (0,L), t \in (0,T),\\
\psi(x,0) = \phi_0 - \omega, \quad \psi_t(x,0) = \phi_1(x), \quad x \in (0,L),\\
\psi(0,t) = 0, \psi'(0,t) = 0, \quad t \in (0,T),\\
\psi(L,t) = 0, \quad \psi'(L,t) = u(t) - \omega'(L), \quad t \in (0,T),
\end{gathered}$$

Let $\phi \in C([0,T], H^4(0,L)) \cap C^1([0,T], H^2(0,L)) \cap C^2([0,T], L^2(0,L))$ be a solution of (1.1) for some $T > 0$. Suppose that $u \in H^2(0,T)$. We introduce

$$\begin{gathered}
\mathcal{E}(t) = \|\phi\|_4^2 + \|\phi_t\|_2^2 + \|\phi_{tt}\|^2,\\
\mathcal{E}_\Gamma(t) = u^2(t) + \dot{u}^2(t) + \ddot{u}^2(t),\\
\mathcal{Q}(t) = \|\phi_t\|^2 + \|\phi''\|^2 + \|\phi_t'\|^2 + \|\phi'''\|^2 + \|\phi_{tt}\|^2 + \|\phi_t''\|^2.
\end{gathered}$$

For solutions of (1.1) near the zero equilibrium, we have the following result.

Theorem 2.1. *Let $\gamma > 0$ be given and ϕ be a solution of* (1.1) *on the interval $[0,T]$ for some $T > 0$ such that the condition*

$$\sup_{0 \le t \le T} \|\phi(t)\|_4 \le \gamma. \tag{2.1}$$

holds. Then there is $c_\gamma > 0$, which depends on the γ but is independent of initial data (ϕ_0, ϕ_1) and boundary functions u, such that

$$\mathcal{Q}(t) \le c_\gamma \mathcal{Q}(0) + c_\gamma \int_0^t \Big[(1 + \mathcal{E}^{1/2}(\tau) + \mathcal{E}(\tau) + \mathcal{E}^{\frac{3}{2}}(\tau))\mathcal{E}(\tau) + \mathcal{E}_\Gamma(\tau)\Big] d\tau. \tag{2.2}$$

and

$$\mathcal{Q}(t) \le \mathcal{E}(t) \le c_\gamma \mathcal{Q}(t) + c_\gamma \mathcal{E}_\Gamma(t), \tag{2.3}$$

for $t \in [0,T]$.

Here we list a few basic properties of Sobolev spaces to be invoked in the sequel.

(i) Let $s_1 > s_2 \geq 0$. For any $\varepsilon > 0$ there is $c_\varepsilon > 0$ such that

$$\|\varphi\|_{s_2}^2 \leq \varepsilon\|\varphi\|_{s_1}^2 + c_\varepsilon\|\varphi\|^2, \quad \forall\varphi \in H^{s_1}(0,L).$$

(ii) If $m > \frac{1}{2}$, then for each $k = 0, 1, \cdots$, we have $H^{m+k}(0,L) \subset C^k([0,L])$ with continuous inclusion.

(iii) If $r := \min\{s_1, s_2, s_1 + s_2 - 1\} \geq 0$, then there is a constant $c > 0$ such that

$$\|fg\|_r \leq c\|f\|_{s_1}\|g\|_{s_2}, \forall f \in H^{s_1}(0,L), g \in H^{s_2}(0,L). \tag{2.4}$$

Lemma 2.2. *(i) Let $f(x,y)$ be a smooth function on $[0,L]\times\mathbb{R}$. Set $F(x) = f(x,\phi)$, for $0 \leq k \leq 4$, then there is $c = c(\sup_{x\in[0,L]}|\phi|) > 0$, such that*

$$\|F\|_k \leq c\sum_{j=0}^{k}(1 + \|\phi\|_4)^j. \tag{2.5}$$

(ii) Let $f(x, y_1, y_2, y_3)$ be a smooth function on $[0,L]\times\mathbb{R}^3$. Set

$$G(x) = f(x, \phi, \phi', \phi''),$$

then there is $c = c\big(\sup_{x\in[0,L]}|\phi|, \sup_{x\in[0,L]}|\phi'|, \sup_{x\in[0,L]}|\phi''|\big) > 0$, such that

$$\|G\| \leq c. \tag{2.6}$$

Proof. (i) Inequality (2.5) is clearly true for $k = 0$. By using (2.4) and the induction of the order k, inequality (2.5) follows.

(ii) A standard method as to the linearly elliptic problem can give inequality (2.6), for example see Taylor [22]. □

Observing the partial differential of the function $a(x,\phi)$ and $b(x,\phi,\phi',\phi'')$, using the formula (2.4) and Lemma 2.2, we have the following lemma.

Lemma 2.3. *Let $\gamma > 0$ be given and ϕ be a solution of (1.1) on the interval [0,T] for some $T > 0$ such that the condition (2.1) holds true. Then there is $c_\gamma > 0$, which depends on the γ, such that*

$$\|a_t\|_2 \leq c_\gamma\mathcal{E}^{1/2}(t), \quad \|a_t'\|_1 \leq c_\gamma\big(\mathcal{E}^{1/2}(t) + \mathcal{E}(t)\big),$$
$$\|a_t''\| \leq c_\gamma\big(\mathcal{E}^{1/2}(t) + \mathcal{E}(t) + \mathcal{E}^{\frac{3}{2}}(t)\big), \quad \|b_t\| \leq c_\gamma\mathcal{E}^{1/2}(t).$$

Lemma 2.4. *Let $\gamma > 0$ be given and ϕ be a solution of (1.1) on the interval [0,T] for some $T > 0$ such that the condition (2.1) holds true. Set*

$$\Upsilon_1(t) = \|\phi_t\|^2 + \|\phi''\|^2, \quad \Upsilon_2(t) = \|\phi_t'\|^2 + \|\phi'''\|^2, \quad \Upsilon_3(t) = \|\phi_{tt}\|^2 + \|\phi_t''\|^2.$$

Then there is $c_\gamma > 0$, which depends on the γ, such that

$$\Upsilon_1(t) \leq c_\gamma\Upsilon_1(0) + c_\gamma\int_0^t \big[(1 + \mathcal{E}^{\frac{1}{2}}(\tau))\mathcal{E}(\tau) + \mathcal{E}_\Gamma(\tau)\big]d\tau, \tag{2.7}$$

$$\begin{aligned}&\Upsilon_2(t) + \Upsilon_3(t)\\ &\leq c_\gamma\big(\Upsilon_2(0) + \Upsilon_3(0)\big) + c_\gamma\int_0^t \Big[\Big(1 + \mathcal{E}^{1/2}(\tau) + \mathcal{E}(\tau) + \mathcal{E}^{\frac{3}{2}}(\tau)\Big)\mathcal{E}(\tau) + \mathcal{E}_\Gamma(\tau)\Big]d\tau\end{aligned} \tag{2.8}$$

for $0 \leq t \leq T$.

Proof. Let

$$P_1(t) = \|\phi_t\|^2 + (a\phi'', \phi'').$$

We obtain

$$\begin{aligned}\dot P_1(t) &= 2(\phi_t, \phi_{tt}) + 2(a\phi'', \phi_t'') + (a_t\phi'', \phi'')\\ &= 2(\phi_t, b - (a\phi'')'') + 2(a\phi'', \phi_t'') + (a_t\phi'', \phi'')\\ &= 2(\phi_t, b) + 2\dot u(t)(a\phi'')(L,t) + (a_t\phi'', \phi'').\end{aligned}$$

It follows that

$$\begin{aligned}\Upsilon_1(t) &\le c_\gamma P_1(t)\\ &\le c_\gamma P_1(0) + c_\gamma\int_0^t (\|\phi_t\|\|b\| + \|a_t\|_2\|\phi''\|^2)dt + c_\gamma\int_0^t (\dot u^2(t) + \phi''^2(L,t))dt\\ &\le c_\gamma \Upsilon_1(0) + c_\gamma\int_0^t \Big[(1+\mathcal{E}^{\frac12}(\tau))\mathcal{E}(\tau) + \mathcal{E}_\Gamma(\tau)\Big]d\tau.\end{aligned}$$

To obtain inequality (2.8), first we differentiate equation (1.1) with respect to x, then we have

$$\phi_{tt}' + (a\phi'')''' = b'. \tag{2.9}$$

Let

$$P_2(t) = \|\phi_t'\|^2 + (a\phi''', \phi'''). \tag{2.10}$$

We obtain

$$\begin{aligned}\dot P_2(t) &= 2(\phi_t', \phi_{tt}') + 2(a\phi''', \phi_t''') + (a_t\phi''', \phi''')\\ &= 2(\phi_t', b' - (a\phi'')''') + 2(a\phi''', \phi_t''') + (a_t\phi''', \phi''')\\ &= 2(\phi_t', b') - 2\dot u(t)b(L,t) + 2\phi_t''(L,t)(a\phi''')(L,t) - 2\phi_t''(0,t)(a\phi''')(0,t)\\ &\quad - 2(\phi_t'', a'\phi''') + 2(\phi_t'', 2a'\phi''' + a''\phi'') + (a_t\phi''', \phi''').\end{aligned}$$

So

$$\begin{aligned}\Upsilon_2(t) &\le c_\gamma P_2(t)\\ &\le c_\gamma P_2(0) + c_\gamma\int_0^t \Big[(1+\mathcal{E}^{1/2}(\tau))\mathcal{E}(\tau) + \dot u^2(\tau)\Big]d\tau\\ &\quad + \varepsilon\int_0^t \big(\phi_t''^2(0,t) + \phi_t''^2(L,t)\big)dt + c_{\gamma,\varepsilon}\int_0^t \big(\phi'''^2(0,t) + \phi'''^2(L,t)\big)dt.\end{aligned} \tag{2.11}$$

Furthermore, we differentiate (1.1) with respect to t, then we have

$$\phi_{ttt} + (a\phi_t'')'' + (a_t\phi'')'' = b_t. \tag{2.12}$$

Let

$$P_3(t) = \|\phi_{tt}\|^2 + (a\phi_t'', \phi_t''). \tag{2.13}$$

We deduce that

$$\begin{aligned}\dot P_3(t) &= 2(\phi_{tt}, \phi_{ttt}) + 2(a\phi_t'', \phi_{tt}'') + (a_t\phi_t'', \phi_t'')\\ &= 2(\phi_{tt}, b_t - (a_t\phi'')'' - (a\phi_t'')'') + 2(a\phi_t'', \phi_{tt}'') + (a_t\phi_t'', \phi_t'')\\ &= 2(\phi_{tt}, b_t) - 2(\phi_{tt}, a_t\phi^{(4)}) - 4(\phi_{tt}, a_t'\phi''') - 2(\phi_{tt}, a_t''\phi'')\\ &\quad + 2\ddot u(t)(a\phi_t'')(L,t) + (a_t\phi_t'', \phi_t'').\end{aligned}$$

So

$$\begin{aligned}\Upsilon_3(t) &\le c_\gamma P_3(t)\\ &\le c_\gamma P_3(0) + c_\gamma \int_0^t \big[(1+\mathcal{E}^{1/2}(\tau)+\mathcal{E}(\tau)+\mathcal{E}^{\frac{3}{2}}(\tau))\mathcal{E}(\tau)\big]d\tau\\ &\quad + \varepsilon \int_0^t \phi_t''^2(L,t)dt + c_{\gamma,\varepsilon}\int_0^t \ddot{u}^2(t)dt.\end{aligned} \tag{2.14}$$

From (2.11) and (2.14), we conclude that

$$\begin{aligned}\Upsilon_2(t)+\Upsilon_3(t) &\le c_\gamma(P_2(0)+P_3(0)) + \varepsilon\int_0^t \big(\phi_t''^2(0,t)+\phi_t''^2(L,t)\big)dt\\ &\quad + c_\gamma\int_0^t \big[(1+\mathcal{E}^{1/2}(\tau)+\mathcal{E}(\tau)+\mathcal{E}^{\frac{3}{2}}(\tau))\mathcal{E}(\tau)+\dot{u}^2(t)\big]d\tau\\ &\quad + c_{\gamma,\varepsilon}\int_0^t \big(\phi'''^2(0,t)+\phi'''^2(L,t)+\ddot{u}^2(t)\big)dt.\end{aligned} \tag{2.15}$$

Next, we estimate the terms $\int_0^t \phi_t''^2(0,t)dt$ and $\int_0^t \phi_t''^2(L,t)dt$.

Differentiating twice (1.1) with respect to x, multiplying the two sides of the equation by $(L-x)\phi'''$ and integrating from 0 to L by parts, we obtain

$$\phi_{tt}'' + (a\phi'')^{(4)} = b'', \tag{2.16}$$

$$\int_0^L \phi_{tt}''(L-x)\phi'''dx = \frac{\partial}{\partial t}(\phi_t'',(L-x)\phi''') + \frac{1}{2}L\phi_t''^2(0,t) - \frac{1}{2}\int_0^L \phi_t''^2 dx,$$

$$\begin{aligned}&\int_0^L (b''-(a\phi'')^{(4)})(L-x)\phi'''dx\\ &= (b'',(L-x)\phi''') - b'(0,t)L\phi'''(0,t) + \ddot{u}_1(t)L\phi'''(0,t) - \frac{1}{2}L(a{\phi^{(4)}}^2)(0,t)\\ &\quad - \frac{1}{2}\int_0^L (a'(L-x)-a){\phi^{(4)}}^2dx + \int_0^L (3a'\phi^{(4)}+3a''\phi'''+a'''\phi'')(L-x)\phi^{(4)}dx\\ &\quad - b(L,t)\phi'''(L,t) + b(0,t)\phi'''(0,t) + \int_0^L (a\phi'')''\phi^{(4)}dx.\end{aligned}$$

Whence

$$\begin{aligned}&\frac{1}{2}L\phi_t''^2(0,t)\\ &= -\frac{\partial}{\partial t}(\phi_t'',(L-x)\phi''') + \frac{1}{2}\int_0^L \phi_t''^2dx + (b'',(L-x)\phi''')\\ &\quad - b'(0,t)L\phi'''(0,t) - \frac{1}{2}L(a{\phi^{(4)}}^2)(0,t) - \frac{1}{2}\int_0^L (a'(L-x)-a){\phi^{(4)}}^2dx\\ &\quad + \int_0^L (3a'\phi^{(4)}+3a''\phi'''+a'''\phi'')(L-x)\phi^{(4)}dx\\ &\quad - b(L,t)\phi'''(L,t) + b(0,t)\phi'''(0,t) + \int_0^L (a\phi'')''\phi^{(4)}dx.\end{aligned} \tag{2.17}$$

It follows that

$$\begin{aligned}\int_0^t \phi_t''^2(0,t)dt &\le c_\gamma\Big(\Upsilon_2(0)+\Upsilon_3(0)+\Upsilon_2(t)+\Upsilon_3(t)\Big)\\ &\quad + c_\gamma\int_0^t \Big[\Big(1+\mathcal{E}^{1/2}(\tau)+\mathcal{E}(\tau)+\mathcal{E}^{\frac32}(\tau)\Big)\mathcal{E}(\tau)\Big]d\tau.\end{aligned} \tag{2.18}$$

Similarly, with respect to the term $\int_0^t \phi_t''^2(L,t)dt$, multiplying the two sides of (2.16) by $x\phi'''$ and integrating from 0 to L by parts, we obtain

$$\int_0^L \phi_{tt}'' x\phi''' dx = \frac{\partial}{\partial t}(\phi_t'', x\phi''') + \frac12 L\phi_t''^2(L,t) - \frac12\int_0^L \phi_t''^2 dx, \tag{2.19}$$

and

$$\begin{aligned}&\int_0^L (b''-(a\phi'')^{(4)})x\phi''' dx\\ &= (b'', x\phi''') - b'(L,t)L\phi'''(L,t) + \ddot u(t)L\phi'''(L,t) + \frac12 L(a{\phi^{(4)}}^2)(L,t)\\ &\quad - \frac12\int_0^L (a'x+a){\phi^{(4)}}^2 dx + \int_0^L (3a'\phi^{(4)}+3a''\phi'''+a'''\phi'')x\phi^{(4)}dx\\ &\quad + b(L,t)\phi'''(L,t) - b(0,t)\phi'''(0,t) - \int_0^L (a\phi'')''\phi^{(4)}dx.\end{aligned} \tag{2.20}$$

Using (2.19) and (2.20) we have

$$\begin{aligned}&\frac12 L\phi_t''^2(L,t)\\ &= -\frac{\partial}{\partial t}(\phi_t'', x\phi''') + \frac12\int_0^L \phi_t''^2 dx + (b'', x\phi''') - b'(L,t)L\phi'''(L,t)\\ &\quad + \ddot u(t)L\phi'''(L,t) + \frac12 L(a{\phi^{(4)}}^2)(L,t) - \frac12\int_0^L (a'x+a){\phi^{(4)}}^2dx\\ &\quad + \int_0^L (3a'\phi^{(4)}+3a''\phi'''+a'''\phi'')x\phi^{(4)}dx\\ &\quad + b(L,t)\phi'''(L,t) - b(0,t)\phi'''(0,t) - \int_0^L (a\phi'')''\phi^{(4)}dx.\end{aligned} \tag{2.21}$$

We deduce the inequality

$$\begin{aligned}\int_0^t \phi_t''^2(L,t)dt &\le c_\gamma\Big(\Upsilon_2(0)+\Upsilon_3(0)+\Upsilon_2(t)+\Upsilon_3(t)\Big)\\ &\quad + c_\gamma\int_0^t \Big[\Big(1+\mathcal{E}^{1/2}(\tau)+\mathcal{E}(\tau)+\mathcal{E}^{\frac32}(\tau)\Big)\mathcal{E}(\tau)+\ddot u^2(t)\Big]d\tau.\end{aligned} \tag{2.22}$$

Finally, Substituting inequality (2.18) and (2.22) in (2.15), choosing $\varepsilon > 0$ small enough such that the term $\varepsilon c_\gamma[\Upsilon_2(t)+\Upsilon_3(t)]$ can be moved to the left hand side of the inequality, then (2.8) is obtained. □

Proof of Theorem 2.1. Lemma 2.4 gives inequality (2.2). To prove inequality (2.3), we notice that $a\phi^{(4)} = b - \phi_{tt} - 2a'\phi''' - a''\phi''$, then there is $c_\gamma > 0$, such that

$$\begin{aligned}\|\phi^{(4)}\|^2 &\le c_\gamma(a\phi^{(4)}, \phi^{(4)})\\ &= c_\gamma(b-\phi_{tt}-2a'\phi'''-a''\phi'', \phi^{(4)})\end{aligned}$$

$$\le c_{\gamma,\varepsilon}(\|b\|^2 + \|\phi_{tt}\|^2 + \|\phi'''\|^2 + \|\phi''\|^2) + \varepsilon\|\phi^{(4)}\|^2.$$

Choosing ε sufficiently small, then there is $c_\gamma > 0$, such that

$$\|\phi^{(4)}\|^2 \le c_\gamma(\|b\|^2 + \|\phi_{tt}\|^2 + \|\phi'''\|^2 + \|\phi''\|^2) \le c_\gamma \mathcal{Q}(t).$$

From the definition of $\mathcal{Q}(t)$ and $\mathcal{E}(t)$ and using the Poincaré inequality, inequality (2.3) holds. □

Using the standard method as for the global solution of partial differential equation, from Theorem 2.1 we have the following proof.

Proof of Theorem 1.1. Clearly, it will suffice to prove Theorem 1.1 for the zero equilibrium $\omega = 0$.

Let $T_1 > 0$ be arbitrary given. We take $\gamma = 1$. Let

$$c_1 = c_\gamma \ge 1 \tag{2.23}$$

be fixed such that the corresponding inequalities (2.2) and (2.3) of Theorem 2.1 hold for t in the existence interval of the solution ϕ.

We shall prove that, if the initial data (ϕ_0, ϕ_1) and boundary value u are compatible and satisfy

$$\mathcal{E}(0) + \max_{0\le t\le T_1} \mathcal{E}_\Gamma(t) + \int_0^{T_1} \mathcal{E}_\Gamma(t)dt \le \frac{1}{16c_1^3} e^{-4c_1^2 T_1}, \tag{2.24}$$

then the solution of problem (1.1) exists at least on the interval $[0, T_1]$.

We set

$$\eta = \frac{1}{4c_1} < \frac{1}{2}. \tag{2.25}$$

Since $\mathcal{E}(0) \le \frac{\eta}{4}$, the solution of short time must satisfy

$$\mathcal{E}(t) \le \eta \le \frac{1}{2} \tag{2.26}$$

for some interval $[0, \delta]$. Let δ_0 be the largest number such that (2.26) is true for $t \in [0, \delta_0)$. We shall prove $\delta_0 \ge T_1$ by contradiction.

Suppose that $\delta_0 < T_1$. In this interval $t \in [0, \delta_0]$, the condition (2.1) is true. We apply Theorem 2.1 and the inequalities (2.2) and (2.3), via (2.23)–(2.26). Then we conclude that

$$\mathcal{E}(t) \le 2c_1^2[\mathcal{E}(0) + \max_{0\le t\le T_1} \mathcal{E}_\Gamma(t) + \int_0^{T_1} \mathcal{E}_\Gamma(t)dt] + 4c_1^2 \int_0^t \mathcal{E}(t)dt, \tag{2.27}$$

for $t \in [0, \delta_0]$. By (2.24) and (2.27), the Gronwall inequality yields

$$\mathcal{E}(\delta_0) \le \eta/2 < \eta.$$

This is a contradiction. □

3. Locally exact controllability

Let $T > 0$ given. The first step of the proof for the local exact controllability depends on the following fact: Let X and Y be Banach spaces and $\Phi : U \to Y$, where U is an open subset of X, be Frechét differentiable. If $\Phi'(x_0) : X \to Y$ is surjective, then there is an open neighborhood of $y_0 = \Phi(x_0)$ contained in the image $\Phi(U)$.

Given an equilibrium $\omega \in H^4(0,L) \cap V(0,L)$, We invoke Theorem 1.1 to define the map for $u \in H^2(0,T)$ by setting

$$F(u) = (\psi(\cdot,T), \psi_t(\cdot,T)),$$

where ψ is the solution to

$$\begin{gathered} \psi_{tt} + (a(x,\psi)\psi'')'' = b(x,\psi,\psi',\psi''), \quad x \in (0,L), t \in (0,T), \\ \psi(x,0) = \omega(x), \quad \psi_t(x,0) = 0, \quad x \in (0,L), \\ \psi(0,t) = 0, \quad \psi'(0,t) = 0, \quad t \in (0,T), \\ \psi(L,t) = 0, \quad \psi'(L,t) = u + \omega'(L), \quad t \in (0,T), \end{gathered} \tag{3.1}$$

Let $\varepsilon_T > 0$ be given by Theorem 1.1. Then

$$F : B(0,\varepsilon_T) \to (H^4(0,L) \cap V(0,L)) \times V(0,L), \tag{3.2}$$

where $B(0,\varepsilon_T) \subset H^2(0,T)$ is the ball with the radius ε_T centered at 0. We note that $F(0) = (\omega(x), 0)$.

We need to evaluate $F'(0)(u) = D_\lambda F(\lambda u)|_{\lambda=0}$. It is easy to check that $F'(0)(u) = (\psi(\cdot,T), \psi_t(\cdot,T))$, where $\psi(x,t)$ is the solution of the linear system

$$\begin{gathered} \psi_{tt} + (p(x)\psi'')'' = b_1(x)\psi + b_2(x)\psi' + b_3(x)\psi'', \quad x \in (0,L), t \in (0,T), \\ \psi(x,0) = 0, \quad \psi_t(x,0) = 0, \quad x \in (0,L), \\ \psi(0,t) = 0, \quad \psi'(0,t) = 0, \quad t \in (0,T), \\ \psi(L,t) = 0, \quad \psi'(L,t) = u(t), \quad t \in (0,T), \end{gathered} \tag{3.3}$$

and

$$\begin{gathered} p(x) = a(x,\omega), \\ b_1(x) = b_{y_1}(x,\omega,\omega',\omega'') - (a_y(x,\omega)\omega'')'', \\ b_2(x) = b_{y_2}(x,\omega,\omega',\omega'') - 2(a_y(x,\omega)\omega'')', \\ b_3(x) = b_{y_3}(x,\omega,\omega',\omega'') - a_y(x,\omega)\omega''. \end{gathered}$$

We now verify that $F'(0)$ is surjective. In the language of control theory surjectivity is just exact controllability, which for a reversible system such as (3.3) is equivalent to null controllability.

Explicitly one has to show that, for $T > 0$ given, given $\psi_0 \in H^4(0,L) \cap V(0,L), \psi_1 \in V(0,L)$, one can find $u \in H^2(0,T)$ such that the solution to

$$\begin{gathered} \psi_{tt} + (p(x)\psi'')'' = b_1(x)\psi + b_2(x)\psi' + b_3(x)\psi'', \quad x \in (0,L), t \in (0,T), \\ \psi(x,0) = \psi_0(x), \quad \psi_t(x,0) = \psi_1(x), \quad x \in (0,L), \\ \psi(0,t) = 0, \quad \psi'(0,t) = 0, \quad t \in (0,T), \\ \psi(L,t) = 0, \quad \psi'(L,t) = u(t), \quad t \in (0,T), \end{gathered} \tag{3.4}$$

satisfies $\psi(\cdot,T) = 0$ and $\psi_t(\cdot,T) = 0$.

Theorem 3.1. *Given an equilibrium $\omega \in H^4(0,L) \cap V(0,L)$. Let $T > 0$ given. Then for any*

$$(\psi^0, \psi^1) \in (H^4(0,L) \cap V(0,L)) \times V(0,L),$$

there is a control $u \in H^2(0,T)$ such that the solution

$$\psi \in C([0,T], H^4(0,L)) \cap C^1([0,T], H^2(0,L)) \cap C^2([0,T], L^2(0,L))$$

of problem (3.4) *satisfies $\psi(\cdot,T) = 0$ and $\psi_t(\cdot,T) = 0$.*

Unfortunately, our proof does not give any information on the time T needed for the controlled motion.

3.1. **Distributed control.** As to the exact controllability of linear systems by distributed control, there is a long history and a lot of results where many approaches are involved. Here the distributed control means that solutions $(\psi(t), \psi_t(t))$ of the controlled system (3.4) are only in the space $L^2(0,L)\times H^{-2}(0,L)$ for $t\in[0,T]$. One of the useful approaches is the multiplier method, which introduced by Ho [6] where Lions [15] provided a key technique of multipliers for observability estimates, to control the linear system by its duality system. In the case of constant coefficients, Lagnese [9, 11], study the controllability of linear beam by using the multiplier method. In this subsection, we will consider the controllability of linear beam with variable coefficients.

As noted in [9], everything is therefore going to rely on a prior inequality

$$\|(\phi_0,\phi_1)\|^2_{H_0^2\times L^2}\le C_T\int_0^T {\phi''}^2(L,t)dt,\quad \forall(\phi_0,\phi_1)\in H_0^2(0,L)\times L^2(0,L) \tag{3.5}$$

where ϕ is the solution of the system

$$\begin{gathered}
\phi_{tt}+(p(x)\phi'')''=b_1(x)\phi-(b_2(x)\phi)'+(b_3(x)\phi)'',\quad x\in(0,L),t\in(0,T),\\
\phi(x,0)=\phi_0(x),\quad \phi_t(x,0)=\phi_1(x),\quad x\in(0,L),\\
\phi(0,t)=0,\phi'(0,t)=0,\quad t\in(0,T),\\
\phi(L,t)=0,\phi'(L,t)=0,\quad t\in(0,T),
\end{gathered}\tag{3.6}$$

Given $\phi_0\in H_0^2(0,L),\phi_1\in L^2(0,L)$. We define ϕ as the solution to (3.6) and we define η as the solution of

$$\begin{gathered}
\eta_{tt}+(p(x)\eta'')''=b_1(x)\eta+b_2(x)\eta'+b_3(x)\eta''\quad x\in(0,L),t\in(0,T),\\
\eta(x,T)=0,\quad \eta_t(x,T)=0,\quad x\in(0,L),\\
\eta(0,t)=0,\quad \eta'(0,t)=0,\quad t\in(0,T),\\
\eta(L,t)=0,\quad \eta'(L,t)=\phi''(L,t),\quad t\in(0,T),
\end{gathered}\tag{3.7}$$

We define the operator $\Lambda: H_0^2(0,L)\times L^2(0,L)\to H^{-2}(0,L)\times L^2(0,L)$ by

$$\lambda(\phi_0,\phi_1)=(\eta_t(0),-\eta(0)) \tag{3.8}$$

The solution η of (3.7) is a weak solution defined by transposition, in such a way that Green's formula makes sense. Therefore

$$\langle\Lambda(\phi_0,\phi_1),(\phi_0,\phi_1)\rangle=\int_0^T p(L){\phi''}^2(L,t)dt. \tag{3.9}$$

Lemma 3.2. *Let ϕ be a solution of (3.6). Then there exist constants $c,c_0>0$ such that for $T\ge t>0$:*

$$e^{-ct}E(0)-N(T)\le E(t)\le[E(0)+N(T)]e^{ct}, \tag{3.10}$$

where we introduced the energy

$$E(t)=\int_0^L\Big(\phi_t^2+p(x){\phi''}^2\Big)dx$$

and set $N(T)=c_0\int_0^T\|\phi\|_1^2dt$.

Proof. Multiplying equation (3.6) by ϕ_t and integrating over $(s,t)\times(0,L)$ by parts in t on the left-hand side we obtain that for all s,t:

$$E(t) = E(s) + 2\int_s^t \int_0^L \phi_t\Big(b_1(x)\phi - (b_2(x)\phi)' + (b_3(x)\phi)''\Big)\,dx\,dt. \tag{3.11}$$

By Schwartz inequality, we obtain for $t \geq s \geq 0$:

$$2\int_s^t \int_0^L \phi_t\Big(b_1(x)\phi - (b_2(x)\phi)' + (b_3(x)\phi)''\Big)\,dx\,dt \leq N(T) + c\int_s^t E(\tau)d\tau.$$

So

$$E(t) \leq [E(s) + N(T)] + c\int_s^t E(\tau)d\tau, \tag{3.12}$$

$$E(s) \leq [E(t) + N(T)] + c\int_s^t E(\tau)d\tau. \tag{3.13}$$

We apply the classical argument of the Gronwall's inequality to (3.12) and (3.13), where we note that the terms into the square brackets are independent of s in (3.13), we thus obtain for $t \geq s \geq 0$:

$$E(t) \leq [E(s) + N(T)]e^{c(t-s)}; \quad E(s) \leq [E(t) + N(T)]e^{c(t-s)}. \tag{3.14}$$

Setting $s = 0$ and thus $t > 0$ in (3.14) yields (3.10). □

The observability inequality for the system (3.6) is covered in the following lemma.

Lemma 3.3. *Let an equilibrium $\omega \in H^4(0,L) \cap V(0,L)$ be given. Then for $T > 0$, Λ is an isomorphism from $H_0^2(0,L)\times L^2(0,L)$ onto $H^{-2}(0,L)\times L^2(0,L)$. In particular, there are constants $C_1 > 0$ and $C_2 > 0$ such that the inequality*

$$C_1\|(\phi_0,\phi_1)\|^2_{H_0^2\times L^2} \leq \int_0^T p(L)\phi''^2(L,t)dt \leq C_2\|(\phi_0,\phi_1)\|^2_{H_0^2\times L^2} \tag{3.15}$$

holds true for $(\phi_0,\phi_1) \in H_0^2(0,L)\times L^2(0,L)$.

Proof. Assume that $\phi_0 \in H_0^2(0,L), \phi_1 \in L^2(0,L)$. Then (3.6) admits a unique solution $\phi \in C([0,T];H_0^2(0,L)) \cap C^1([0,T];L^2(0,L))$ and the energy $E(t)$ of the system (3.6) satisfies

$$E(0) = E(\phi_0,\phi_1) = \int_0^L (\phi_1^2 + p(x)\phi_0''^2)dx.$$

That is, we are going to prove that there exist two constants $C_1, C_2 > 0$ such that:

$$C_1E(0) \leq \int_0^T p(L)\phi''^2(L,t)dt \leq C_2E(0). \tag{3.16}$$

We always have the right side of (3.16), so we just need to prove that

$$\int_0^T p(L)\phi''^2(L,t)dt \geq C_1E(0).$$

We use multiplier $h(x)\phi'$, where $h(x)$ is a function on the interval $[0,L]$, and we obtain

$$\int_0^L\int_0^T \phi_{tt}h(x)\phi'\,dx\,dt = (h(x)\phi',\phi_t)|_0^T + \frac{1}{2}\int_0^T\int_0^L h'(x)\phi_t^2\,dx\,dt.$$

$$\begin{aligned}&\int_0^T\int_0^L (p(x)\phi'')''h(x)\phi'\,dx\,dt\\&= -\frac{1}{2}\int_0^T p(x)h(x)\phi''^2|_0^L dt + \frac{3}{2}\int_0^T\int_0^L p(x)h'(x)\phi''^2\,dx\,dt\\&\quad -\frac{1}{2}\int_0^T\int_0^L p'(x)h(x)\phi''^2\,dx\,dt - \frac{1}{2}\int_0^T\int_0^L (p(x)h''(x))'\phi'^2\,dx\,dt.\end{aligned}$$

$$\begin{aligned}&\int_0^T\int_0^L (b_1(x)\phi - (b_2(x)\phi)' + (b_3(x)\phi)'')h\phi'\,dx\,dt\\&= \int_0^T\int_0^L \Big(m_1(x)\phi^2 + m_2(x)\phi'^2\Big)\,dx\,dt.\end{aligned}$$

These equalities give

$$\begin{aligned}&(h(x)\phi',\phi_t)|_0^T + \frac{1}{2}\int_0^T\int_0^L h'(x)\phi_t^2\,dx\,dt + \frac{3}{2}\int_0^T\int_0^L p(x)h'(x)\phi''^2\,dx\,dt\\&- \frac{1}{2}\int_0^T\int_0^L p'(x)h(x)\phi''^2\,dx\,dt - \frac{1}{2}\int_0^T\int_0^L (p(x)h''(x))'\phi'^2\,dx\,dt\\&- \frac{1}{2}\int_0^T p(x)h(x)\phi''^2|_0^L dt\\&= \int_0^T\int_0^L \Big(m_1(x)\phi^2 + m_2(x)\phi'^2\Big)\,dx\,dt.\end{aligned}$$

Next we use another multiplier $h'(x)\phi$. Integrating by part on $[0,T]\times[0,L]$, we obtain

$$\begin{aligned}&\int_0^L\int_0^T \phi_{tt}h'(x)\phi\,dx\,dt = (h'(x)\phi,\phi_t)|_0^T - \int_0^T\int_0^L h'(x)\phi_t^2\,dx\,dt.\\&\int_0^L\int_0^T (p(x)\phi'')''h'(x)\phi\,dx\,dt\\&= \int_0^T\int_0^L p(x)h'(x)(\phi'')^2\,dx\,dt + \frac{1}{2}\int_0^T\int_0^L (p(x)h'''(x))''\phi^2\,dx\,dt\\&\quad - \int_0^T\int_0^L \Big(p(x)h'''(x) + (p(x)h''(x))'\Big)\phi'^2\,dx\,dt\,,\end{aligned}$$

$$\begin{aligned}&\int_0^T\int_0^L \Big(b_1(x)\phi - (b_2(x)\phi)' + (b_3(x)\phi)''\Big)h'(x)\phi\,dx\,dt\\&= \int_0^T\int_0^L \Big(m_3(x)\phi^2 + m_4(x)\phi'^2\Big)\,dx\,dt,\end{aligned}$$

where functions $m_i(x)(i = 1,2,3,4)$ are the function of $b_1(x)$, $b_2(x)$, $b_3(x)$, $h(x)$ and their derivatives. Considering the fact that functions $m_i(x)(i = 1,2,3,4)$ are all lower order term, we omit specific functions form here.

By using the above equalities, we have

$$
\begin{aligned}
&\frac{1}{2}\int_0^T p(x)h(x)\phi''^2|_0^L dt \\
&= \Big(h(x)\phi' - \frac{1}{2}h'(x)\phi, \phi_t\Big)\Big|_0^T + \int_0^T\int_0^L h'(x)\phi_t^2\,dx\,dt \\
&\quad + \frac{1}{2}\int_0^T\int_0^L p(x)h'(x)\phi''^2\,dx\,dt \\
&\quad + \frac{1}{2}\int_0^T\int_0^L (p(x)h'(x) - p'(x)h(x))\phi''^2\,dx\,dt \\
&\quad - \int_0^T\int_0^L [d_1(x)\phi^2 + d_2(x)\phi'^2]\,dx\,dt,
\end{aligned} \tag{3.17}
$$

Where functions $d_1(x)$, $d_2(x)$ are the function of $b_1(x)$, $b_2(x)$, $b_3(x)$, $h(x)$ and their derivatives.

Let $h(x)$ be the solution of the problem

$$
\begin{gathered}
h_x = \frac{b}{p}h + 1, \\
h(0) = 0.
\end{gathered} \tag{3.18}
$$

where $b(x) = \max(p_x, 0)$, e.g.,

$$
h(x) = e^{\int_0^x \frac{b}{p}dx}\int_0^x e^{-\int_0^s \frac{b}{p}d\tau}ds, \quad 0 < x < L.
$$

Therefore, if we introduce

$$
Y = \Big(h(x)\phi' - \frac{1}{2}h'(x)\phi, \phi_t\Big)\Big|_0^T.
$$

From (3.17), we obtain

$$
\frac{1}{2}\int_0^T p(L)h(L)\phi''^2(L,t)dt \geq \int_0^T E(t)dt - |Y| - C\int_0^T\int_0^L (\phi^2 + \phi'^2)\,dx\,dt. \tag{3.19}
$$

For $|Y|$, we have

$$
\begin{aligned}
|Y| &\leq \varepsilon\int_0^L {\phi_t}^2 dx|_0^T + C_\varepsilon\int_0^L (\phi^2 + \phi'^2)dx|_0^T \\
&\leq \varepsilon\Big(E(T) + E(0)\Big) + C_\varepsilon\Big(\|\phi(T)\|_{H^1}^2 + \|\phi(0)\|_{H^1}^2\Big).
\end{aligned}
$$

Using the inequality in Lemma 3.2, we compute

$$
\int_0^T E(t)dt \geq \Big(\int_0^T e^{-ct}dt\Big)E(0) - TN(T) \geq k_T[E(0) + E(T)] - \frac{N(T)}{2}, \tag{3.20}
$$

where $k_T = (\int_0^T e^{-ct}dt)\frac{e^{-cT}}{2}$ and $E(T) \geq e^{-cT}E(0) - N(T)$.

Using inequalities (3.19)-(3.20) and (3.15), we obtain

$$
\begin{aligned}
&\frac{1}{2}\int_0^T p(L)h(L)\phi''^2(L,t)dt \\
&\geq k_T[E(0) + E(T)] - \frac{N(T)}{2} - C\int_0^T\int_0^L (\phi^2 + \phi'^2)\,dx\,dt
\end{aligned}
$$

$$
\begin{aligned}
&- \varepsilon\Big\{\Big(E(T)+E(0)\Big)+C_\varepsilon\Big(\|\phi(T)\|_{H^1}^2+\|\phi(0)\|_{H^1}^2\Big)\Big\}\\
&\geq (k_T-\varepsilon)[E(0)+E(T)]-C_T\|\phi\|^2_{L^\infty([0,T];H^1(0,L))}\\
&\geq (k_T-\varepsilon)[E(0)+e^{-cT}E(0)-N(T)]-C_T\|\phi\|^2_{L^\infty([0,T];H^1(0,L))}.
\end{aligned}
$$

Choosing $\varepsilon > 0$ small enough, such that $k_T - \varepsilon > 0$, we obtain

$$
\alpha E(0) \leq \int_0^T p(L)h(L)\phi''^2(L)dt + C_T\|\phi\|^2_{L^\infty([0,T];H^1(0,L))}, \tag{3.21}
$$

where $\alpha = 2(k_T - \varepsilon)(1 + e^{-CT})$.

By using a compactness and uniqueness argument similar to Lagnese [11], we can easily prove that there exist constant $C > 0$,such that

$$
\|\phi\|^2_{L^\infty(0,T;H^1(0,L))} \leq C\int_0^T \phi''^2(L,t)dt.
$$

So the proof is complete. □

Therefore, Λ defines an isomorphism from $H_0^2(0,L)\times L^2(0,L)$ onto $H^{-2}(0,L)\times L^2(0,L)$. For $\psi_0 \in L^2(0,L), \psi_1 \in H^{-2}(0,L)$ given , we solve

$$
\Lambda(\overline{\phi}_0, \overline{\phi}_1) = (\psi_1, -\psi_0).
$$

Then we define $\overline{\phi}$ as the corresponding solution of (3.6) and we take

$$
u(t) = \overline{\phi''}(L,t) \in L^2(0,T),
$$

which is the control driving the system to rest at time $t = T$.

Let us now introduce a new norm

$$
\|(\phi_0,\phi_1)\|^2_{\text{new}} = \int_0^T p(L)\phi''^2(L,t)dt.
$$

It follows from (3.15) that for $T > 0$, $\|(\phi_0,\phi_1)\|_{\text{new}}$ is a norm which is equivalent to the $H_0^2(0,L)\times L^2(0,L)$ norm.

However, the above control strategy only gives distributed control functions because solutions (η,η_t) of the controlled system (3.7) are only in $L^2\times H^{-2}$ no matter (ϕ_0,ϕ_1) are smooth or not.

3.2. **Smooth control.** Smooth control has been considered by Lasiecka and Triggiani [13, 14], Tataru [23]. Here we shall modify the above control strategy to obtain smooth control to meet the need of Theorem 3.1 by induction on the order of the space.

Firstly, we define operator B by

$$
Bu = -(p(x)u'')'' + b_1(x)u - (b_2(x)u)' + (b_3(x)u)'', \quad \forall u \in H^4(0,L). \tag{3.22}
$$

Let $T > 0$ be given. We assume that $z \in C^\infty(-\infty,\infty)$ is such that $0 \leq z(t) \leq 1$ with

$$
z(t) = \begin{cases} 0, & t \geq T \\ 1, & t \leq 0 \end{cases} \tag{3.23}
$$

For $(\phi_0,\phi_1) \in (H^4(0,L)\cap H_0^2(0,L))\times H_0^2(0,L)$ given, we solve (3.6) and then, instead of (3.7), we solve the problem

$$\begin{gathered}
\eta_{tt} + (p(x)\eta'')'' = b_1\eta + b_2\eta' + b_3\eta'', \quad x\in(0,L), t\in(0,T),\\
\eta(x,T)=0, \quad \eta_t(x,T)=0, \quad x\in(0,L),\\
\eta(0,t)=0, \quad \eta'(0,t)=0, \quad t\in(0,T),\\
\eta(L,t)=0, \quad \eta'(L,t)=z(t)\phi''(L,t), \quad t\in(0,T),
\end{gathered} \tag{3.24}$$

Let Λ be given by (3.8) where η is the solution of (3.24) this time. It is easy to check that, for any $(\phi_0,\phi_1),(\varphi_0,\varphi_1)\in H_0^2(0,L)\times L^2(0,L)$,

$$\langle \Lambda(\phi_0,\phi_1),(\varphi_0,\varphi_1)\rangle_{L^2\times L^2} = \int_0^T z(t)p(L)\phi''(L,t)\varphi''(L,t)dt, \tag{3.25}$$

where ϕ and φ are solutions of (3.6) with initial data (ϕ_0,ϕ_1) and (φ_0,φ_1), respectively.

We shall show that problem (3.24) provides smooth controls to Theorem 3.1 by the following lemma.

Lemma 3.4. *Let Λ be given by* (3.8) *where η is the solution of* (3.24). *Then there are constants $c_1>0$ and $c_2>0$ such that*

$$\begin{aligned}
c_1\|(\phi_0,\phi_1)\|_{H^4(0,L)\times H^2(0,L)} &\le \|\Lambda(\phi_0,\phi_1)\|_{L^2(0,L)\times H^2(0,L)}\\
&\le c_2\|(\phi_0,\phi_1)\|_{H^4(0,L)\times H^2(0,L)},
\end{aligned} \tag{3.26}$$

for all $(\phi_0,\phi_1)\in (H^4(0,L)\cap H_0^2(0,L))\times H_0^2(0,L)$,

Proof. From the previous argument,

$$\begin{aligned}
c_1\|(\phi_0,\phi_1)\|_{H^2(0,L)\times L^2(0,L)} &\le \|\Lambda(\phi_0,\phi_1)\|_{H^{-2}(0,L)\times L^2(0,L)}\\
&\le c_2\|(\phi_0,\phi_1)\|_{H^2(0,L)\times L^2(0,L)}
\end{aligned} \tag{3.27}$$

holds. Let $(\phi_0,\phi_1)\in (H^4(0,L)\cap H_0^2(0,L))\times H_0^2(0,L)$ be given, Suppose that ϕ is the solution of problem (3.6) corresponding to the initial data (ϕ_0,ϕ_1).

For $(\varphi_0,\varphi_1)\in (H^6(0,L)\cap H_0^2(0,L))\times(H^4(0,L)\cap H_0^2(0,L))$, let φ is the solution of (3.6) corresponding to the initial data (φ_0,φ_1). Then φ_t and φ_{tt} are the solutions of (3.6) corresponding to the initial data $(\varphi_1,B\varphi_0)$ and $(B\varphi_0,B\varphi_1)$, respectively.

Using the initial data (ϕ_0,ϕ_1) and $(B\varphi_0,B\varphi_1)$ in the formula (3.25), we obtain

$$(\eta(0),B\varphi_1)-(\eta_t(0),B\varphi_0) = -\int_0^T z(t)p(L)\phi''(L,t)\varphi_{tt}''(L,t)dt. \tag{3.28}$$

On the one hand, integrating by parts with respect to the variable t on $[0,T]$, we obtain

$$\begin{aligned}
&-\int_0^T z(t)p(L)\phi''(L,t)\varphi_{tt}''(L,t)dt\\
&= p(L)\phi_0''(L)\varphi_1''(L) + p(L)\int_0^T z_t(t)\phi''(L,t)\varphi_t''(L,t)dt\\
&\quad + p(L)\int_0^T z(t)\phi_t''(L,t)\varphi_t''(L,t)dt
\end{aligned}$$

On the other hand, using (3.22),

$$(\eta(0),B\varphi_1)$$

$$
\begin{aligned}
&= -\int_0^L \eta(0)(p(x)\varphi_1'')''dx + \int_0^L \eta(0)\Big(b_1(x)\varphi_1 - (b_2(x)\varphi_1)' + (b_3(x)\varphi_1)''\Big)dx \\
&= p(L)\phi_0''(L)\varphi_1''(L) - \int_0^L (p(x)\eta''(0))''\varphi_1 dx \\
&\quad + \int_0^L \Big(b_1(x)\eta(0) + b_2(x)\eta'(0) + b_3(x)\eta''(0)\Big)\varphi_1 dx
\end{aligned}
$$

then

$$
\begin{aligned}
&(\eta(0), B\varphi_1) - (\eta_t(0), B\varphi_0) \\
&= p(L)\phi_0''(L)\varphi_1''(L) - \int_0^L (p(x)\eta''(0))''\varphi_1 dx \\
&\quad + \int_0^L \Big(b_1(x)\eta(0) + b_2(x)\eta'(0) + b_3(x)\eta''(0)\Big)\varphi_1 dx - (\eta_t(0), B\varphi_0).
\end{aligned}
$$

So we have the identity

$$
I(\phi,\varphi) = (B^*\eta(0), \varphi_1) - (\eta_t(0), B\varphi_0), \tag{3.29}
$$

where

$$
\begin{gathered}
I(\phi,\varphi) = p(L)\int_0^T z_t(t)\phi''(L,t)\varphi_t''(L,t)dt + p(L)\int_0^T z(t)\phi_t''(L,t)\varphi_t''(L,t)dt, \\
B^*\eta(0) = -(p(x)\eta''(0))'' + b_1(x)\eta(0) + b_2(x)\eta'(0) + b_3(x)\eta''(0).
\end{gathered}
$$

Since $(H^6(0,L)\cap H_0^2(0,L)) \times (H^4(0,L)\cap H_0^2(0,L))$ is dense in $(H^4(0,L)\cap H_0^2(0,L)) \times H_0^2(0,L)$, the identity (3.29) is true for all $(\varphi_0,\varphi_1) \in (H^4(0,L)\cap H_0^2(0,L)) \times H_0^2(0,L)$. Letting $\varphi_0 = 0$ in (3.29), we obtain

$$
I(\phi,\varphi) = (B^*\eta(0), \varphi_1), \tag{3.30}
$$

for $\varphi_1 \in H_0^2(0,L)$ where φ is the solution of (3.6) for the initial data $(0,\varphi_1)$.

Moreover, by inequality (3.15), we have the estimate

$$
\begin{aligned}
&I(\phi,\varphi)| \\
&= p(L)\int_0^T z_t(t)\phi''(L,t)\varphi_t''(L,t)dt + p(L)\int_0^T z(t)\phi_t''(L,t)\varphi_t''(L,t)dt \\
&\le c\Big[\int_0^T p(L)(\phi''(L,t))^2dt + \int_0^T p(L)(\phi_t''(L,t))^2dt\Big]^{1/2} \\
&\quad \times \Big(\int_0^T p(L)(\varphi_t''(L,t))^2dt\Big)^{1/2} \\
&\le c\Big(E(\phi_0,\phi_1) + E(\phi_1, B\phi_0)\Big)^{1/2}\Big(\int_0^T p(L)(\varphi_t''(L,t))^2dt\Big)^{1/2} \\
&\le c\Big(\|\phi_0\|_4^2 + \|\phi_1\|_2^2\Big)^{1/2}\|\varphi_1\|_2.
\end{aligned} \tag{3.31}
$$

In terms of (3.30)-(3.31), we obtain

$$
\|B^*\eta(0)\|_{-2} \le \sup_{\|\varphi_1\|_2=1}\Big(B^*\eta(0), \varphi_1\Big) \le c(\|\phi_0\|_4^2 + \|\phi_1\|_2^2)^{1/2}. \tag{3.32}
$$

Now using the ellipticity of the operator B^* and from (3.32), we have

$$\begin{aligned}\|\eta(0)\|_2 &\le c\Big(\|\eta(0)\|_0 + \|B^*\eta(0)\|_{-2}\Big)\\ &\le c\Big(\|\eta(0)\|_0 + (\|\phi_0\|_4^2 + \|\phi_1\|_2^2)^{1/2}\Big)\\ &\le c(\|\phi_0\|_4^2 + \|\phi_1\|_2^2)^{1/2}.\end{aligned}$$

where $\|\eta(0)\|_0 \le c(\|\phi_0\|_2^2 + \|\phi_1\|_0^2)^{1/2}$ is used. A similar argument yields

$$\|\eta_t(0)\|_0 \le c(\|\phi_0\|_4^2 + \|\phi_1\|_2^2)^{1/2}, \tag{3.33}$$

after we let $\varphi_0 \in (H^4(0,L) \cap H_0^2(0,L)), \varphi_1 = 0$ in (3.29).

Next, let us prove the left hand side of inequality (3.26). We set $\phi_0 = \varphi_0$ and $\phi_1 = \varphi_1$ in (3.29),

$$\begin{aligned}I(\phi,\phi) &\ge \int_0^{T_1} p(L)(\phi_t''(L,t))^2dt - c_\varepsilon \int_0^T p(L)(\phi''(L,t))^2dt\\ &\quad - \varepsilon\int_0^T p(L)(\phi_t''(L,t))^2]dt\\ &\ge \int_0^{T_1} p(L)(\phi_t''(L,t))^2dt - c_\varepsilon E(\phi_0,\phi_1) - \varepsilon E(\phi_1, B\phi_0)\\ &\ge C_1E(\phi_1,B\phi_0) - c_\varepsilon(\|\phi_0\|_2^2 + \|\phi_1\|_0^2).\end{aligned} \tag{3.34}$$

Since

$$\begin{aligned}&(B^*\eta(0),\phi_1)\\ &= -\int_0^L p(x)\eta''(0)\phi_1''dx + \int_0^L \Big(b_1(x)\eta(0) + b_2(x)\eta'(0) + b_3(x)\eta''(0)\Big)\phi_1 dx,\end{aligned}$$

we have

$$\begin{aligned}&(B^*\eta(0),\phi_1) - (\eta_t(0), B\phi_0)\\ &\le c(\|\eta(0)\|_2\|\phi_1\|_2 + \|\eta_t(0)\|_0\|B\phi_0\|_0)\\ &\le c\Big(\|\eta(0)\|_2^2 + \|\eta_t(0)\|_0^2\Big)^{1/2}\Big(\|\phi_1\|_2^2 + \|B\phi_0\|_0^2\Big)^{1/2}.\end{aligned} \tag{3.35}$$

Using (3.34)-(3.35) and the induction assumption

$$c(\|\phi_0\|_2^2 + \|\phi_1\|_0^2) \le \|\eta(0)\|_0^2 + \|\eta_t(0)\|_{-2}^2,$$

it follows that

$$\begin{aligned}\|\phi_0\|_4^2 + \|\phi_1\|_2^2 &\le cE(\phi_1,B\phi_0) + c\Big(\|\phi_0\|_2^2 + \|\phi_1\|_0^2\Big)\\ &\le c\Big(\|\eta(0)\|_2^2 + \|\eta_t(0)\|_0^2\Big) + c\Big(\|\phi_0\|_2^2 + \|\phi_1\|_0^2\Big)\\ &\le c\Big(\|\eta(0)\|_2^2 + \|\eta_t(0)\|_0^2\Big).\end{aligned}$$

The proof is complete. □

A similar argument can used for establishing the inequality

$$\begin{aligned}c_1\|(\phi_0,\phi_1)\|_{H^6(0,L)\times H^4(0,L)} &\le \|\Lambda(\phi_0,\phi_1)\|_{H^2(0,L)\times H^4(0,L)}\\ &\le c_2\|(\phi_0,\phi_1)\|_{H^6(0,L)\times H^4(0,L)}.\end{aligned} \tag{3.36}$$

for all $(\phi_0,\phi_1) \in (H^6(0,L)\cap H_0^2(0,L)) \times (H^4(0,L)\cap H_0^2(0,L))$.

Proof of Theorem 3.1. It follows that the operator

$$\Lambda : (H^6(0,L) \cap H_0^2(0,L)) \times (H^4(0,L) \cap H_0^2(0,L)) \to V(0,L) \times (H^4(0,L) \cap V(0,L))$$

is surjective. Let $(\psi_0, \psi_1) \in (H^4(0,L) \cap V(0,L)) \times V(0,L)$ be given, then there is $(\phi_0, \phi_1) \in (H^6(0,L) \cap H_0^2(0,L)) \times (H^4(0,L) \cap H_0^2(0,L))$ such that the control

$$u(t) = z(t)\phi''(L,t)$$

which drives system (3.4) to rest at the time T, where ϕ is the solution of (3.6) with the initial data (ϕ_0, ϕ_1).

Since ϕ_t, ϕ_{tt} is the solution of (3.6) with the initial data $(\phi_1, B\phi_0)$ and $(B\phi_0, B\phi_1)$ respectively, we conclude that $\phi_t''(L,t)$,$\phi_{tt}''(L,t) \in L^2(0,T)$ from Lemma 3.3. Then $u \in H^2(0,T)$. $\square$

If we change the boundary control condition into $\phi(0,t) = 0$, $\phi''(0,t) = 0$, $\phi(L,t) = 0$, $\phi''(L,t) = u(t)$ or others, the methods in this article still be applicable.

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Xiao-Min Cao
School of Mathematical Sciences, Shanxi University, Taiyuan 030006, China
E-mail address: caoxm@sxu.edu.cn