Electronic Journal of Differential Equations, Vol. 2015 (2015), No. 240, pp. 1–10.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

EXISTENCE OF SOLUTIONS FOR FOUR-POINT RESONANCE BOUNDARY-VALUE PROBLEMS ON TIME SCALES

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Abstract. By using Mawhin's continuation theorem, we prove the existence of solutions for a class of multi-point boundary-value problem under different resonance conditions on time scales.

1. Introduction

Consider multi-point boundary-value problem on time scales

$$x^{\Delta\nabla}(t) = f(t, x(t), x^{\Delta}(t)) + e(t), t \in (0,1) \cap T, \tag{1.1}$$

subject to boundary condition

$$x(0) = \alpha x(\eta), x(1) = \beta x(\xi), \tag{1.2}$$

where T is a time scale such that $0, 1 \in T, \eta, \xi \in (0,1) \cap T$, $f : T \times R^2 \to \mathbb{R}$ is a continuous function and $e(t) \in L^1[0,1].\alpha, \beta \in R$ hold

$$\alpha = 1, \beta = 1 \tag{1.3}$$

or

$$\alpha\eta(1-\beta) + (1-\alpha)(1-\beta\xi) = 0. \tag{1.4}$$

By condition (1.3) or (1.4), the differential operator in (1.1) is not invertible, which is called problem at resonance. The study on multi-point nonlocal boundary-value problems for linear second-order ordinary differential equations was initiated by Il'in and Moiseev [6, 7]. Since then some existence results were obtained for general boundary-value problems by several authors. We refer the reader to some recent results, such as [3, 5, 9, 10, 11] at non-resonance, and [4, 13, 14, 15] at resonance and reference therein.

For problem (1.1), (1.2) in the continuous setting, by using upper and lower solution method Rachünková [13, 14, 15] obtained excellent results about the problem

$$x''(t) = f(t, x(t), x'(t)) + e(t), \quad t \in (0,1), \tag{1.5}$$

subject to boundary condition

$$x(0) = x(\eta), x(1) = x(\xi) \tag{1.6}$$

2010 *Mathematics Subject Classification.* 34N05, 34B10.
Key words and phrases. Boundary-value problem; time scale; coincidence degree.

Submitted March 6, 2015. Published September 21, 2015.

Bai [2] developed the upper and lower solution method and obtained existence and multiplicity results for the resonance problem

$$x''(t) = f(t, x(t), x'(t)) + e(t), \quad t \in (0,1), \tag{1.7}$$

subject to boundary condition

$$x(0) = \alpha x(\eta), \quad x(1) = \beta x(\xi) \tag{1.8}$$

where condition (1.4) is satisfied. But the existence results for resonance cases on time scales is rare (see [8]). Motivated by works above, we apply coincidence degree theory [12] to establish existence results for resonance problems (1.1)–(1.4). Considering that the main tools we used are different from [2, 13, 14, 15], the results we establish follows are new even in the continuous setting.

The article is organized as follows. Some basic definitions and conclusions on time scales and the main tools we used are introduced in section 2. In section 3, we discuss the existence of solutions under condition (1.3) and the existence results of case (1.4) is considered in section 4.

2. Preliminaries

First we present some basic definitions on time scales (see [1]). A time scale T is a closed nonempty subset of $\mathbb{R}$. For $t < \sup T$ and $r > \inf T$, we define the forward jump operator σ and the backward jump operator ρ respectively by

$$\sigma(t) = \inf\{\tau \in T | \tau > t\}, \quad \rho(t) = \sup\{\tau \in T | \tau < t\},$$

for all $t \in T$. If $\sigma(t) > t$, t is said to be right scattered, and if $\sigma(t) = t$, t is said to be right dense. If $\rho(t) < t$, t is said to be left scattered, and if $\rho(t) = t$, t is said to be left dense. A function f is left-dense continuous, if f is continuous at each left dense point in T and its right-sided limits exists at each right dense points in T.

For $u : T \to \mathbb{R}$ and $t \in T$, we define the delta-derivative of $u(t)$, $u^{\Delta}(t)$, to be the number (when it exists), with the property that for each $\varepsilon > 0$, there is a neighborhood, U, of t such that for all $s \in U$,

$$|u(\sigma(t)) - u(s) - u^{\Delta}(t)(\sigma(t) - s)| \le \varepsilon|\sigma(t) - s|.$$

For $u : T \to \mathbb{R}$ and $t \in T$, we define the nabla-derivative of $u(t), u^{\nabla}(t)$, to be the number (when it exists), with the property that for each $\varepsilon > 0$, there is a neighborhood, U, of t such that for all $s \in U$,

$$|u(\rho(t)) - u(s) - u^{\nabla}(t)(\rho(t) - s)| \le \varepsilon|\sigma(t) - s|.$$

Then we recall some notations and an abstract existence result briefly.

Let X, Y be real Banach spaces and let $L : \operatorname{dom} L \subset X \to Y$ be a Fredholm operator with index zero, here $\operatorname{dom} L$ denotes the domain of L. This means that $\operatorname{Im} L$ is closed in Y and $\dim \ker L = \dim(Y/\operatorname{Im} L) < +\infty$. Consider the supplementary subspaces X_1 and Y_1 such that $X = \ker L \oplus X_1$ and $Y = \operatorname{Im} L \oplus Y_1$ and let $P : X \to \ker L$ and $Q : Y \to Y_1$ be the natural projections. Clearly, $\ker L \cap (\operatorname{dom} L \cap X_1) = \{0\}$, thus the restriction $L_p := L|_{\operatorname{dom} L \cap X_1}$ is invertible. Denote by K the inverse of L_p.

Let Ω be an open bounded subset of X with $\operatorname{dom} L \cap \Omega \neq \emptyset$. A map $N : \overline{\Omega} \to Y$ is said to be L-compact in $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded and the operator $K(I-Q)N : \overline{\Omega} \to X$ is compact. We first give the famous Mawhin continuation theorem.

Lemma 2.1 (Mawhin [12]). *Suppose that X and Y are Banach spaces, and $L : \operatorname{dom} L \subset X \to Y$ is a Fredholm operator with index zero.Furthermore, $\Omega \subset X$ is an open bounded set and $N : \overline{\Omega} \to Y$ is L-compact on $\overline{\Omega}$. If*

(1) *$Lx \neq \lambda Nx$ for all $x \in \partial\Omega \cap \operatorname{dom} L, \lambda \in (0,1)$;*
(2) *$Nx \notin \operatorname{Im} L$ for all $x \in \partial\Omega \cap \ker L$;*
(3) *$\deg\{JQN, \Omega \cap \ker L, 0\} \neq 0$, where $J : \ker L \to \operatorname{Im} Q$ is an isomorphism,*

then the equation $Lx = Nx$ has a solution in $\overline{\Omega} \cap \operatorname{dom} L$.

3. Existence results under condition (1.3)

In this section problem (1.1), (1.2) is considered under the assumption that $\alpha = 1$ and $\beta = 1$. We introduce the space

$$X = \{x : [0,1] \to \mathbb{R} : x^{\Delta} \in AC[0,1], x^{\Delta\nabla} \in L^1[0,1]\}$$

endowed with the norm $\|x\| = \sup\{\|x\|_0, \|x^{\Delta}\|_0\}$, where $\|x\|_0 = \sup_{t\in[0,1]} |x(t)|$. Let $Y = L^1[0,1]$ with the norm

$$\|x\|_1 = \int_0^1 |x(t)|\nabla t.$$

Let a linear mapping $L : \operatorname{dom} L \subset X \to Y$ with

$$\operatorname{dom} L = \{x \in X : x(0) = x(\eta), x(1) = x(\xi)\}$$

be defined by $Lx = x^{\Delta\nabla}(t)$. Define the mapping $N : X \to Y$ by

$$Nx(t) = f(t, x(t), x^{\Delta}(t)) + e(t)$$

Then problem (1.1), (1.2) can be written as $Lx = Nx$, here L is a linear operator.

Lemma 3.1. *If $\beta = 1$ and $\alpha = 1$ then*

(1)
$$\operatorname{Im} L = \{y(t)| \int_0^{\eta} (\xi-s)y(s)\nabla s + \int_{\eta}^{\xi} \eta(\xi-1)y(s)\nabla s - \int_0^1 \eta(1-s)y(s)\nabla s = 0\}; \tag{3.1}$$

(2) *$L : \operatorname{dom} L \subset X \to Y$ is a Fredholm operator with index zero;*
(3) *Define projector operator $P : X \to \ker L$ as $Px = x(0)$, then the generalized inverse of operator L, $K_P : \operatorname{Im} L \to \operatorname{dom} L \cap \ker P$ can be written as*
$$K_P(y) = -\frac{t}{\eta} \int_0^{\eta} (\eta - s)y(s)\nabla s + \int_0^t (t-s)y(s)\nabla s, \tag{3.2}$$
satisfying $\|K_P(y(t))\|_1 \leq 2\|y\|_1$.

Proof. (1) First we show that

$$\operatorname{Im} L = \{y(t) \in Y| \int_0^{\eta} (\xi-s)y(s)\nabla s + \int_{\eta}^{\xi} \eta(\xi-1)y(s)\nabla s - \int_0^1 \eta(1-s)y(s)\nabla s = 0\}.$$

First suppose $y(t) \in \operatorname{Im} L$, then there exists $x(t)$ such that $Lx = y$. That is

$$x(t) = \int_0^t (t-s)y(s)\nabla s + x^{\Delta}(0)t + x(0).$$

Condition $x(0) = x(\eta)$, $x(1) = x(\xi)$ imply that

$$\int_0^{\eta} (\xi - s)y(s)\nabla s + \int_{\eta}^{\xi} \eta(\xi - 1)y(s)\nabla s - \int_0^1 \eta(1-s)\nabla s = 0.$$

Next we suppose that

$$y(t) \in \{y(t)| \int_0^{\eta}(\xi - s)y(s)\nabla s + \int_{\eta}^{\xi} \eta(\xi - 1)y(s)\nabla s - \int_0^1 \eta(1 - s)\nabla s = 0\}.$$

Let $x(t) \in X$, where

$$x(t) = \int_0^t (t - s)y(s)\nabla s - \frac{t}{\eta}\int_0^{\eta}(\eta - s)y(s)\nabla s\,.$$

Then $Lx = x^{\Delta\nabla} = y(t)$. because

$$y(t) \in \{y(t)| \int_0^{\eta}(\xi - s)y(s)\nabla s + \int_{\eta}^{\xi} \eta(\xi - 1)y(s)\nabla s - \int_0^1 \eta(1 - s)y(s)\nabla s = 0\},$$

by a simple computation we have $x(0) = x(\eta), x(1) = x(\xi)$. Thus $y(t) \in \operatorname{Im} L$. Summing up two steps above we obtain

$$\operatorname{Im} L = \{y(t) \in Y| \int_0^{\eta}(\xi - s)y(s)\nabla s + \int_{\eta}^{\xi} \eta(\xi - 1)y(s)\nabla s - \int_0^1 \eta(1 - s)y(s)\nabla s = 0\}.$$

(2) We claim that L is a Fredholm operator with index zero. It is easy to see that $\ker L = R$. Next we prove that $Y = \operatorname{Im} L \oplus \ker L$. Suppose $y(t) \in Y$, define the projector operator Q as

$$Q(y) = \frac{\int_0^{\eta}(\xi - s)y(s)\nabla s + \int_{\eta}^{\xi} \eta(\xi - 1)y(s)\nabla s - \int_0^1 \eta(1 - s)y(s)\nabla s}{\int_0^{\eta}(\xi - s)\nabla s + \int_{\eta}^{\xi} \eta(\xi - 1)\nabla s - \int_0^1 \eta(1 - s)\nabla s}$$

Let $y^* = y(t) - Q(y(t))$, by a simple computation, $y^* \in \operatorname{Im} L$. Hence $Y = \operatorname{Im} L + \ker L$, furthermore considering $\operatorname{Im} L \cap \ker L = \{0\}$, we have $Y = \operatorname{Im} L \oplus \ker L$. Thus $\dim\ker L = \operatorname{codim}\operatorname{Im} L$, which means L is a Fredholm operator with index zero.

(3) Define the projector operator $P : X \to \ker L$ as $Px = x(0)$, for $y(t) \in \operatorname{Im} L$,

$$(LK_P)(y(t)) = y(t),$$

and for $x(t) \in \operatorname{dom} L \cap \ker P$, we know

$$\begin{aligned}(K_P L)(x(t)) &= K_P(x^{\Delta\nabla}(t)) \\ &= -\frac{t}{\eta}\int_0^{\eta}(\eta - s)x^{\Delta\nabla}(s)\nabla s + \int_0^t\int_0^s x^{\Delta\nabla}(\tau)\nabla\tau\nabla s \\ &= x(t).\end{aligned}$$

This shows that $K_P = (L_{\operatorname{dom} L\cap\ker P})^{-1}$. Furthermore from the definition of the norms in the X, Y,

$$\|K_P(y(t))\|_1 \le 2\|y\|_1.$$

The above arguments complete the proof. □

Theorem 3.2. *Let $f : [0, 1] \times R^2 \to \mathbb{R}$ be a continuous function. Assume that following conditions are satisfied:*

(C1) *There exist functions $a, b, c, r \in L^1[0, 1]$ and constant $\theta \in [0, 1)$ such that for all $(x, y) \in R^2$, either*

$$|f(t, x, y)| \le a(t)|x| + b(t)|y| + c(t)|y(t)|^{\theta} + r(t), \tag{3.3}$$

or

$$|f(t, x, y)| \le a(t)|x| + b(t)|y| + c(t)|x(t)|^{\theta} + r(t). \tag{3.4}$$

(C2) *There exists a constant $M > 0$ such that for $x \in \operatorname{dom} L, |x(\rho(t))| > M$, for all $t \in [0,1]$,*

$$\begin{aligned} &\int_0^\eta (\xi - s)(f(s,x,x^\Delta) + e(s))\nabla s + \int_\eta^\xi \eta(\xi - 1)(f(s,x,x^\Delta) + e(s))\nabla s \\ &- \int_0^1 \eta(1-s)(f(s,x,x^\Delta) + e(s))\nabla s \neq 0 \end{aligned} \tag{3.5}$$

(C3) *There exists $M^* > 0$ such that for $d \in R$, if $|d| > M^*$, then either*

$$d \times \int_0^\eta (f(t,d,0) + e(t))\nabla t > 0 \tag{3.6}$$

or else

$$d \times \int_0^\eta (f(t,d,0) + e(t))\nabla t < 0 \tag{3.7}$$

Then for each $e \in L^1[0,1]$, resonance problem (1.1), (1.2) with $\alpha = 1$ and $\beta = 1$ has at least one solution provided that

$$\|a\|_1 + \|b\|_1 < \frac{1}{3}.$$

Proof. We divide the proof into two steps.

Step 1. Let

$$\Omega_1 = \{x \in \text{dom } L \setminus \ker L : Lx = \lambda Nx\} \quad \text{for some } \lambda \in [0,1],$$

Then Ω_1 is bounded. Suppose that $x \in \Omega_1$, $Lx = \lambda Nx$, thus $\lambda \neq 0$, $Nx \in \operatorname{Im} L = \ker Q$, hence

$$\begin{aligned} &\int_0^\eta (\xi - s)(f(s,x,x^\Delta) + e(s))\nabla s + \int_\eta^\xi \eta(\xi - 1)(f(s,x,x^\Delta) + e(s))\nabla s \\ &- \int_0^1 \eta(1-s)(f(s,x,x^\Delta) + e(s))\nabla s = 0 \end{aligned} \tag{3.8}$$

Then (3.8) and condition (C2) imply that there exists $t_0 \in T$ such that $|x(t_0)| < M$. In view of $x(0) = x(\rho(t_0)) - \int_0^{t_0} x^\Delta(s)\nabla s$, we obtain that

$$|x(0)| < M + \|x^\Delta\|_0, \quad t \in T. \tag{3.9}$$

For $x(1) = x(\eta)$, there exists $t_1 \in (\eta, 1) \cap T$ such that $x^\Delta(t_1) = 0$. Then from

$$x^\Delta(t) = x^\Delta(t_1) + \int_{t_1}^t x^{\Delta\nabla}(s)\nabla s,$$

we have

$$\|x^\Delta(t)\|_0 \leq \|Nx\|_1 \tag{3.10}$$

Hence, from (3.9) and (3.10), we have

$$|x(0)| \leq M + \|Nx\|_1.$$

Again for $x \in \Omega_1, x \in \operatorname{dom} L \setminus \ker L$, then $(I-P)x \in \operatorname{dom} L \cap \ker P$, $LPx = 0$, thus from Lemma 3.1, we have

$$\|(I-P)x\| = \|K_P L(I-P)x\| \leq 2\|L(I-P)x\|_1 = 2\|Lx\|_1 \leq 2\|Nx\|_1. \tag{3.11}$$

Then

$$\|x\| \leq \|Px\| + \|(I-P)\| \leq M + 3\|Nx\|_1.$$

If assumption (3.3) holds, we obtain

$$
\begin{aligned}
\|x\| &\le M + 3\|Nx\|_1 = M + 3|f(t,x(t),x^{\Delta}(t)) + e(t)|_1 \\
&\le M + 3(\|a\|_1|x| + \|b\|_1|x^{\Delta}| + \|c\|_1|x^{\Delta}|^{\theta} + \|r\|_1 + \|e\|_1) \\
&\le 3\|a\|_1\|x\|_0 + 3[\|b\|_1\|x^{\Delta}\|_0 + \|c\|_1\|x^{\Delta}\|_0^{\theta} + \|r\|_1 + \|e\|_1 + \frac{M}{3}].
\end{aligned}
$$

Thus

$$
\|x\|_0 \le \frac{3}{1-3\|a\|_1}[\|b\|_1\|x^{\Delta}\|_0 + \|c\|_1\|x^{\Delta}\|_0^{\theta} + \|r\|_1 + \|e\|_1 + \frac{M}{3}].
$$

Then

$$
\begin{aligned}
\|x^{\Delta}\|_0 \le \|x\| &\le [\frac{3\|a\|_1}{1-3\|a\|_1} + 1]3[\|b\|_1\|x^{\Delta}\|_0 + \|c\|_1\|x^{\Delta}\|_0^{\theta} + \|r\|_1 + \|e\|_1 + \frac{M}{3}] \\
&= 3\frac{\|b\|_1}{1-3\|a\|_1}\|x^{\Delta}\|_0 + \frac{3}{1-3\|a\|_1}[\|c\|_1\|x^{\Delta}\|_0^{\theta} + \|r\|_1 + \|e\|_1 + \frac{M}{3}]
\end{aligned}
$$

By a simple computation,

$$
\|x^{\Delta}\|_0 \le \frac{3\|c\|_1}{1-3\|a\|_1-3\|b\|_1}\|x^{\Delta}\|_0^{\theta} + \frac{3}{1-3\|a\|_1-3\|b\|_1}[\|r\|_1 + \|e\|_1 + \frac{M}{3}].
$$

Since $\theta \in [0,1)$ and $\|a\|_1 + \|b\|_1 < 1/3$, there exists positive constant M_1 such that $\|x^{\Delta}\|_0 \le M_1$. Then from (3.9), there exists positive constant M_2 such that $\|x\|_0 \le M_1$. Hence $\|x\| = \max\{\|x\|_0, \|x^{\Delta}\|_0\} \le \max\{M_1, M_2\}$, which means that Ω_1 is bounded. If (3.4) hold, similar to the above argument, we can prove Ω_1 is also bounded.

Step 2 The set $\Omega_2 = \{x \in \ker L : Nx \in \operatorname{Im} L\}$ is bounded. The fact that $x \in \Omega_2$ implies that $x = d$, $N(x) = f(t,d,0) + e(t)$ and $QNx = 0$, thus

$$
\begin{aligned}
Q(Nx) = &\Big(\int_0^{\eta}(\xi - s)(f(s,d,0)+e(s))\nabla s + \int_{\eta}^{\xi}\eta(\xi-1)(f(s,d,0)+e(s))\nabla s \\
&- \int_0^1 \eta(1-s)(f(s,d,0)+e(s))\nabla s\Big) \\
&\div \Big(\int_0^{\eta}(\xi-s)\nabla s + \int_{\eta}^{\xi}\eta(\xi-1)\nabla s - \int_0^1\eta(1-s)\nabla s\Big) = 0
\end{aligned} \tag{3.12}
$$

So $|d| < M^*$, thus $x = d$ is bounded. The proof of step 2 is complete.

Let $\Omega = \{x \in X : \|x\| < N_1\}$, where $N_1 > \max\{M_1, M_2, M^*\}$. Then $\overline{\Omega}_1 \subset \Omega$ and $\overline{\Omega}_2 \subset \Omega$. From argument above, it is obviously that conditions (1), (2) of Lemma 2.1 are satisfied. Furthermore, by using Ascolli-Arezela theorem, it is easy to see that $K_P(I-Q)N : \overline{\Omega} \to Y$ is compact, thus N is L-compact on $\overline{\Omega}$.

Next, we claim that condition (3) of Lemma 2.1 is also satisfied. If the first part of condition (C3) is satisfied, define the isomorphism $J : \operatorname{Im} Q \to \ker L$ by $J(a) = a$ and let

$$
H(\lambda, x) = \lambda x + (1-\lambda)JQNx, (\lambda, x) \in \Omega \times [0,1].
$$

By a simple calculation, we obtain, for $(\lambda, x) \in \partial(\Omega \cap \ker L) \times [0,1]$,

$$
xH(\lambda, x) = \lambda x^2 + (1-\lambda)xQNx > 0.
$$

Thus $H(\lambda, x) \neq 0$.

If the second part of condition (C3) is satisfied, define

$$H(\lambda, x) = -\lambda x + (1-\lambda)JQNx, (\lambda, x) \in \Omega \times [0,1],$$

where $J : \operatorname{Im} Q \to \ker L$ defined by $J(a) = -a$, similar with above, we can obtain that $H(\lambda, x) \neq 0$. Thus

$$\begin{aligned}\deg(JQN, \Omega \cap \ker L, 0) &= \deg(H(x,0), \Omega \cap \ker L, 0)\\ &= \deg(H(x,1), \Omega \cap \ker L, 0)\\ &= \deg(I, \Omega \cap \ker L, 0) \neq 0.\end{aligned}$$

Then by Lemma 2.1, $Lx = Nx$ has at least one solution in $\operatorname{dom} L \cap \Omega$, which means resonance problem (1.1), (1.2) has at least one solution. The proof is complete. □

4. Existence result under condition (1.4)

This section we consider problem (1.1),(1.2) under the assumption that

$$\alpha\eta(1-\beta) + (1-\alpha)(1-\beta\xi) = 0.$$

The normed spaces X, Y and operators L, N are defined as in Section 3.

Lemma 4.1. *If $\alpha\eta(1-\beta) + (1-\alpha)(1-\beta\xi) = 0$, then*

(1)
$$\begin{aligned}\operatorname{Im} L = \{y(t) | \alpha(1-\beta)\int_0^\eta (\eta - s)y(s)\nabla s + (1-\alpha)\beta \int_0^\xi (\xi - s)y(s)\nabla s\\ - (1-\alpha)\int_0^1 (1-s)y(s)\nabla s = 0\};\end{aligned}$$

(2) *$L : \operatorname{dom} L \subset X \to Y$ is a Fredholm operator with index zero.*

(3) *Define projector operator $P : X \to \ker L$ as $Px = x(0)$, then the generalized inverse of operator L, $K_P : \operatorname{Im} L \to \operatorname{dom} L \cap \ker P$ can be written as*

$$K_P(y) = \int_0^t (t-s)y(s)\nabla s + \frac{t}{1-\beta\xi}[\beta \int_0^\xi (\xi - s)y(s)\nabla s - \int_0^1 (1-s)y(s)\nabla s],$$

satisfying

$$\|K_P(y(t))\|_1 \le (1 + |\frac{\beta+1}{1-\beta\xi}|)\|y\|_1.$$

Proof. (1) First we show that

$$\begin{aligned}\operatorname{Im} L = \Big\{y(t) | \alpha(1-\beta)\int_0^\eta (\eta - s)y(s)\nabla s + (1-\alpha)\beta \int_0^\xi (\xi - s)y(s)\nabla s\\ - (1-\alpha)\int_0^1 (1-s)y(s)\nabla s = 0\Big\}\end{aligned}$$

First suppose $y(t) \in \operatorname{Im} L$, then there exists $x(t)$ such that $Lx = y$. That is

$$x(t) = \int_0^t (t-s)y(s)\nabla s + u^\Delta(0)t + u(0).$$

Then boundary condition $x(0) = \alpha x(\eta), x(1) = \beta x(\xi)$ together with $\alpha\eta(1-\beta) + (1-\alpha)(1-\beta\xi) = 0$ imply that

$$\alpha(1-\beta)\int_0^\eta (\eta - s)y(s)\nabla s + (1-\alpha)\beta \int_0^\xi (\xi - s)y(s)\nabla s - (1-\alpha)\int_0^1 (1-s)y(s)\nabla s = 0.$$

Next we assume that

$$y(t) \in \Big\{y(t)|\alpha(1-\beta)\int_0^\eta(\eta-s)y(s)\nabla s+(1-\alpha)\beta\int_0^\xi(\xi-s)y(s)\nabla s \\ -(1-\alpha)\int_0^1(1-s)y(s)\nabla s=0\Big\}.$$

Let $x(t)\in X$, where

$$x(t)=\int_0^t(t-s)y(s)\nabla s+\frac{t}{1-\beta\xi}[\beta\int_0^\xi(\xi-s)y(s)\nabla s-\int_0^1(1-s)y(s)\nabla s].$$

Then $Lx=x^{\Delta\nabla}=y(t)$. Since

$$y(t) \in \Big\{y(t)|\alpha(1-\beta)\int_0^\eta(\eta-s)y(s)\nabla s+(1-\alpha)\beta\int_0^\xi(\xi-s)y(s)\nabla s \\ -(1-\alpha)\int_0^1(1-s)y(s)\nabla s=0\Big\},$$

by a simple computation we have

$$x(0)=\alpha x(\eta),\quad x(1)=\beta x(\xi)$$

Thus $y(t)\in\operatorname{Im}L$. Summing up the two steps above we obtain that: (1)

$$\operatorname{Im}L=\{y(t)|\alpha(1-\beta)\int_0^\eta(\eta-s)y(s)\nabla s+(1-\alpha)\beta\int_0^\xi(\xi-s)y(s)\nabla s \\ -(1-\alpha)\int_0^1(1-s)y(s)\nabla s=0\}.$$

(2) Next we claim that L is a Fredholm operator with index zero. It is easy to see that $\ker L=R$. Next we prove that $Y=\operatorname{Im}L\oplus\ker L$. Suppose $y(t)\in Y$, define the projector operator Q as

$$Q(y)=\Big(\alpha(1-\beta)\int_0^\eta(\eta-s)y(s)\nabla s+(1-\alpha)\beta\int_0^\xi(\xi-s)y(s)\nabla s \\ -(1-\alpha)\int_0^1(1-s)y(s)\nabla s\Big) \\ \div\Big(\alpha(1-\beta)\int_0^\eta(\eta-s)\nabla s+(1-\alpha)\beta\int_0^\xi(\xi-s)\nabla s-(1-\alpha)\int_0^1(1-s)\nabla s\Big)$$

Let $y^*=y(t)-Q(y(t))$, by a simple computation, $y^*\in\operatorname{Im}L$.Hence $Y=\operatorname{Im}L+\ker L$, furthermore considering $\operatorname{Im}L\cap\ker L=\{0\}$, we have $Y=\operatorname{Im}L\oplus\ker L$. Thus

$$\dim\ker L=\operatorname{codim}\operatorname{Im}L,$$

which means L is a Fredholm operator with index zero.

(3) Define the projector operator $P:X\to\ker L$ as $Px=x(0)$, for $y(t)\in\operatorname{Im}L$, we have

$$(LK_P)(y(t))=y(t),$$

and for $x(t)\in\operatorname{dom}L\cap\ker P$, we know

$$(K_PL)(x(t)) \\ =K_P(x^{\Delta\nabla}(t))$$

$$
\begin{aligned}
&= \int_0^t (t-s)x^{\Delta\nabla}\nabla s + \frac{t}{1-\beta\xi}[\beta\int_0^\xi (\xi-s)x^{\Delta\nabla}\nabla s - \int_0^1 (1-s)x^{\Delta\nabla}\nabla s] \\
&= x(t)
\end{aligned}
$$

These shows that $K_P = (L_{\operatorname{dom} L\cap\ker P})^{-1}$. Furthermore from the definition of the norms in the X, Y, we have

$$
\|K_P(y(t))\|_1 \le (1+|\frac{\beta+1}{1-\beta\xi}|)\|y\|_1.
$$

□

Theorem 4.2. *Let $f:[0,1]\times R^2\to\mathbb{R}$ be a continuous function. Condition C_1, C_2 are satisfied and:*

- (C5) *There exists a constant $M>0$ such that for $x\in\operatorname{dom} L$, $|x(\rho(t))|>M$, for all $t\in[0,1]$,*

$$
\begin{aligned}
&\alpha(1-\beta)\int_0^\eta (\eta-s)(f(s,x,x^\Delta)+e(s))\nabla s \\
&+(1-\alpha)\beta\int_0^\xi (\xi-s)(f(s,x,x^\Delta)+e(s))\nabla s \\
&-(1-\alpha)\int_0^1 (1-s)(f(s,x,x^\Delta)+e(s))\nabla s \ne 0.
\end{aligned} \tag{4.1}
$$

Then for each $e\in L^1[0,1]$, resonance problem (1.1), (1.2) with $\alpha\eta(1-\beta)+(1-\alpha)(1-\beta\xi)=0$ has at least one solution provided that

$$
\|a\|_1+\|b\|_1 < \frac{1}{2+|\frac{\beta+1}{1-\beta\xi}|}.
$$

The proof is similar to that of Theory 3.1 and it is omitted here.

5. An example

In this section we give an example to illustrate the main results of this article. Let $T=\{0\}\cup\{\frac{1}{2^{n+1}}\}\cup[\frac{1}{2},1]$, $(n=1,2,\dots)$. We consider the four-point boundary-value problem

$$
x^{\Delta\nabla}(t) = \frac{1}{18}x(t)+\frac{1}{18}x^\Delta(t)+\frac{1}{12}\sin(x^\Delta(t))^{1/5}, \quad t\in(0,1)\cap T, \tag{5.1}
$$

subject to the boundary condition

$$
x(0)=x(\frac{1}{3}), \quad x(1)=x(\frac{1}{2}), \tag{5.2}
$$

It is easy to see that $\alpha=\beta=1$, $\eta=1/3$, $\xi=1/2$, $a(t)=b(t)=\frac{1}{18}$, $c(t)=1/12$, $\|a(t)\|+\|b(t)\|<1/3$ and

$$
f(t,\ x,\ y) \le \frac{1}{18}|x|+\frac{1}{18}|y|+\frac{1}{12}|y|^{1/5}, \quad f(t,\ x,\ y) \ge \frac{1}{18}|x|-\frac{1}{18}|y|-\frac{1}{12}.
$$

Then all conditions of Theorem 3.2 hold. Hence, the problem has at least one nontrival solution at resonance.

Acknowledgements. The authors would like to thank the anonymous referees for their careful reading of this manuscript and for suggesting useful changes. This work was sponsored by the NSFC (11201109), Anhui Provincial Natural Science Foundation (1408085QA07), the Higher School Natural Science Project of Anhui Province (KJ2014A200) and the outstanding talents plan of Anhui High school.

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