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EXISTENCE OF BOUNDED SOLUTIONS FOR NONLINEAR HYPERBOLIC PARTIAL DIFFERENTIAL EQUATIONS

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Abstract. In this article we first establish a new representation formula for bounded solutions to a class of nonlinear second-order hyperbolic partial differential equations. Next, we use of our newly-established representation formula to establish the existence of bounded solutions to these nonlinear partial differential equations.

1. Introduction

Aziz and Meyers [2] established the existence, uniqueness, and continuous dependence on the initial data of periodic solutions to the class of nonlinear second-order hyperbolic partial differential equations

$$\begin{gathered}\frac{\partial^2 u}{\partial x \partial t} + a(t,x)\frac{\partial u}{\partial x} + b(t,x)\frac{\partial u}{\partial t} + c(t,x)u = f(t,x,u), \quad \text{in } \mathbb{R}\times[0,T], \\ u(t,0) = \theta(t), \quad \text{for all } t \in \mathbb{R},\end{gathered} \tag{1.1}$$

where $a, b, c : \mathbb{R}\times[0,T] \to \mathbb{R}$ and $f : \mathbb{R}\times[0,T]\times\mathbb{R} \to \mathbb{R}$ are p-periodic functions and $\theta : \mathbb{R} \to \mathbb{R}$ is a p-periodic continuously differentiable function. The main tool utilized by Aziz and Meyers is a representation formula presented by Picone [8]. Some years ago, Al-Islam [1] used the same representation formula to study the existence and uniqueness of pseudo-almost periodic solutions to (1.2) under some appropriate conditions.

The use of Picone's representation formula is somewhat tedious as it is expressed in terms of three functions α, β, and γ, which are solutions to some other partial differential equations. The first objective of this paper consists of using operator theory tools to establish a new representation formula for bounded solutions to (1.1) in the special case $\theta(t) \equiv 0$; that is,

$$\begin{gathered}\frac{\partial^2 u}{\partial x \partial t} + a(t,x)\frac{\partial u}{\partial x} + b(t,x)\frac{\partial u}{\partial t} + c(t,x)u = f(t,x,u), \quad \text{in } \mathbb{R}\times[0,T], \\ u(t,0) = 0, \quad \text{for all } t \in \mathbb{R}.\end{gathered} \tag{1.2}$$

Our second objective consists of using our newly-established representation formula to study the existence of bounded (respectively, pseudo-almost automorphic)

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solutions to (1.2) when the coefficients $a, b, c, a_x : \mathbb{R} \times [0, T] \to \mathbb{R}$ are bounded (respectively, almost automorphic) and the forcing term $f : \mathbb{R} \times [0, T] \times \mathbb{R} \to \mathbb{R}$ is bounded (respectively, pseudo-almost automorphic in $t \in \mathbb{R}$ uniformly with respect to the two other variables).

One should point out that other slightly different versions of (1.2) have been considered in the literature. In particular, Poorkarimi and Wiener [9] studied bounded and almost periodic solutions to a slightly modified version of (1.2), which in fact represents a mathematical model for the dynamics of gas absorption. However, the study of pseudo-almost automorphic solutions to (1.2) is an untreated original question, which constitutes the main motivation of this article.

The study of periodic, almost periodic, almost automorphic, pseudo-almost periodic, weighted pseudo-almost periodic, and pseudo-almost automorphic solutions to differential differential equations constitutes one of the most relevant topics in qualitative theory of differential equations mainly due to their applications. Some contributions on pseudo-almost automorphic solutions to differential and partial differential equations have recently been made in [3, 4, 6, 7, 10, 11]. Here we study the existence of bounded (respectively, pseudo-almost automorphic) solutions to (1.2) under some appropriate assumptions. One should point out that the case $\theta \not\equiv 0$ makes the operators involved in our study nonlinear. Such a case will be left for future investigations.

The article is organized as follows: Section 2 is devoted to preliminaries and notations from operator theory as well as from the concept of pseudo-almost automorphy. In Section 3, we establish a representation formula. Section 4 is devoted to the main result. In Section 5, we give an example to illustrate our main result.

2. Preliminaries

Notation. Let $(\mathbb{X}, \|\cdot\|)$ and $(\mathbb{Y}, \|\cdot\|_{\mathbb{Y}})$ be Banach spaces. Let $BC(\mathbb{R}, \mathbb{X})$ (respectively, $BC(\mathbb{R} \times \mathbb{Y}, \mathbb{X})$) denote the collection of all $\mathbb{X}$-valued bounded continuous functions (respectively, the class of jointly bounded continuous functions $F : \mathbb{R} \times \mathbb{Y} \to \mathbb{X}$). The space $BC(\mathbb{R}, \mathbb{X})$ equipped with its natural norm, that is, the sup norm defined by

$$\|u\|_\infty = \sup_{t \in \mathbb{R}} \|u(t)\|,$$

is a Banach space. Furthermore, $C(\mathbb{R}, \mathbb{Y})$ (respectively, $C(\mathbb{R} \times \mathbb{Y}, \mathbb{X})$) denotes the class of continuous functions from $\mathbb{R}$ into $\mathbb{Y}$ (respectively, the class of jointly continuous functions $F : \mathbb{R} \times \mathbb{Y} \to \mathbb{X}$).

If A is a linear operator upon $\mathbb{X}$, then the notations $D(A)$ and $\rho(A)$ stand respectively for the domain and the resolvent of A. The space $B(\mathbb{X}, \mathbb{Y})$ denotes the collection of all bounded linear operators from $\mathbb{X}$ into $\mathbb{Y}$ equipped with its natural uniform operator topology $\|\cdot\|$. We also set $B(\mathbb{Y}) = (\mathbb{Y}, \mathbb{Y})$ whose corresponding norm will be denoted $\|\cdot\|$.

Pseudo-Almost Automorphic Functions.

Definition 2.1. *A function $f \in C(\mathbb{R}, \mathbb{X})$ is said to be almost automorphic if for every sequence of real numbers $(s'_n)_{n\in\mathbb{N}}$, there exists a subsequence $(s_n)_{n\in\mathbb{N}}$ such that*

$$g(t) := \lim_{n\to\infty} f(t + s_n)$$

is well defined for each $t \in \mathbb{R}$, and

$$\lim_{n\to\infty} g(t - s_n) = f(t)$$

for each $t \in \mathbb{R}$.

If the convergence above is uniform in $t \in \mathbb{R}$, then f is almost periodic. Denote by $AA(\mathbb{X})$ the collection of such almost automorphic functions. Note that $AA(\mathbb{X})$ equipped with the sup-norm $\|\cdot\|_\infty$ is a Banach space.

Definition 2.2. A jointly continuous function $F : \mathbb{R} \times \mathbb{Y} \to \mathbb{X}$ is said to be almost automorphic in $t \in \mathbb{R}$ if $t \to F(t, x)$ is almost automorphic for all $u \in K$ ($K \subset \mathbb{Y}$ being any bounded subset). Equivalently, for every sequence of real numbers $(s'_n)_{n\in\mathbb{N}}$, there exists a subsequence $(s_n)_{n\in\mathbb{N}}$ such that

$$H(t, u) := \lim_{n\to\infty} F(t + s_n, u)$$

is well defined in $t \in \mathbb{R}$ and for each $u \in K$, and

$$\lim_{n\to\infty} H(t - s_n, u) = F(t, u)$$

for all $t \in \mathbb{R}$ and $u \in K$. The collection of such functions will be denoted by $AA(\mathbb{Y}, \mathbb{X})$.

Define

$$PAP_0(\mathbb{R}, \mathbb{X}) := \big\{f \in BC(\mathbb{R}, \mathbb{X}) : \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \|f(s)\| ds = 0\big\}.$$

Similarly, $PAP_0(\mathbb{Y}, \mathbb{X})$ will denote the collection of all bounded continuous functions $F : \mathbb{R} \times \mathbb{Y} \to \mathbb{X}$ such that

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \|F(s, x)\| ds = 0$$

uniformly in $x \in K$, where $K \subset \mathbb{Y}$ is any bounded subset.

Definition 2.3 ([6, 10])**.** A function $f \in BC(\mathbb{R}, \mathbb{X})$ is called pseudo-almost automorphic if it can be expressed as $f = g + \phi$, where $g \in AA(\mathbb{X})$ and $\phi \in PAP_0(\mathbb{X})$. The collection of such functions will be denoted by $PAA(\mathbb{X})$.

The functions g and ϕ appearing in Definition 2.3 are respectively called the *almost automorphic* and the *ergodic perturbation* components of f.

Definition 2.4. A bounded continuous function $F : \mathbb{R} \times \mathbb{Y} \to \mathbb{X}$ is said to be pseudo-almost automorphic whenever it can be expressed as $F = G + \Phi$, where $G \in AA(\mathbb{Y}, \mathbb{X})$ and $\Phi \in PAP_0(\mathbb{Y}, \mathbb{X})$. The collection of such functions will be denoted by $PAA(\mathbb{Y}, \mathbb{X})$.

Theorem 2.5 ([10])**.** *The space $PAA(\mathbb{X})$ equipped with the supremum norm $\|\cdot\|_\infty$ is a Banach space.*

Theorem 2.6. *Suppose $F : \mathbb{R} \times \mathbb{Y} \to \mathbb{X}$ belongs to $PAA(\mathbb{Y}, \mathbb{X})$; $F = G + H$, with $u \to G(t, u)$ being uniformly continuous on any bounded subset K of $\mathbb{Y}$ uniformly in $t \in \mathbb{R}$. Furthermore, we suppose that there exists $L > 0$ such that*

$$\|F(t, u) - F(t, v)\| \le L\|u - v\|_{\mathbb{Y}}$$

for all $u, v \in \mathbb{Y}$ and $t \in \mathbb{R}$. Then the function defined by $h(t) = F(t, \varphi(t))$ belongs to $PAA(\mathbb{X})$ provided $\varphi \in PAA(\mathbb{Y})$.

Theorem 2.7 ([10]). *If $F : \mathbb{R} \times \mathbb{Y} \to \mathbb{X}$ belongs to $PAA(\mathbb{Y}, \mathbb{X})$ and if $u \to F(t, u)$ is uniformly continuous on any bounded subset K of $\mathbb{Y}$ for each $t \in \mathbb{R}$, then the function defined by $h(t) = F(t, \varphi(t))$ belongs to $PAA(\mathbb{X})$ provided $\varphi \in PAA(\mathbb{Y})$.*

For more on pseudo-almost automorphic functions and related issues, we refer the reader to the book by Diagana [4].

3. Representation formula for bounded solutions of (1.2)

Let $\mathcal{C}_T = C[0, T]$ be the Banach space of all continuous functions from $[0, T]$ to $\mathbb{R}$ equipped with the sup norm defined by

$$\|\varphi\|_T := \sup_{x \in [0,T]} |\varphi(x)|$$

for all $\varphi \in \mathcal{C}_T$.

To study (1.2) our first task consists of using operator theory tools to establish a new representation formula. For that, if $q : [0, T] \to \mathbb{R}$ is a measurable function, we consider the linear operators A and B defined on $\mathcal{C}_T$ by

$$D(A) = \big\{\varphi \in \mathcal{C}_T : \varphi_x = \frac{d\varphi}{dx} \in \mathcal{C}_T \text{ and } \varphi(0) = 0\big\}, \quad A\varphi = \frac{d\varphi}{dx}, \text{ for all } \varphi \in D(A),$$
$$D(B_q) = \big\{\varphi \in \mathcal{C}_T : q\varphi \in \mathcal{C}_T\big\}, \quad B_q\varphi = q\varphi.$$

Obviously, if $q \in \mathcal{C}_T$, then $D(B_q) = \mathcal{C}_T$. Moreover, using the above-mentioned operators, one can easily see that (1.2) can be rewritten as follows

$$(A + B_b)\frac{\partial u}{\partial t} + (B_a A + B_c)u = f. \tag{3.1}$$

To study (3.1), we consider the differential equation

$$L\frac{dv}{dt} + Mv = g, \tag{3.2}$$

where $L = A + B_\beta$ and $M = B_\alpha A + B_\gamma$ with $\alpha, \beta, \gamma : [0, T] \to \mathbb{R}$ being continuous functions. Notice that L and M are respectively defined by

$$D(L) = D(A) \cap D(B_\beta) = D(A) \quad \text{and} \quad Lv = \frac{dv}{dx} + \beta v, \quad \text{for all } v \in D(A)$$

and

$$D(M) = D(B_\alpha A) \cap D(B_\gamma) = D(A) \quad \text{and} \quad Mv = \alpha\frac{dv}{dx} + \gamma v, \quad \text{for all } v \in D(A).$$

The next lemma shows that (3.2) in fact is not a singular differential equation ($0 \in \rho(L)$), which makes our computations less tedious.

Lemma 3.1. *If the function $\beta : [0, T] \to \mathbb{R}$ is continuous, then the operator L is invertible and its inverse L^{-1} is given for all $w \in \mathcal{C}_T$ by*

$$L^{-1}w(x) := \int_0^x K(x, y)w(y)dy,$$

where the kernel K is defined by

$$K(x, y) := e^{-\int_y^x \beta(r)dr}$$

for all $0 \le y \le x \le T$. Furthermore, if $\beta_* := \inf_{y \in [0,T]} \beta(y) > 0$, then $\|L^{-1}\| \le T$.

Proof. First of all, we need to solve the differential equation

$$\frac{du}{dy} + \beta u = v \tag{3.3}$$

where $u \in D(A)$ and $v \in \mathcal{C}_T$. For that, multiplying both sides of (3.3) by the function $R(y) = e^{\int_0^y \beta(r)dr}$ and integrating on $[0, x]$, we obtain

$$\begin{aligned} u(x) &= e^{-\int_0^x \beta(r)dr} \int_0^x e^{\int_0^y \beta(r)dr} v(y)dy \\ &= \int_0^x K(x,y)v(y)dy \end{aligned}$$

where $K(x,y) = e^{-\int_y^x \beta(r)dr}$ for all $0 \le y \le x \le T$. Therefore,

$$L^{-1}v(x) := \int_0^x K(x,y)v(y)dy$$

for all $v \in \mathcal{C}_T$.

Now, using the fact $K(x,y) \le e^{-\beta_*(x-y)} \le 1$ for $0 \le y \le x \le T$, one can easily see that

$$\|L^{-1}v(x)\| \le \|v\|_T \int_0^x |K(x,y)|dy \le T\|v\|_T$$

and hence $\|L^{-1}\| \le T$. □

Let $\mathbb{Z}_T$ (respectively $\mathbb{Y}_T$) be the Banach space of all bounded (jointly) continuous functions from $\mathbb{R} \times [0,T]$ to $\mathbb{R}$ (respectively, from $[0,T] \times \mathbb{R}$ to $\mathbb{R}$) equipped with the sup norm defined for each $u \in \mathbb{Z}_T$ (respectively, $u \in \mathbb{Y}_T$) by

$$\|u\|_{T,\infty} := \sup_{t\in\mathbb{R}, x\in[0,T]} |u(t,x)|.$$

Moreover, we set

$$\begin{gathered} K_t(x,y) := e^{-\int_y^x b(t,r)dr}, \\ H(t,x) = \frac{\partial a}{\partial x}(t,x) + a(t,x)b(t,x) - c(t,x) \end{gathered}$$

for all $t \in \mathbb{R}$ and $x, y \in [0,T]$. Let us point out that the quantity H given above is also known as the Euler's invariant, see for instance Ibragimov [5].

The proof of the main results of this paper requires the following assumptions:

- (H1) There exists $\delta > 0$ such that $a(t,x) \ge \delta$ for all $t \in \mathbb{R}$ and $x \in [0,T]$.
- (H2) The function $f : \mathbb{R} \times [0,T] \times \mathbb{R} \to \mathbb{R}$ is Lipschitz in the third variable uniformly in the first and second variables; that is, there exists $C > 0$ such that
$$\big|f(t,x,u) - f(t,x,v)\big| \le C|u - v| \tag{3.4}$$
for all $u, v \in \mathbb{R}$ uniformly in $t \in \mathbb{R}$ and $x \in [0,T]$.
- (H3) The function $f = g + h \in PAA(\mathbb{Y}_T, \mathbb{R})$ (g being the almost automorphic component while h represents the ergodic part). Moreover, $g : \mathbb{Y}_T \to \mathbb{R}$, $(x,u) \to g(t,x,u)$ is uniformly continuous on bounded subset of $\mathbb{Y}_T$ uniformly in $t \in \mathbb{R}$.
- (H4) The functions $(t,x) \to a(t,x), \frac{\partial a}{\partial x}(t,x), b(t,x), c(t,x)$ are jointly continuous and almost automorphic in $t \in \mathbb{R}$ uniformly in $x \in [0,T]$.

Under (H4), we set

$$C_\infty := \sup_{t\in\mathbb{R}, x\in[0,T]} |H(t,x)| = \sup_{t\in\mathbb{R}, x\in[0,T]} \Big|\frac{\partial a}{\partial x}(t,x) + a(t,x)b(t,x) - c(t,x)\Big|,$$
$$B_\infty := \sup_{s\in\mathbb{R}, x\in[0,T]} \Big(\int_0^x e^{-\int_y^x b(s,r)dr} dy\Big).$$

We have the following representation formula for bounded solutions of (1.2).

Theorem 3.2. *Assume* (H1)–(H2) *and the functions* $a, b, c : \mathbb{R} \times [0,T] \to \mathbb{R}$ *are jointly bounded continuous. Then* (1.2) *has a unique bounded continuous solution* $\widetilde{u}$ *whenever* $C + C_\infty < \delta B_\infty^{-1}$. *Furthermore,* $\widetilde{u}$ *is given by the new representation formula*

$$\widetilde{u}(t,x) = \int_{-\infty}^t e^{-\int_s^t a(\sigma,x)d\sigma} G\widetilde{u}(s,x)ds \tag{3.5}$$

where

$$\begin{aligned} G\widetilde{u}(t,x) &= \int_0^x \Big[\frac{\partial a}{\partial y}(t,y) + a(t,y)b(t,y) - c(t,y)\Big] K_t(x,y)\widetilde{u}(t,y)dy \\ &\quad + \int_0^x K_t(x,y) f(t,y,\widetilde{u}(t,y))dy. \end{aligned}$$

Proof. Replacing α by a, β by b, and γ by c, in the previous setting and using the fact L^{-1} exists (Lemma 3.1), it follows that the solvability of (1.2) is equivalent to that of the following first-order partial differential equation

$$\frac{\partial u}{\partial t} = -L^{-1}Mu + L^{-1}f. \tag{3.6}$$

Notice that the operator $L^{-1}M$ can be explicitly computed. Indeed, for each $v \in D(A)$, we have

$$\begin{aligned} L^{-1}Mv(x) &= L^{-1}\Big(a\frac{dv}{dx} + cv\Big)(x) \\ &= \int_0^x K_t(x,y)a(t,y)\frac{dv}{dy}dy + \int_0^x K_t(x,y)c(t,y)v(y)dy \\ &= \Big[a(t,y)K_t(x,y)v(y)\Big]_0^x - \int_0^x \frac{\partial}{\partial y}[a(t,y)K_t(x,y)]v(y)dy \\ &\quad + \int_0^x K_t(x,y)c(t,y)v(y)dy \\ &= a(t,x)v(x) - \int_0^x \Big[\frac{\partial a}{\partial y}(t,y) + a(t,y)b(t,y) - c(t,y)\Big] K_t(x,y)v(y)dy. \end{aligned}$$

Using the expression of $L^{-1}M$, one can easily see that (3.6) is equivalent to

$$\frac{\partial u}{\partial t} = -a(t,x)u + Gu(t,x) \tag{3.7}$$

where

$$\begin{aligned} Gu(t,x) &= \int_0^x \Big[\frac{\partial a}{\partial y}(t,y) + a(t,y)b(t,y) - c(t,y)\Big] K_t(x,y)u(t,y)dy \\ &\quad + \int_0^x K_t(x,y) f(t,y,u(t,y))dy. \end{aligned}$$

Clearly, bounded solutions to (3.7) are given by

$$u(t,x)=\int_{-\infty}^{t}\exp\Big\{-\int_{s}^{t}a(\sigma,x)d\sigma\Big\}Gu(s,x)ds.$$

Setting

$$\Gamma u(t,x):=\int_{-\infty}^{t}e^{-\int_s^t a(\sigma,x)d\sigma}Gu(s,x)ds,$$

one can easily see that Γ maps $\mathbb{Z}_T$ into itself.

In addition, it is easy to see that

$$\|\Gamma u-\Gamma v\|_{T,\infty}\le B_\infty\delta^{-1}\big(C+C_\infty\big)\|u-v\|_{T,\infty}.$$

Therefore, the nonlinear integral operator Γ has a unique fixed point $\widetilde{u}\in\mathbb{Z}_T$ whenever $C+C_\infty<\delta B_\infty^{-1}$. In this event, the function $\widetilde{u}$ is the only bounded continuous solution to (1.2). □

4. Existence of pseudo-almost automorphic solutions

Theorem 4.1. *Assume* (H1)–(H4) *and that* $b_*:=\inf_{t\in\mathbb{R},x\in[0,T]}b(t,x)>0$. *Then* (1.2) *has a unique pseudo almost automorphic solution* $\widetilde{u}$ *whenever* $C+\mathcal{C}_\infty<\delta B_\infty^{-1}$.

Proof. Let $u=u_1+u_2\in PAA(\mathbb{Z}_T)$ and let $f=g+h\in PAA(\mathbb{Y}_T,\mathbb{R})$ where u_1 and g are the almost automorphic components while u_2 and h represent the ergodic part. Consequently, G can be rewritten as $Gu=G_1u+G_2u$, where

$$\begin{aligned}G_1u(t,x)&=\int_0^x\Big[\frac{\partial a}{\partial y}(t,y)+a(t,y)b(t,y)-c(t,y)\Big]K_t(x,y)u_1(t,y)dy\\&\quad+\int_0^x K_t(x,y)g(t,y,u(t,y))dy\end{aligned}$$

and

$$\begin{aligned}G_2u(t,x)&=\int_0^x\Big[\frac{\partial a}{\partial y}(t,y)+a(t,y)b(t,y)-c(t,y)\Big]K_t(x,y)u_2(t,y)dy\\&\quad+\int_0^x K_t(x,y)h(t,y,u(t,y))dy\end{aligned}$$

Since $t\to b(t,x)$ is almost automorphic uniformly in $x\in[0,T]$, then for every sequence of real numbers $(s_n')_{n\in\mathbb{N}}$ there exists a subsequence $(s_n)_{n\in\mathbb{N}}$ such that

$$b_1(t,r):=\lim_{n\to\infty}b(t+s_n,r)$$

is well defined for each $t\in\mathbb{R}$ uniformly in $r\in[0,T]$, and

$$b(t,r)=\lim_{n\to\infty}b_1(t-s_n,r)$$

for each $t\in\mathbb{R}$ uniformly in $r\in[0,T]$.

Now

$$-\int_y^x b_1(t,r)dr=-\int_y^x\lim_{n\to\infty}b(t+s_n,r)dr=-\lim_{n\to\infty}\int_y^x b(t+s_n,r)dr$$

is well defined for each $t\in\mathbb{R}$ uniformly in $x,y\in[0,T]$, and

$$-\int_y^x b(t,r)dr=-\int_y^x\lim_{n\to\infty}b_1(t-s_n,r)dr=-\lim_{n\to\infty}\int_y^x b_1(t-s_n,r)dr$$

for each $t\in\mathbb{R}$ uniformly in $x,y\in[0,T]$.

Using the continuity of the exponential function it follows that

$$K^1_t(x,y) := \lim_{n\to\infty} K_{t+s_n}(x,y)$$

is well defined for each $t\in\mathbb{R}$ uniformly in $x,y\in[0,T]$, and

$$K_t(x,y) = \lim_{n\to\infty} K^1_{t-s_n}(x,y)$$

for each $t\in\mathbb{R}$ uniformly in $x,y\in[0,T]$, and hence $t\to K_t(x,y)$ is almost automorphic uniformly in $x,y\in[0,T]$.

Clearly, $t\to H(t,y)K_t(x,y)u_1(t,y)$ and $t\to K_t(x,y)g(t,y,u(t,y))$ are almost automorphic functions for all $x,y\in[0,T]$ as products of almost automorphic functions. It easily follows that $t\to G_1u(t,x)$ is almost automorphic uniformly in $x\in[0,T]$.

Now

$$\begin{aligned}
&\frac{1}{2r}\int_{-r}^{r}|G_2u(t,x)|dt\\
&= \frac{1}{2r}\int_{-r}^{r}\Big|\int_0^x H(t,y)K_t(x,y)u_2(t,y) + \int_0^x K_t(x,y)h(t,y,u(t,y))dy\Big|dt\\
&\le \frac{C_\infty e^{Tb_*}}{2r}\int_{-r}^{r}\int_0^x |u_2(t,y))|\,dy\,dt + \frac{e^{Tb_*}}{2r}\int_{-r}^{r}\int_0^x |h(t,y,u(t,y))|\,dy\,dt\\
&\le C_\infty e^{Tb_*}\int_0^x\Big(\frac{1}{2r}\int_{-r}^{r}|u_2(t,y))|dt\Big)dy + e^{Tb_*}\int_0^x\Big(\frac{1}{2r}\int_{-r}^{r}|h(t,y,u(t,y))|dt\Big)dy,
\end{aligned}$$

and thus

$$\lim_{T\to\infty}\frac{1}{2r}\int_{-r}^{r}|G_2u(t,x)|dt = 0$$

uniformly in $x\in[0,T]$. Therefore $t\to Gu(t,x)\in PAA(\mathbb{Z}_T)$ uniformly in $x\in[0,T]$.

Now

$$\Gamma u(t,x) := \int_{-\infty}^{t} e^{-\int_s^t a(\sigma,x)d\sigma}Gu(s,x)ds = \Gamma_1u(t,x) + \Gamma_2u(t,x),$$

where

$$\Gamma_ju(t,x) := \int_{-\infty}^{t} e^{-\int_s^t a(\sigma,x)d\sigma}G_ju(s,x)ds,\quad j=1,2.$$

Since $s\to e^{-\int_s^t a(\sigma,x)d\sigma}G_1u(s,x)$ is almost automorphic and that

$$\|\Gamma_1u\|_{T,\infty} \le \|G_1u\|_{T,\infty}\delta^{-1} < \infty$$

it follows that $t\mapsto\Gamma_1u(t,x)$ is almost automorphic uniformly in $x\in[0,T]$.

Now

$$\begin{aligned}
\frac{1}{2r}\int_{-r}^{r}|\Gamma_2u(t,x)|dt &\le \frac{1}{2r}\int_{-r}^{r}\int_{-\infty}^{t} e^{-\delta(t-s)}|G_2u(s,x)|\,ds\,dt\\
&= \int_0^\infty e^{-\delta\sigma}\Big(\frac{1}{2r}\int_{-r}^{r}|G_2u(t-\sigma,x))|dt\Big)d\sigma.
\end{aligned}$$

Since $PAP_0(\mathbb{Z}_T)$ is translation invariant and $G_2\in PAP_0(\mathbb{Z}_T)$ it follows that

$$\lim_{r\to\infty}\frac{1}{2r}\int_{-r}^{r}|G_2u(t-\sigma,x))|dt = 0$$

for each $\sigma\in\mathbb{R}$, uniformly in $x\in[0,T]$.

Using the Lebesgue's Dominated Convergence Theorem it follows that

$$\lim_{r\to\infty}\frac{1}{2r}\int_{-r}^{r}|\Gamma_2 u(t,x)|dt=0$$

uniformly in $x\in[0,T]$.

In view of the above, it follows that $t\to\Gamma u(t,x)$ is pseudo-almost automorphic uniformly in $x\in[0,T]$. Therefore, Γ maps $PAA(\mathbb{Z}_T)$ into itself. Moreover, from Theorem 3.2, we have

$$\|\Gamma u-\Gamma v\|_{T,\infty}\le B_\infty(C+C_\infty)\delta^{-1}\|u-v\|_{T,\infty}.$$

Therefore Γ has a unique fixed point $\widetilde{u}\in\mathbb{Z}_T$ whenever $C+C_\infty<\delta B_\infty^{-1}$. In this event, the function $\widetilde{u}$ is the only pseudo-almost automorphic solution to (1.2). $\square$

5. An example

Fix $\delta_0>0$. Consider the system of nonlinear hyperbolic partial differential equations (1.2) in which

$$\begin{gathered}a(t,x)=\delta_0(2+\sin t)(2+\cos\frac{x}{\delta_0}),\quad b(t,x)=2-\sin t,\\ c(t,x)=\delta_0(4-\sin^2 t)(2+\cos\frac{x}{\delta_0}),\quad f(t,x,u)=\frac{1}{2}\big(u\sin t+e^{-|t|}\sin u\big)\end{gathered}$$

for all $t\in\mathbb{R}$, $x\in[0,1]$, and $u\in\mathbb{R}$. For all $u,v\in\mathbb{R}$, $t\in\mathbb{R}$ and $x\in[0,1]$, we have

$$\begin{gathered}|f(t,x,u)-f(t,x,v)|=\frac{1}{2}|(u-v)\sin t+e^{-|t|}(\sin u-\sin v)|\le|u-v|,\\ a(t,x)=\delta_0(2+\sin t)(2+\cos\frac{x}{\delta_0})\ge\delta_0>0,\\ b_*:=\inf_{t\in\mathbb{R},x\in[0,1]}b(t,x)=\inf_{t\in\mathbb{R},x\in[0,1]}(2-\sin t)=1>0.\end{gathered}$$

Clearly, assumptions (H1)–(H4) are satisfied with $\delta=\delta_0$ and $C=1$. From

$$\begin{aligned}\int_0^x e^{-\int_y^x b(s,r)dr}dy&=\int_0^x e^{-\int_y^x(2-\sin s)dr}dy\\ &=\int_0^x e^{-(2-\sin s)(x-y)}dy\\ &=\frac{1-e^{x(\sin s-2)}}{2-\sin s},\end{aligned}$$

we deduce that

$$B_\infty=\sup_{s\in\mathbb{R},x\in[0,1]}\Big(\frac{1-e^{x(\sin s-2)}}{2-\sin s}\Big)\le 1-e^{-3}.$$

Similarly,

$$C_\infty=\sup_{t\in\mathbb{R},x\in[0,1]}|H(t,x)|=\sup_{t\in\mathbb{R},x\in[0,1]}\Big|-(2+\sin t)\sin\frac{x}{\delta_0}\Big|\le 3.$$

In view of the above, $B_\infty(C+C_\infty)\le 4(1-e^{-3})$. Therefore, using Theorem 4.1, it follows that (1.2) with the above-mentioned coefficients has a unique pseudo-almost automorphic solution whenever δ_0 is chosen so $\delta_0>4(1-e^{-3})$.

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