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EXISTENCE OF POSITIVE PERIODIC SOLUTIONS FOR NEUTRAL LIÉNARD DIFFERENTIAL EQUATIONS WITH A SINGULARITY

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Abstract. By applying Mawhin's continuation theorem, we study the existence of positive periodic solutions for a second-order neutral functional differential equation

$$((x(t)-cx(t-\sigma)))''+f(x(t))x'(t)+g(t,x(t-\delta))=e(t),$$

where g has a strong singularity at $x=0$ and satisfies a small force condition at $x=\infty$, which is different from the corresponding ones known in the literature.

1. Introduction

In recent years, the existence of periodic solutions for the second order differential equations with a singularity have been studied in many literature. See [1]-[15] and the references therein.

Wang [15] studied the Liénard equation with a singularity and a deviating argument

$$x''(t)+f(x(t))x'(t)+g(t,x(t-\sigma))=0, \tag{1.1}$$

where $0\le\sigma<T$ is a constant, $f:\mathbb{R}\to\mathbb{R}$, $g:\mathbb{R}\times(0,+\infty)\to\mathbb{R}$ is an L^2-Carathéodory function, $g(t,x)$ is a T-periodic function in the first argument and can be singular at $x=0$, i. e., $g(t,x)$ can be unbounded as $x\to 0^+$.

Let (1.1) be of repulsive type and set

$$\overline{g}(x)=\frac{1}{T}\int_0^T g(t,x)dt,\ x>0.$$

Assume that

$$\varphi(t)=\lim_{x\to+\infty}\sup\frac{g(t,x)}{x},$$

exists uniformly for a. e. $t\in[0,T]$, i.e., for any $\varepsilon>0$, there is $g_\varepsilon\in L^2(0,T)$ such that

$$g(t,x)\le(\varphi(t)+\varepsilon)x+g_\varepsilon,$$

for all $x>0$ and a. e. $t\in[0,T]$. Assume that $\varphi\in C(\mathbb{R},\mathbb{R})$ and $\varphi(t+T)=\varphi(t)$, $t\in\mathbb{R}$.

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Wang established the following theorem.

Theorem 1.1. *Assume that the following conditions are satisfied:*

(H1) *(Balance) There exist constants $0 < D_1 < D_2$ such that if x is a positive continuous T-periodic function satisfying*
$$\int_0^T g(t, x(t))dt = 0,$$
then
$$D_1 \le x(\tau) \le D_2, \text{ for some } \tau \in [0, T].$$
(H2) *(Degree) $\overline{g}(x) < 0$ for all $x \in (0, D_1)$, and $\overline{g}(x) > 0$ for all $x > D_2$.*
(H3) *(Decomposition) $g(t, x) = g_0(x) + g_1(t, x)$, where $g_0 \in C((0, +\infty), \mathbb{R})$ and $g_1 : [0, T] \times [0, +\infty) \to \mathbb{R}$ is an L^2-Carathéodory function, i. e., g_1 is measurable with respect to the first variable, continuous with respect to the second one, and for any $b > 0$ there is $h_b \in L^2((0, T); [0, +\infty))$ such that $|g_1(t, x)| \le h_b(t)$ for a.e. $t \in [0, T]$ and all $x \in [0, b]$.*
(H4) *(Strong force at $x = 0$) $\int_0^1 g_0(x)dx = -\infty$.*
(H5) *(Small force at $x = \infty$)*
$$\|\varphi\|_\infty < (\frac{\sqrt{\pi}}{T})^2.$$

Then (1.1) *has at least one positive T-periodic solution.*

Meanwhile, the problem of the existence of periodic solutions to the neutral functional differential equation was studied in many papers, see [16]-[11] and the references therein. For example, in [7], Liu and Huang studied the following neutral functional differential equation
$$(u(t) + Bu(t - \tau))' = g_1(t, u(t)) + g_2(u(t - \tau_1)) + p(t).$$
And in in [11], Lu and Ge studied the existence of periodic solutions for a kind of second-order neutral functional differential equation of the form
$$\begin{aligned} &\frac{d^2}{dt^2}\Big(u(t) - \sum_{j=1}^{n} c_j u(t - r_j)\Big) \\ &= f(u(t))u'(t) + \alpha(t)g(u(t)) + \sum_{j=1}^{n} \beta_j g(u(t - \gamma_j)) + p(t), \end{aligned}$$
where f, $g \in C(\mathbb{R}; \mathbb{R})$, $a(t)$, $p(t)$, $\beta_j(t)$, $\gamma_j(t)$ ($j = 1, 2, \dots, n$) are continuous periodic functions defined on $\mathbb{R}$ with period $T > 0$, c_j, $r_j \in \mathbb{R}$ are constants with $r_j > 0$ ($j = 1, 2, \dots, n$). By using the continuation theorem of coincidence degree theory and some new analysis techniques, the authors obtained some new results on the existence of periodic solution.

However, to the best of our knowledge, the studying of positive periodic solutions for the neutral functional differential equation with a singularity is relatively infrequent. As we know, in order to establish the existence of positive periodic solutions, a key condition is that the greatest lower bound must be estimated because of the singularity. However, it is difficult to verify the greatest lower bound, especially for the neutral functional differential equations with a singularity. Besides, because of the singularity, the third condition of Mawhin's continuation theorem is not easy to verify.

Inspired by the above facts, in this paper, we consider the following neutral Liénard differential equation with a singularity and a deviating argument

$$((x(t)-cx(t-\sigma)))''+f(x(t))x'(t)+g(t,x(t-\delta))=e(t), \tag{1.2}$$

where c is a constant with $|c|<1$, $0\le\sigma,\ \delta<T$, $f:\mathbb{R}\to\mathbb{R}$ is continuous, $g:[0,T]\times(0,+\infty)\to\mathbb{R}$ is a continuous function and can be singular at $u=0$, i. e., $g(t,u)$ can be unbounded as $u\to 0^+$. $e(t)$ is T-periodic with $\int_0^T e(t)dt=0$. And we can easily see that when $c=0$, the (1.2) transforms into (1.1). To sum up, our results are essentially new.

The rest of this paper is organized as follows. In Section 2, we state some necessary definitions and lemmas. In Section 3, we prove the main result. At last, we will give an example of an application in section 4.

2. Preliminaries

To prove the announced result, we state the following necessary definitions and lemmas. Denote the operator A by

$$A:C_T\to C_T,\quad (Ax)(t)=x(t)-cx(t-\sigma),$$

where $C_T=\{\varphi\in C(\mathbb{R},\mathbb{R}),\ \varphi(t+T)=\varphi(t)\}$, with norm $||\varphi||_0=\max_{t\in[0,T]}|\varphi(t)|$. Clearly, C_T is a Banach space. Define the operator

$$L:D(L)\subset X\to Y,\quad Lx=(Ax)',$$

where $D(L)=\{x\in C^1(\mathbb{R},\mathbb{R}),\ x(t)=x(t+T)\}$. Define

$$N:C_T\to C_T,\quad (Nx)(t)=-f(x(t))x'(t)-g(t,x(t-\delta))+e(t).$$

Then (1.2) can be rewritten by $Lx=Nx$.

Lemma 2.1 ([10])**.** *If $|c|<1$ then A has continuous inverse on C_T and*

1. $\|A^{-1}x\|\le\frac{\|x\|_0}{|1-|c||}$ *for all* $x\in C_T$*;*
2. $\int_0^T|(A^{-1}f)(t)|dt\le\frac{1}{|1-|c||}\int_0^T|f(t)|dt$ *for all* $f\in C_T$*;*
3. $\int_0^T|A^{-1}f|^2(t)dt\le\frac{1}{(1-|c|)^2}\int_0^T f^2(t)dt$ *for all* $f\in C_T$*.*

From Hale's terminology [4], a solution of the (1.2) is $x\in C(\mathbb{R},\mathbb{R})$ such that $Ax\in C^1(\mathbb{R},\mathbb{R})$ and (1.2) is satisfied on $\mathbb{R}$. In general, x is not from $C^1(\mathbb{R},\mathbb{R})$. Nevertheless, it is easy to see that $(Au)'=Au'$. Thus, a T-periodic solution x of the (1.2) must be from $C^1(\mathbb{R},\mathbb{R})$. According to Lemma 2.1, we can easily obtain that $\ker L=\mathbb{R}$, $\operatorname{Im}L=\{x:x\in X,\int_0^T x(s)ds=0\}$. Thus L is a Fredholm operator with index zero.

Let the projections P and Q be

$$P:C_T\to\ker L, Px=\frac{1}{T}\int_0^T x(s)ds;$$

$$Q:C_T\to C_T\setminus\operatorname{Im}L, Qy=\frac{1}{T}\int_0^T x(s)ds.$$

Let $L_p=L|_{D(L)\cap\ker P}:C_T\cap\ker P\to\operatorname{Im}L$. Then L_P has continuous inverse L_p^{-1} on $\operatorname{Im}L$ defined by

$$(L_p^{-1}y)(t)=A^{-1}\Big(\int_0^T G(t,s)y(s)ds\Big),$$

where

$$G_k(t) = \begin{cases} \frac{s-T}{T}, & 0 \le t \le s; \\ \frac{s}{T}, & s \le t \le T. \end{cases}$$

Lemma 2.2 ([2]). *Let X and Y be two real Banach spaces, and Ω is an open and bounded set of X, and $L : D(L) \subset X \to Y$ is a Fredholm operator of index zero and the operator $N : \bar{\Omega} \subset X \to Y$ is said to be L-compact in $\bar{\Omega}$. In addition, if the following conditions hold:*

(1) *$Lx \neq \lambda Nx$ for all $(x, \lambda) \in \partial\Omega \times (0, 1)$;*
(2) *$QNx \neq 0$ for all $x \in \ker L \cap \partial\Omega$;*
(3) *$\deg\{JQN, \Omega \cap \ker L, 0\} \neq 0$, where $J : ImQ \to \ker L$ is a homeomorphism.*

Then $Lx = Nx$ has at least one solution in $D(L) \cap \bar{\Omega}$.

For the sake of convenience, we list the following assumptions:

(H1) There exist positive constants D_1 and D_2 with $D_1 < D_2$ such that
 (1) for each positive continuous T-periodic function $x(t)$ satisfying $\int_0^T g(t, x(t))dt = 0$, there exists a positive point $\tau \in [0, T]$ such that
$$D_1 \le x(\tau) \le D_2;$$
 (2) $\overline{g}(x) < 0$ for all $x \in (0, D_1)$ and $\overline{g}(x) > 0$ for all $x > D_2$, where $\overline{g}(x) = \frac{1}{T}\int_0^T g(t, x)dt$, $x > 0$.

(H2) $g(t, x) = g_1(t, x) + g_0(x)$, where $g_1 : [0, T] \times (0, +\infty) \to \mathbb{R}$ is a continuous function and
 (1) there exist positive constants m_0 and m_1 such that
$$g(t, x) \le m_0 x + m_1, \quad \text{for all } (t, x) \in [0, T] \times (0, +\infty);$$
 (2) $\int_0^1 g_0(x)dx = -\infty$.

3. Main results

Theorem 3.1. *Suppose that the conditions (H1)-(H2) hold, $|c| < 1$ and*

$$\frac{|c|(1 + |c|) + m_0 T^2}{(1 - |c|)^2} < 1,$$

then the (1.2) *has at least one positive T-periodic solution.*

Proof. Consider the operator equation

$$Lx = \lambda Nx, \quad \lambda \in (0, 1).$$

Let $\Omega_1 = \{x \in \overline{\Omega}, Lx = \lambda Nx, \lambda \in (0, 1)\}$. If $x \in \Omega_1$, then x must satisfy

$$((Au)'(t))' + \lambda f(u(t))u'(t) + \lambda g(t, u(t - \delta)) = \lambda e(t). \tag{3.1}$$

Integrating (3.1) on the interval $[0, T]$, we have

$$\int_0^T g(t, u(t - \delta))dt = 0. \tag{3.2}$$

It follows from (H1)(1) that there exist positive constants D_1, D_2 and $\tau \in [0, T]$ such that

$$D_1 \le u(\tau) \le D_2. \tag{3.3}$$

Then, we obtain

$$\|u\|_0 = \max_{t\in[0,T]} |u(t)| \le \max_{t\in[0,T]} |u(\tau) + \int_\tau^t u'(s)ds| \le D_2 + \int_0^T |u'(s)|ds. \tag{3.4}$$

Multiplying the both sides of (3.1) by $u(t)$ and integrating on the interval $[0, T]$, we obtain

$$\begin{aligned}
&\int_0^T ((Au)'(t))'u(t)dt \\
&= -\lambda \int_0^T f(u(t))u'(t)u(t)dt - \lambda \int_0^T g(t, u(t-\delta))u(t)dt + \lambda \int_0^T e(t)u(t)dt \\
&= -\lambda \int_0^T g(t, u(t-\delta))u(t)dt + \lambda \int_0^T e(t)u(t)dt.
\end{aligned} \tag{3.5}$$

Furthermore,

$$\begin{aligned}
\int_0^T ((Au)'(t))'u(t)dt &= -\int_0^T (Au)'(t)u'(t)dt \\
&= -\int_0^T (Au)'(t)[u'(t) - cu'(t-\sigma) + cu'(t-\sigma)]dt \\
&= -\int_0^T (Au)'(t)[(Au')(t) + cu'(t-\sigma)]dt \\
&= -\int_0^T |(Au')(t)|^2 dt - \int_0^T cu'(t-\sigma)(Au)'(t)dt.
\end{aligned} \tag{3.6}$$

Substituting (3.6) in (3.5), we obtain

$$\begin{aligned}
&\int_0^T |(Au')(t)|^2 dt \\
&= -\int_0^T cu'(t-\sigma)(Au)'(t)dt + \lambda \int_0^T g(t, u(t-\delta))u(t)dt - \lambda \int_0^T e(t)u(t)dt \\
&\le |c| \int_0^T |u'(t-\sigma)||(Au)'(t)|dt + \int_0^T |g(t, u(t-\delta))||u(t)|dt + \int_0^T |e(t)||u(t)|dt.
\end{aligned}$$

It follows from (H2)(1) that

$$\begin{aligned}
\int_0^T |(Au')(t)|^2 dt \le |c| \int_0^T |u'(t-\sigma)||(Au)'(t)|dt + m_0 \int_0^T |u(t)|^2 dt \\
+ m_1 \int_0^T |u(t)|dt + \int_0^T |e(t)||u(t)|dt.
\end{aligned} \tag{3.7}$$

Moreover, by applying Hölder inequality and Minkowski inequality, we can have

$$\begin{aligned}
&\int_0^T |u'(t-\sigma)||(Au)'(t)|dt \\
&\le \Big(\int_0^T |(Au)'(t)|^2 dt\Big)^{1/2} \Big(\int_0^T |u'(t-\sigma)|^2 dt\Big)^{1/2} \\
&= \Big(\int_0^T |(Au)'(t)|^2 dt\Big)^{1/2} \Big(\int_0^T |u'(t)|^2 dt\Big)^{1/2}
\end{aligned}$$

$$
\begin{aligned}
&= \Big[\Big(\int_0^T |u'(t) - cu'(t-\sigma)|^2 dt\Big)^{1/2}\Big]\Big(\int_0^T |u'(t)|^2 dt\Big)^{1/2} \\
&\le \Big[\Big(\int_0^T |u'(t)|^2 dt\Big)^{1/2} + \Big(\int_0^T |cu'(t-\sigma)|^2 dt\Big)^{1/2}\Big]\Big(\int_0^T |u'(t)|^2 dt\Big)^{1/2} \\
&\le \Big[\Big(\int_0^T |u'(t)|^2 dt\Big)^{1/2} + |c|\Big(\int_0^T |u'(t)|^2 dt\Big)^{1/2}\Big]\Big(\int_0^T |u'(t)|^2 dt\Big)^{1/2} \\
&= (1+|c|)\int_0^T |u'(t)|^2 dt. \qquad (3.8)
\end{aligned}
$$

Substituting (3.8) into (3.7) and by (3.4), we can obtain

$$
\begin{aligned}
\int_0^T |(Au')(t)|^2 dt &\le |c|(1+|c|)\int_0^T |u'(t)|^2 dt + m_0 T\|u\|_0^2 \\
&\quad + (m_1 + \|e\|_0)T\|u\|_0 \\
&\le |c|(1+|c|)\int_0^T |u'(t)|^2 dt + m_0 T(D_2 + \int_0^T |u'(s)|ds)^2 \\
&\quad + (m_1 + \|e\|_0)T(D_2 + \int_0^T |u'(s)|ds).
\end{aligned} \tag{3.9}
$$

By applying the third part of Lemma (2.1), we have

$$
\int_0^T |u'(t)|^2 dt = \int_0^T |(A^{-1}A)u'(t)|^2 dt \le \frac{1}{(1-|c|)^2}\int_0^T |(Au')(t)|^2 dt. \tag{3.10}
$$

Substituting (3.10) into (3.9) and by applying Hölder inequality, we obtain

$$
\begin{aligned}
&\int_0^T |(Au')(t)|^2 dt \\
&\le [|c|(1+|c|) + m_0 T^2]\int_0^T |u'(t)|^2 dt \\
&\quad + [2m_0 D_2 + m_1 + \|e\|_0]T\sqrt{T}\Big(\int_0^T |u'(t)|^2 dt\Big)^{1/2} + m_0 T D_2^2 + (m_1 + \|e\|_0)TD_2 \\
&\le \frac{|c|(1+|c|) + m_0 T^2}{(1-|c|)^2}\int_0^T |(Au')(t)|^2 dt \\
&\quad + \frac{[2m_0 D_2 + m_1 + \|e\|_0]T\sqrt{T}}{1-|c|}\Big(\int_0^T |(Au')(t)|^2 dt\Big)^{1/2} + m_0 T D_2^2 \\
&\quad + (m_1 + \|e\|_0)TD_2.
\end{aligned}
$$

It follows from $\frac{|c|(1+|c|)+m_0T^2}{(1-|c|)^2} < 1$ that there exist a positive constant M such that

$$
\int_0^T |(Au')(t)|^2 dt \le M,
$$

which combining with (3.10) gives

$$
\int_0^T |u'(t)|^2 dt \le \frac{M}{(1-|c|)^2}. \tag{3.11}
$$

Then by (3.4), we obtain

$$\|u\|_0 \le D_2 + \frac{\sqrt{TM}}{1-|c|} := M_1. \tag{3.12}$$

Since $[Au](t)$ is T-periodic, there exists $t_0 \in [0,T]$ such that $[Au'](t_0) = 0$. Hence, we have that, for $t \in [0,T]$,

$$\begin{aligned} |[Au'](t)| &= |[Au'](t_0) + \int_{t_0}^{t} ([Au'](s))' ds| \\ &\le \lambda \int_0^T |f(u(t))||u'(t)|dt + \lambda \int_0^T |g(t,u(t-\delta))|dt. \end{aligned} \tag{3.13}$$

Set $F_{M_1} = \max_{|u|\le M_1} |f(u)|$, then by (3.11) we obtain

$$\int_0^T |f(u(t))||u'(t)|dt \le F_{M_1} \int_0^T |u'(t)|dt \le \frac{F_{M_1}\sqrt{TM}}{1-|c|}. \tag{3.14}$$

Write

$$I_+ = \{t \in [0,T] : g(t,u(t-\delta)) \ge 0\}; \quad I_- = \{t \in [0,T] : g(t,u(t-\delta)) \le 0\}.$$

Then it follows from (3.2) and (H2)(1) that

$$\begin{aligned} \int_0^T |g(t,u(t-\delta))|dt &= \int_{I_+} g(t,u(t-\delta))dt - \int_{I_-} g(t,u(t-\delta))dt \\ &= 2\int_{I_+} g(t,u(t-\delta))dt \\ &\le 2m_0 \int_0^T u(t-\delta)dt + 2\int_0^T m_1 dt \\ &\le 2m_0 T\|u\|_0 + 2Tm_1. \end{aligned} \tag{3.15}$$

According to (3.14) and (3.15), we have

$$\begin{aligned} \|Au'\|_0 &\le \lambda \int_0^T |f(u(t))||u'(t)|dt + \lambda \int_0^T |g(t,u(t-\delta))|dt \\ &\le \lambda\Big(\frac{F_{M_1}\sqrt{TM}}{1-|c|} + 2m_0TM_1 + 2Tm_1\Big), \end{aligned}$$

which combining with the first part of Lemma 2.1, we see that

$$\begin{aligned} |u'(t)| = |[A^{-1}Au'](t)| &\le \frac{\|Au'\|_0}{|1-|c||} \\ &\le \frac{\frac{F_{M_1}\sqrt{TM}}{1-|c|} + 2m_0TM_1 + 2Tm_1}{|1-|c||}, \end{aligned}$$

i. e.,

$$\|u'\|_0 \le \frac{\frac{F_{M_1}\sqrt{TM}}{1-|c|} + 2m_0TM_1 + 2Tm_1}{|1-|c||} := A_3. \tag{3.16}$$

On the other hand, it follows from (3.1) and (H2) that

$$\begin{aligned} ((Au)'(t+\delta))' = &-\lambda f(u(t+\delta))u'(t+\delta) - \lambda[g_1(t+\delta,u(t)) + g_0(u(t))] \\ &+ \lambda e(t+\delta). \end{aligned} \tag{3.17}$$

Multiplying both sides of (3.17) by $u'(t)$, we have

$$
\begin{aligned}
((Au)'(t+\delta))'u'(t) &= -\lambda f(u(t+\delta))u'(t+\delta)u'(t) \\
&\quad - \lambda[g_1(t+\delta,u(t)) + g_0(u(t))]u'(t) + \lambda e(t+\delta)u'(t).
\end{aligned} \tag{3.18}
$$

Let $\tau \in [0,T]$ be as in (3.3). For any $t \in [\tau,T]$, integrating (3.18) on the interval $[\tau,T]$, we have

$$
\begin{aligned}
\lambda \int_{u(\tau)}^{u(t)} g_0(u)du &= \lambda \int_\tau^t g_0(u(t))u'(t)dt \\
&= -\int_\tau^t ((Au)'(t+\delta))'u'(t)dt - \lambda \int_\tau^t f(u(t+\delta))u'(t+\delta)u'(t)dt \\
&\quad - \lambda \int_\tau^t g_1(t+\delta,u(t))u'(t)dt + \lambda \int_\tau^t e(t+\delta)u'(t)dt.
\end{aligned}
$$

Set $G_{M_1} = \max_{|u|\le M_1} |g_1(t,u)|$, then from the inequality above, we obtain

$$
\begin{aligned}
&\lambda| \int_{u(\tau)}^{u(t)} g_0(u)du| \\
&= \lambda| \int_\tau^t g_0(u(t))u'(t)dt| \\
&\le \int_0^T |((Au)'(t+\delta))'u'(t)|dt + \lambda \int_0^T |f(u(t+\delta))||u'(t+\delta)||u'(t)|dt \\
&\quad + \lambda \int_0^T |g_1(t+\delta,u(t))||u'(t)|dt + \lambda \int_0^T |e(t+\delta)||u'(t)|dt \\
&\le \|u'\|_0 \int_0^T |((Au)'(t+\delta))'|dt + \lambda F_{M_1}\|u'\|_0^2 T + \lambda G_{M_1}\|u'\|_0 T + \lambda \|e\|_0\|u'\|_0 T,
\end{aligned}
$$

i. e.,

$$
\begin{aligned}
\lambda| \int_{u(\tau)}^{u(t)} g_0(u)du| \le \|u'\|_0 \int_0^T |((Au)'(t+\delta))'|dt + \lambda F_{M_1}\|u'\|_0^2 T \\
+ \lambda G_{M_1}\|u'\|_0 T + \lambda \|e\|_0\|u'\|_0 T.
\end{aligned} \tag{3.19}
$$

Moreover,

$$
\begin{aligned}
\int_0^T |((Au)'(t+\delta))'|dt &= \int_0^T |((Au)'(t))'|dt \\
&\le \lambda\Big(\int_0^T |f(u(t))||u'(t)|dt + \int_0^T |g(t,u(t-\delta))|dt\Big),
\end{aligned}
$$

which combining with (3.14) and (3.15) yields

$$
\begin{aligned}
\int_0^T |((Au)'(t))'|dt &\le \lambda\Big(\int_0^T |f(u(t))||u'(t)|dt + \int_0^T |g(t,u(t-\delta))|dt\Big) \\
&\le \lambda\Big(\frac{F_{M_1}\sqrt{TM}}{1-|c|} + 2m_0TM_1 + 2Tm_1\Big).
\end{aligned} \tag{3.20}
$$

Substituting (3.20) into (3.19) and combining with (3.16), obtain

$$
| \int_{u(\tau)}^{u(t)} g_0(u)du| \le A_3\Big(\frac{F_{M_1}\sqrt{TM}}{1-|c|} + 2m_0TM_1 + 2Tm_1\Big) + F_{M_1}A_3^2 T
$$

$$+ G_{M_1} A_3 T + \|e\|_0 A_3 T < +\infty.$$

According to (H2)(2), we can see that there exists a constant $M_2 > 0$ such that, for $t \in [\tau, T]$,

$$u(t) \geq M_2. \tag{3.21}$$

For the case $t \in [0, \tau]$, we can handle similarly.

Let us define

$$0 < A_1 = \min\{D_1, M_2\},$$
$$A_2 = \max\{D_2, M_1\}.$$

Then by (3.3), (3.12) and (3.21), we obtain

$$A_1 \leq u(t) \leq A_2. \tag{3.22}$$

Set

$$\Omega = \{x = (u, v)^\top \in X : \frac{A_1}{2} < u(t) < A_2 + 1, ||u'||_0 < A_3 + 1\}.$$

Then condition (1) of Lemma 2.2 is satisfied.

Assume that there exists $x \in \partial\Omega \cap \ker L$ such that $QNx = \frac{1}{T}\int_0^T Nx(s)ds = 0$, i. e.,

$$\frac{1}{T}\int_0^T [-f(u(t))u'(t) - g(t, u(t-\delta)) + e(t)]dt = 0, \tag{3.23}$$

then we have

$$\frac{1}{T}\int_0^T g(t, u(t-\delta))dt = 0.$$

It follows from the (H1)(1) we can see that

$$\frac{A_1}{2} < D_1 \leq u(t) \leq D_2 < A_2 + 1,$$

which contradicts the assumption $x \in \partial\Omega$. So for all $x \in \ker L \cap \partial\Omega$, we have $QNx \neq 0$. Therefore, condition (2) of Lemma 2.2 is satisfied.

Finally, we prove that condition (3) of Lemma 2.1 is also satisfied. Let

$$z = Kx = x - \frac{A_1 + A_2}{2},$$

then, we have

$$x = z + \frac{A_1 + A_2}{2}.$$

Define $J : ImQ \to \ker L$ is a linear isomorphism with $J(u) = u$, and define

$$H(\mu, x) = \mu Kx + (1-\mu)JQNx, \ \forall (x, \mu) \in \Omega \times [0, 1].$$

Then,

$$H(\mu, x) = \mu x - \frac{\mu(A_1 + A_2)}{2} + \frac{1-\mu}{T}\int_0^T g(t, x)dt. \tag{3.24}$$

Now we claim that $H(\mu, x)$ is a homotopic mapping. Assume, by way of contradiction, i. e., there exists $\mu_0 \in [0, 1]$ and $x_0 \in \partial\Omega$ such that $H(\mu_0, x_0) = 0$.

Substituting μ_0 and x_0 into (3.24), we have

$$H(\mu_0, x_0) = \mu_0 x_0 - \frac{\mu_0(A_1 + A_2)}{2} + (1-\mu_0)\overline{g}(x_0). \tag{3.25}$$

It follows $H(\mu_0, x_0) = 0$ that $x_0 = A_1$ or A_2. Furthermore, If $x_0 = A_1$, it follows from (H1)(2) that $\overline{g}(x_0) < 0$, then we have

$$\mu_0 x_0 - \frac{\mu_0(A_1 + A_2)}{2} + (1 - \mu_0)\overline{g}(x_0) < \mu_0(x_0 - \frac{A_1 + A_2}{2}) < 0. \tag{3.26}$$

If $x_0 = A_2$, it follows from (H1)(2) that $\overline{g}(x_0) > 0$, then we have

$$\mu_0 x_0 - \frac{\mu_0(A_1 + A_2)}{2} + (1 - \mu_0)\overline{g}(x_0) > \mu_0(x_0 - \frac{A_1 + A_2}{2}) > 0. \tag{3.27}$$

Combining with (3.26) and (3.27), we can see that $H(\mu_0, x_0) \neq 0$, which contradicts the assumption. Therefore $H(\mu, x)$ is a homotopic mapping and $x^\top H(\mu, x) \neq 0$, for all $(x, \mu) \in (\partial\Omega \cap \ker L) \times [0, 1]$, then

$$\begin{aligned}
\deg(\mathrm{JQN}, \Omega \cap \ker \mathrm{L}, 0) &= \deg(H(0, x), \Omega \cap \ker \mathrm{L}, 0) \\
&= \deg(H(1, x), \Omega \cap \ker \mathrm{L}, 0) \\
&= \deg(Kx, \Omega \cap \ker \mathrm{L}, 0) \\
&= \sum_{x \in K^{-1}(0)} \operatorname{sgn}|K'(x)| \\
&= 1 \neq 0.
\end{aligned}$$

Thus, condition (3) of Lemma 2.2 is also satisfied.

Therefore, by applying Lemma 2.1, we can conclude that (1.2) has at least one positive T-periodic solution. □

4. Example

In this section, we provide an example to illustrate results from the previous sections.

Example 4.1. Consider the neutral Liénard differential equation with a singularity and a deviating argument,

$$\begin{aligned}
&((u(t) - 0.1u(t - \pi))')' + \Big(\frac{u^2(t)}{3 + u(t)} + 9\Big)u'(t) \\
&+ \frac{1}{2}(1 + \frac{1}{2}\sin 8t)u(t - \delta) - \frac{1}{u(t - \delta)} = \sin 8t.
\end{aligned} \tag{4.1}$$

Corresponding to Theorem 3.1 and (1.2), we have

$$\begin{gathered}
f(u(t)) = \frac{u^2(t)}{3 + u(t)} + 9, \quad e(t) = \sin 8t, \\
g(t, u(t - \delta)) = \frac{1}{2}\Big(1 + \frac{1}{2}\sin 8t\Big)u(t - \delta) - \frac{1}{u(t - \delta)}.
\end{gathered}$$

Then, we choose

$$\sigma = \pi, \quad c = 0.1, \quad T = \frac{\pi}{4}, \quad m_0 = \frac{3}{4}, \quad D_1 = 2, \quad D_2 = 3.$$

Thus, $|c| < 1$ and the conditions (H1) and (H2) are satisfied. Meanwhile, we have

$$\frac{|c|(1 + |c|) + m_0 T^2}{(1 - |c|)^2} \approx 0.706 < 1.$$

Hence, by applying Theorem 3.1, we can see that (4.1) has at least one positive $\frac{\pi}{4}$-periodic solution.

Remark 4.2. Since only a few papers consider positive periodic solutions for the neutral Liénard equation. One can easily see that all the results in [1]-[4] and the references therein are not applicable to (4.1) for obtaining positive periodic solutions with period $\frac{\pi}{4}$. This implies that the results in this paper are essentially new.

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References

[1] A. Fonda, R. Manásevich, F. Zanolin; *Subharmonic solutions for some second order differential equations with singularities*, SIAM J. Math. Anal. 24 (1993) 1294–1311.
[2] R. Gaines, J. Mawhin; *Coincidence Degree and Nonlinear Differential Equations*, Springer Verlag, Berlin, 1977.
[3] R. Hakl, P. Torres, M. Zamora; *Periodic solutions to singular second order differential equations: the repulsive case*, Nonlinear Anal. TMA. 74 (2011) 7078–7093.
[4] J. Hale; *Theory of Functional Differential Equations*, Springer, New York, 1977.
[5] I. Kiguradze, B. Shekhter; *Singular boundary value problems for ordinary second-order differential equations*, J. Sov. Math. 43(2) (1988) 2340–2417.
[6] A. Lazer, S. Solimini; *On periodic solutions of nonlinear differential equations with singularities*, Proc. Amer. Math. Soc. 88 (1987) 109–114.
[7] B. Liu, L. Huang; *Existence and uniqueness of periodic solutions for a kind of first order neutral functional differential equation*, J. Math. Anal. Appl. 322 (2006) 121–132.
[8] X. Li, Z. Zhang; *Periodic solutions for second order differential equations with a singular nonlinearity*, Nonlinear Anal. TMA. 69 (2008) 3866–3876.
[9] S. Lu, W. Ge; *On the existence of periodic solutions for neutral functional differential equation*, Nonlinear Anal. TMA. 54 (2003) 1285–1306.
[10] S. Lu, W. Ge; *Periodic solutions for a kind of second order differential equations with multiple deviating arguments*, Appl. Math. Comput. 146 (1) (2003) 195–209.
[11] S. Lu, W. Ge; *Existence of periodic solutions for a kind of second-order neutral functional differential equation*, Appl. Math. Comput. 157 (2004) 433–448.
[12] P. Torres; *Existence of one-signed periodic solutions of some second order differential equations via a Krasnoselskii fixed point theorem*, J. Differential Equations. 190 (2003) 643–662.
[13] Z. Wang; *Periodic solutions of the second order differential equations with singularities*, Nonlinear Anal. TMA. 58 (2004) 319–331.
[14] Z. Wang, T. Ma; *Existence and multiplicity of periodic solutions of semilinear resonant Duffing equations with singularities*, Nonlinearity. 25 (2012) 279–307.
[15] Z. Wang; *Periodic solutions of Liénard equation with a singularity and a deviating argument*, Nonlinear Anal. Real World Appl. 16 (2014) 227–234.
[16] M. Zhang; *Periodic solutions to linear and quasilinear neutral functional differential equation*, J. Math. Anal. Appl. 189 (1995) 378–392.
[17] M. Zhang; *Periodic solutions of Liénard equations with singular forces of repulsive type*, J. Math. Anal. Appl. 203 (1996) 254–269.

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