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AN INVERSE COEFFICIENT PROBLEM FOR A NONLINEAR REACTION DIFFUSION EQUATION WITH A NONLINEAR SOURCE

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Abstract. In this article, we consider the problem of identifying an unknown coefficient in a nonlinear diffusion equation. Under appropriate conditions, we prove the existence and the uniqueness of solution for the inverse problem. For the numerical solution of the inverse problem, a numerical method based on discretization of the minimization problem, steepest descent method and least squares approach is proposed. A numerical example is given to illustrate applicability and high accuracy of the proposed method.

1. Introduction

We consider the following n-dimensional nonlinear inverse reaction-diffusion problem

$$\begin{gathered}
u_t = \nabla\cdot(a(u)\nabla u) + f(u), \quad (x,t)\in\Omega_T,\\
u(x,0)=0, \quad x\in\overline{\Omega},\\
-a(u(x,t))\nabla u(x,t) = \vec{g}(x,t), \quad x\in B_0^1,\ t\in[0,T],\\
u_{x_i}(x,t)=0, \quad x\in B_0^i,\ t\in[0,T],\ i=2,\dots,n,\\
u_{x_i}(x,t)=0, \quad x\in B_1^i,\ t\in[0,T],\ i=1,\dots,n,\\
u(x,t)=f_1(x,t), \quad x\in B_0^1,\ t\in[0,T],
\end{gathered} \tag{1.1}$$

where $\Omega := [0,1]^n$ and $\Omega_T := \Omega\times(0,T)$ are two domains in $\mathbb{R}^n$ and $\mathbb{R}^{n+1}$ respectively, $x=(x_1,x_2,\dots,x_n)\in\Omega$, $T>0$ is a final time, $B_0^i=\{(x_1,x_2,\dots,x_i=0,x_{i+1},\dots,x_n)\}$ and $B_1^i=\{(x_1,x_2,\dots,x_i=1,x_{i+1},\dots,x_n)\}$. In this problem, we assume that the compatibility condition $f_1(0,0)=0$ is satisfied. The last Dirichlet condition in (1.1) is used as an additional condition.

The parabolic equation in (1.1) has many applications. For instance, it is used to describe the spread of populations in space [9, 10]. It is also used in modeling chemical and bio-chemical reactions [7, 14]. In general the nonlinear source term $f(u)$ is a smooth function and it describes processes with really change the present u, i.e. something happens to it (birth, death, chemical reactions, etc.) not just diffuse in the space. Also in the context of heat conduction and diffusion when u

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represents temperature and concentration, $f(u)$ is interpreted as a heat and material source respectively.

It is known that the direct problem, i.e the problem (1.1) without the additional condition, has a unique solution if $a(u)$ satisfies certain conditions [6]. The inverse problem here consists of determining the unknown coefficient $a(u)$ in the problem (1.1). Nonlinear inverse problems similar to (1.1) have been previously treated by many authors [1, 2, 3, 4, 5, 12, 13]. In this article, we consider the existence and uniqueness of the solution of a higher dimensional inverse reaction-diffusion problem with a general nonlinear source. We prove that the inverse problem has a unique solution in the class of admissible coefficients.

Now we provide some preliminary material. First we define the following function spaces:

$$
\begin{gathered}
|u|_D = \sup \big\{ u(s), s \in D \big\}, \\
H_\alpha(u) = \sup \Big\{ \frac{u(p) - u(q)}{d(p,g)^\alpha} : p, q \in D, p \neq q \Big\}, \\
|u|_\alpha = |u|_D + H_\alpha(u), \\
|u|_{1+\alpha} = |u|_\alpha + \sum_{i=1}^{n} \big| \frac{\partial u}{\partial x_i} \big|_\alpha, \\
|u|_{2+\alpha} = |u|_\alpha + \sum_{i=1}^{n} \big| \frac{\partial u}{\partial x_i} \big|_\alpha + \sum_{i,j=1}^{n} \big| \frac{\partial^2 u}{\partial x_i \partial x_j} \big|_\alpha + \big| \frac{\partial u}{\partial t} \big|_\alpha,
\end{gathered}
$$

where $D = \Omega_T$, $d(p,q)$ is usual Euclidean metric for the points p and q in D and $\alpha > 0$ is a constant. The space of all functions u for which $|u|_{2+\alpha} < \alpha$ is denoted by $C_{2+\alpha}(D)$. In [6], it is proved that the space $C_{2+\alpha}(D)$ is a Banach space with the corresponding norm.

Definition 1.1. A set $\mathcal{A}$ satisfying the following conditions is called the class of admissible coefficients in optimal control and inverse problems:

(1) $a \in C_{2+\alpha}(I)$ with $|a|_{2+\alpha} \leq c$;
(2) $\nu \leq a \leq \mu$ and $a'(s) > 0$, for $s \in I$;
(3) $|a'| \leq \delta$ and $|a''| \leq \delta$ for $s \in I$;

where $\alpha \in (0,1)$, I is a closed interval, $a : I \to \mathbb{R}$ and c, ν, μ, δ are positive constants.

This article is organized as follows. In section 2 the inverse problem (1.1) is reduced to an equivalent auxiliary problem and existence and uniqueness of the inverse problem is proved. We present our numerical method for the numerical solution of the inverse problem in Section 3. A numerical example is also given to show efficiency of the method.

2. Existence and uniqueness for the inverse problem

In this section we prove that the inverse problem (1.1) has a unique solution. We use the well-known Kirchoff's transformation

$$T_a(u) = \int_0^u a(s)\, ds,$$

where $a \in \mathcal{A}$ and $u > 0$. Let $u = u(x,t)$ be a solution of (1.1). Then define $v(x,t)$ as

$$v(x,t) = T_a(u(x,t)) = \int_0^{u(x,t)} a(s)\,ds. \tag{2.1}$$

From (2.1), we reduce the inverse problem (1.1) to the auxiliary problem

$$\begin{gathered} v_t = a(T_a^{-1}(v))\Delta v + a(T_a^{-1}(v))f(T_a^{-1}(v)), \quad (x,t) \in \Omega_T, \\ v(x,0) = 0, \quad x \in \overline{\Omega}, \\ -\nabla v(x,t) = \vec{g}(x,t), \quad x \in B_0^1,\ t \in [0,T], \\ v_{x_i}(x,t) = 0, \quad x \in B_0^i,\ t \in [0,T],\ i = 2,\dots,n, \\ v_{x_i}(x,t) = 0, \quad x \in B_1^i,\ t \in [0,T],\ i = 1,\dots,n, \\ v(x,t) = F(x,t), \quad x \in B_0^1,\ t \in [0,T], \end{gathered} \tag{2.2}$$

where $F(x,t) := \int_0^{f_1(x,t)} a(s)\,ds$. We note that $\frac{d}{du}T_a(u) \geq \nu > 0$ implies that $T_a(u)$ is invertible. Now, we prove the following comparison theorem.

Theorem 2.1. *Let $f \in C^1(\Omega_T)$, $\vec{g}(x,t)$ and F be continuous functions. In addition assume that $\vec{g}_t(x,t)$ and $\frac{\partial F}{\partial t}$ are positive and continuous functions. Then,*

$$w_\nu \leq v \leq w_\mu, \tag{2.3}$$

where v is the solution of (2.2), w_ν and w_μ are solutions of the following problem for $\lambda = \nu$ and $\lambda = \mu$ respectively:

$$\begin{gathered} L_\lambda w := \lambda \Delta w + \lambda f(T_a^{-1}(w)) - w_t = 0, \quad (x,t) \in \Omega_T, \\ w(x,0) = 0, \quad x \in \overline{\Omega}, \\ -\nabla w(x,t) = \vec{g}(x,t), \quad x \in B_0^1,\ t \in [0,T], \\ w_{x_i}(x,t) = 0, \quad x \in B_0^i,\ t \in [0,T],\ i = 2,\dots,n, \\ w_{x_i}(x,t) = 0, \quad x \in B_1^i,\ t \in [0,T],\ i = 1,\dots,n, \\ w(x,t) = F(x,t), \quad x \in B_0^1,\ t \in [0,T]. \end{gathered} \tag{2.4}$$

Proof. Let $\tilde{a} = a(T_a^{-1}(v))$. Now, we estimate $L_{\tilde{a}}(w_\mu) - L_{\tilde{a}}(w_\mu)$. Since, $w_{\mu_t} = \mu[\Delta w_\mu + f(T_a^{-1}(w_\mu))]$ and $v_t = \tilde{a}[\Delta v + f(T_a^{-1}(v))]$, we obtain

$$L_{\tilde{a}}(w_\mu) - L_{\tilde{a}}(v) = (\tilde{a} - \mu)\,[\Delta w_\mu + f(T_a^{-1}(w_\mu))]. \tag{2.5}$$

To use the maximum principle on [11, page 177], we need to show that $[\Delta w_\mu + f(T_a^{-1}(w_\mu))] \geq 0$. For this purpose let $r = \frac{\partial w_\mu}{\partial t}$. Then $r(x,t)$ satisfies

$$\begin{gathered} r_t = \big[\Delta r + f'(T_a^{-1}(w_\mu))\frac{1}{a'(w_\mu)} r\big], \quad (x,t) \in \Omega_T, \\ r(x,0) = 0, \quad x \in \overline{\Omega}, \\ -\nabla r(x,t) = \vec{g}_t(x,t), \quad x \in B_0^1,\ t \in [0,T], \\ r_{x_i}(x,t) = 0, \quad x \in B_0^i,\ t \in [0,T],\ i = 2,\dots,n, \\ r_{x_i}(x,t) = 0, \quad x \in B_1^i,\ t \in [0,T],\ i = 1,\dots,n, \\ r(x,t) = \frac{\partial}{\partial t}F(x,t), \quad x \in B_0^1,\ t \in [0,T]. \end{gathered} \tag{2.6}$$

Employing the maximum principle on [11, page 177], we conclude that $r \geq 0$, which implies that $[\Delta w_\mu + f(T_a^{-1}(w_\mu))] \geq 0$. Thus, $L_{\tilde{a}}(w_\mu) - L_{\tilde{a}}(v) \leq 0$. By the maximum principle [11, page 172], we conclude that $w_\mu \geq v$. The proof for the other side of the inequality (2.3) is similar. □

Now, we state and prove an existence theorem.

Theorem 2.2. *Under the conditions of Theorem 2.1, the inverse problem* (1.1) *has a solution for each $a \in \mathcal{A}$.*

Proof. Let $z_0 = 0$ and z_n, $n = 1, 2, \dots$, be solution of the problem

$$\begin{gathered}
(z_n)_t = a(T_a^{-1}(z_{n-1}))[\Delta z_n + f(T_a^{-1}(z_{n-1}))], \quad (x,t) \in \Omega_T, \\
z_n(x,0) = 0, \quad x \in \overline{\Omega}, \\
-\nabla z_n(x,t) = \vec{g}(x,t), \quad x \in B_0^1,\ t \in [0,T], \\
(z_n)_{x_i}(x,t) = 0, \quad x \in B_0^i,\ t \in [0,T],\ i = 2, \dots, n, \\
(z_n)_{x_i}(x,t) = 0, \quad x \in B_1^i,\ t \in [0,T],\ i = 1, \dots, n, \\
z_n(x,t) = F(x,t), \quad x \in B_0^1,\ t \in [0,T].
\end{gathered} \tag{2.7}$$

Then z_n is a bounded sequence in $C_{2+\alpha}(\Omega_T)$ [6]. Now we show that z_n is monotone increasing. For this we employ induction. If we put $n = 1$ in (2.7) and note that $z_0 = 0$ we obtain

$$(z_1)_t = a(T_a^{-1}(0)[\Delta z_1 + f(T_a^{-1}(0) = a(0)[\Delta z_1 + f(0)]. \tag{2.8}$$

This says that z_1 is a solution of (2.2) for $\lambda = a(0)$. Using Theorem 2.1 we deduce that $z_1 \geq z_0$. Now suppose that $z_{n-1} \leq z_n$. Applying the same method in Theorem 2.1 for z_{n+1} and z_n we find that $z_n \leq z_{n+1}$ which shows that $\{z_n\}$, is a monotone increasing sequence. Applying a simple version of Lemma 1 in [1] we deduce that there is a $z \in C_{2+\alpha}(\Omega_T)$ such that

$$\begin{gathered}
\Delta z_n \to \Delta z, \quad \text{as } n \to \infty, \\
z_n \to z, \quad \text{as } n \to \infty.
\end{gathered}$$

Passing to the limit in the first equation of (2.7) as $n \to \infty$ and observing that z satisfies all conditions in (2.2) we find that z satisfies the problem (2.2). □

As z is a solution of (2.2) and the operator T_a is invertible, $u = T_a^{-1} z$ is a solution of the problem (1.1).

Theorem 2.3. *Under the assumptions of Theorems 2.1 and 2.2, the problem* (1.1) *has a unique solution.*

Proof. Let $u(x,t)$ and $v(x,t)$ be two solutions of (2.2) and let $z(x,t) = v(x,t) - u(x,t)$. Then

$$\begin{aligned}
z_t = v_t - u_t &= [a(T_a^{-1}(v))\Delta v - a(T_a^{-1}(u))\Delta u] \\
&\quad + [a(T_a^{-1}(v)) f(T_a^{-1}(v)) - a(T_a^{-1}(u)) f(T_a^{-1}(u))].
\end{aligned} \tag{2.9}$$

Now, we estimate the term in the first bracket on the right hand side of (2.9). For this, add and subtract the term $a(T_a^{-1}(v))\Delta u$. Then, we have

$$a(T_a^{-1}(v))\Delta v - a(T_a^{-1}(u))\Delta u = a(T_a^{-1}(v))\Delta z + [a(T_a^{-1}(v)) - a(T_a^{-1}(u))]\Delta u.$$

Using smoothness of the functions a and T_a^{-1}, we conclude that

$$a(T_a^{-1}(v)) - a(T_a^{-1}(u)) = (C(x,t)\Delta u)\, z, \tag{2.10}$$

where

$$C(x,t) = \frac{a'\big(p_a(T_a^{-1}(v(x,t)), T_a^{-1}(u(x,t)))\big)}{a(q_a(v(x,t)), u(x,t))}$$

and $p_a(y_1,y_2)$, $q_a(y_1,y_2)$ are two numbers between y_1 and y_2.

Next, we estimate the term in the second bracket on the right hand side of (2.9). Let $h(s) = a(s)f(s)$. Then

$$\begin{aligned} &a(T_a^{-1}(v))f(T_a^{-1}(v)) - a(T_a^{-1}(u))f(T_a^{-1}(u)) \\ &= h(T_a^{-1}(v)) - h(T_a^{-1}(u)) = \frac{h'(T_a^{-1}(\tilde u))}{a(q_a(v(x,t),u(x,t)))} z, \end{aligned} \tag{2.11}$$

where $\tilde u$ is a number between $T_a^{-1}(v)$ and $T_a^{-1}(u)$.

Combining (2.10), (2.11) we conclude that $z(x,t)$ satisfies the equation

$$z_t = a(T_a^{-1}(v))\Delta z + C_*(x,t)z,$$

where

$$C_*(x,t) = C(x,t)\Delta u + \frac{h'(T_a^{-1}(\tilde u))}{a(q_a(v(x,t),u(x,t)))}.$$

Moreover, $z(x,t)$ satisfies the initial and boundary conditions

$$\begin{gathered} z(x,0) = 0, \quad x \in \overline{\Omega}, \\ -\nabla z(x,t) = \vec{0}, \quad x \in B_0^1,\ t \in [0,T], \\ z_{x_i}(x,t) = 0, \quad x \in B_0^i,\ t \in [0,T],\ i = 2,\dots,n, \\ z_{x_i}(x,t) = 0, \quad x \in B_1^i,\ t \in [0,T],\ i = 1,\dots,n, \\ z(x,t) = 0, \quad x \in B_0^1,\ t \in [0,T]. \end{gathered}$$

Employing the maximum principle [11, page 177] for $z(x,t)$, we conclude that $z(x,t) \equiv 0$, which concludes the proof. □

3. Numerical solution of the inverse problem

In this section, we present our numerical method for the solution of the inverse problem. For simplicity, we consider only one dimensional case in space. In this case, the inverse problem (1.1) becomes

$$\begin{gathered} u_t = (a(u)u_x)_x + f(u), \quad (x,t) \in \Omega_T, \\ u(x,0) = 0, \quad x \in \overline{\Omega}, \\ -a(u(0,t))u_x(0,t) = g(t), \quad t \in [0,T], \\ u_x(1,t) = 0, \quad t \in [0,T], \\ u(0,t) = f_1(t), \quad t \in [0,T], \end{gathered} \tag{3.1}$$

where $\Omega := [0,1]$ and $\Omega_T := \Omega \times (0,T)$.

We note that the same method is used in [13]. For the completeness of the content, we explain the main steps of the method. The essence of the method is to approximate the unknown coefficient $a(u)$ by polynomials. Since the unknown diffusion coefficient $a(u)$ is continuous on a compact domain Ω_T, there exists a sequence of polynomials converging to $a(u)$. Our starting point is that the correct

$a(u)$ will yield the solution satisfying the condition $u(0,t) = f_1(t)$, hence $a(u)$ will minimize the functional

$$F(c) = \|u(c,0,t) - f_1(t)\|_2^2,$$

where $u(c,x,t)$ is the solution of the direct problem with the diffusion coefficient $c(u)$ and $\|\cdot\|_2$ is the L_2 norm on Ω. Hence, our strategy is to find a polynomial of degree n that minimizes $F(c)$, i.e, n^{th} degree polynomial approximation of $a(u)$ for the desired n. From now on we take $c(u) = c_0 + c_1 u + \dots + c_n u^n$ as $c = (c_0, \dots, c_n)$, hence $F(c)$ is a function of n variables. To overcome the ill-posedness of the inverse problem, Tikhonov regularization is applied. A regularization term with a regularization parameter λ is added to $F(c)$

$$G(c) = \|u(c,0,t) - f_1(t)\|_2^2 + \lambda \|c\|^2,$$

where $\|c\|$ denotes the Euclidean norm of c. From now on, we fix n and λ.

The method for minimizing $G(c)$ depends on the properties of $F(c)$, e.g., convexity, differentiability etc. In our case, the convexity or differentiability of $F(c)$ is not clear due to the term $u(c,x,t)$. However, we do not envision a major drawback in assuming the differentiability of $F(c)$ in numerical implementations. For this reason, we proceed the minimization of $G(c)$ by the steepest descent method which will utilize the gradient of F. In this method, the algorithm starts with an initial point b_0, then the point providing the minimum is approximated by the points

$$b_{i+1} = b_i + \triangle b_i,$$

where $\triangle b_i$ is the feasible direction which minimizes

$$E(\triangle b) = G(b_i + \triangle b).$$

This procedure is repeated until a stop criterion is satisfied, i.e, $\|\triangle b_i\| < \epsilon$ or $|G(b_{i+1}) - G(b_i)| < \epsilon$ or a certain number of iterations. In the minimization of $E(\triangle b)$, we use the following estimate on $u(b_i + \triangle b, 0, t)$:

$$u(b_i + \triangle b, 0, t) \simeq u(b_i, 0, t) + \nabla u(b_i, 0, t) \cdot \triangle b,$$

where ∇ denotes the gradient of $u(b,0,t)$ with respect to b. Hence $E(\triangle b)$ turns out to be

$$E(\triangle b) = \|\nabla u(b_i,0,t) \cdot \triangle b + u(b_i,0,t) - f_1(t)\|_2^2 + \lambda \|\triangle b\|_2^2.$$

In numerical calculations, we note that $\|\cdot\|_2$ can be discretized by using a finite number of points in $[0,T]$, i.e., for $t_1 = 0 < t_2 < \dots < t_q = T$, hence $E(\triangle b)$ has its new form as

$$E(\triangle b) \simeq \sum_{k=1}^{q} (u(b_i,0,t_k) + \nabla u(b_i,0,t_k) \cdot \triangle b - f_1(t_k))^2 + \lambda \|\triangle b\|_2^2. \tag{3.2}$$

Now the minimization of this problem is a least squares problem whose solution leads to the normal equation (see [8])

$$(\lambda I + A^T A)\triangle b = A^T K,$$

where

$$A = [\nabla u(b_i,0,t_1)^T \dots \nabla u(b_i,0,t_q)^T],$$
$$K = [u(b_i,0,t_1) - f_1(t_1) \dots u(b_i,0,t_q) - f_1(t_q)]^T.$$

Now the optimal direction is found by

$$\triangle b = (\lambda I + A^T A)^{-1} A^T K. \tag{3.3}$$

In forming A, the computation (or estimation) of s^{th} component of the vector $\nabla u(b_i, 0, t_k)$ can be achieved by

$$\frac{u(b_i + he_s, 0, t_k) - u(b_i, 0, t_k)}{h}, \tag{3.4}$$

where e_s is the standard unit vector whose s^{th} component is 1 and h is the differential step.

The algorithm can be summarized by following steps:

Step 1. Set b_0, n, λ and a stopping criterion k or ϵ (iteration number less than k or size of $\|\triangle b_i\| \leq \epsilon$).

Step 2. Calculate $\triangle b_i$ using (3.3) and set $b_{i+1} = b_i + \triangle b_i$.

Step 3. Stop when the criterion is achieved.

Example. In this example, we solve the inverse problem (3.1) for $f(u) = Du\big(1 - \frac{u}{K}\big)$, where D the constant growth rate, and K is the carrying capacity as limitation of growth for population dynamics model. For simplicity, we take $D = K = 1$, hence $f(u) = u(1-u)$. We also take $g(t) = \sin(t)$. The additional data $u(0,t) = f_1(t)$ is found numerically. The correct solution is $a(u) = 1 + 2u + 3u^2 + u^3$. See Table 1. We note that all computations have been carried out in MATLAB. In solving the direct problem for each value of c, MATLAB PDE solver is used.

Because of the discretization of the problem, many variables appear in computations. These variables and their values in our computations are listed below:

(1) The degree of the polynomial $c(u)$: $n = 2, 3, 4, 5$ are taken.
(2) Initial guess for the coefficients of $c(u)$: All initial guesses for the coefficients are taken to be vectors composed of 1's in order to get an objective observation.
(3) Differential step h: $h = 0.1, h = 0.01$ are taken.
(4) Number of t points: $q = 10$ and $q = 100$ are taken.
(5) Number of (x,t) points in mesh grid used in Matlab PDE solver: taken to be $q \times q$ where q is already determined in (4).
(6) Stopping criterion: $\epsilon = 0.01$ or maximum iteration number: $k = 100$.
(7) Regularization parameter: λ is taken to be zero in the noise-free examples, but an optimal λ is searched to deal with noisy data. Since the problem is highly nonlinear, we seek the best regularization parameter empirically.

In the applications, the additional data $u(0,t)$ is generally given with a noise, i.e., $u(0,t) + \gamma u(0,t)$ where γ is called noise level and is generally less than 0.1. The example is now tested with $u(0,t)$ plus some noise. The algorithm is run for the best choices of h, q and the initial guesses in the previous calculations, i.e., $h = 0.1, q = 100$. The noise levels are taken as $\gamma = +0.02, -0.04$. Table 2 and Table 3 shows the results. In this table, we also give the relative errors which is defined as

$$\frac{\|u - u_a\|_\infty}{\|u\|_\infty},$$

where $\|\cdot\|_\infty$ denotes maximum norm, u and u_a are the solutions corresponding to the correct $a(u)$ and observed $a(u)$ respectively. The relative error provides a gauge to compare the results for noisy data for different regularization parameters. See Table 4.

TABLE 1. Initial guesses and results for $n = 2, 3$,4,5.

Initial guess	$h = 0.1, q = 10$
(1,1)	(0.9486, 2.5990)
(1,1,1)	(0.9862, 2.1632, 1.6133)
(1,1,1,1)	(1.0176, 1.7199, 2.9978, -0.5290)
(1,1,1,1,1)	(1.0194, 1.5602, 4.7561, -6.1976, 6.1034)
	$h = 0.01, q = 10$
(1,1)	(0.9521, 2.5653)
(1,1,1)	(0.9996, 2.0289, 1.7452)
(1,1,1,1)	(0.9922, 2.1623, 1.0803, 1.6373)
(1,1,1,1,1)	(1.0101, 1.7365, 3.6651, -3.9461, 4.5934)
	$h = 0.1, q = 100$
(1,1)	(0.9506, 2.5998)
(1,1,1)	(1.0008, 2.0268, 1.8113)
(1,1,1,1)	(1.0095, 1.8579, 2.4673, 0.0554)
(1,1,1,1,1)	(1.0028, 1.9976, 1.6339, 1.8159, -0.4184)
	$h = 0.01, q = 100$
(1,1)	(0.9611, 2.5334)
(1,1,1)	(0.9941, 2.0922, 1.6526)
(1,1,1,1)	(1.0006, 2.0090, 1.7427, 0.8832)
(1,1,1,1,1)	(0.9961, 2.1296, 0.8972, 2.9274, -0.8337)

TABLE 2. The results for given γ values.

Initial guess	$\gamma = +0.02$	Relative error
(1,1)	(0.9247, 2.4262)	0.0531
(1,1,1)	(0.9541, 2.0427, 1.3576)	0.0350
(1,1,1,1)	(0.9694, 1.7606, 2.5081, -0.6971)	0.0290
(1,1,1,1,1)	(0.9598, 1.9843, 1.0394, 2.8274, -2.2120)	0.0256

TABLE 3. The results for given γ values.

Initial guess	$\gamma = -0.04$	Relative error
(1,1)	(1.0024, 2.9468,)	0.0298
(1,1,1)	(1.0941, 1.9951, 2.7186)	0.0076
(1,1,1,1)	(1.0897, 2.0524, 2.3855, 1.5605)	0.0078
(1,1,1,1,1)	(1.0889, 2.0244, 2.8228, -0.2071, 3.1688)	0.0078

The above experiment clearly indicates that the initial guess, q (the number of t points) and n (the degree of the polynomial $c(u)$) are the main factors affecting the accuracy of the solutions. The changes in differential step h is observed to have a negligible effect in finding feasible directions. In our experiments $h = 0.1$ appears

Table 4. Regularization parameters and relative errors for different noise levels.

	$\gamma = +0.02$	$\gamma = -0.04$
(1,1)	(0.9250, 2.4252)	(1.0029, 2.9955)
λ	10^{-5}	10^{-4}
Relative error	0.0530	0.0299
(1,1,1)	(0.9635, 1.9784, 1.4416)	(1.0910, 2.0231, 2.6732)
λ	10^{-4}	10^{-5}
Relative error	0.0350	0.0077
(1,1,1,1)	(1.0105, 1.7411, 1.3630, 1.1535)	(1.0845, 2.1218, 2.1640, 1.7523)
λ	10^{-3}	10^{-4}
Relative error	0.0281	0.0078
(1,1,1,1,1)	(0.9661, 1.9976, 1.1769, 0.8555, 0.8037)	(1.0782, 2.1933, 2.0096, 1.6764, 1.4226)
λ	10^{-4}	10^{-4}
Relative error	0.0243	0.0078

to be good enough for a satisfactory solution. The initial guesses have to be chosen close enough to the coefficients of the correct solution. However, it is hard to give a radius of the trust region around the expected coefficients. One way to overcome this problem is to start with $n = 1$ with several initial guesses then choose the best one for it (call it x_0) then make it $n = 2$, use the solution $(x_0, 1)$ as an initial guess and repeat it for the other dimensions. Although the initial guesses in the above experiment have not been determined with this procedure, that approach also has been observed to work well in the example. It is observed that q has a significant impact on the solution. However, the way how it affects the algorithm is not very clear. It appears that in $q = 100$ works better. As we mentioned above, we solve the direct problem by MATLAB PDE solver which uses Finite Element Method (FEM). In general, increase of mesh points will also increase the accuracy of the solution. This might be the fact behind the result $q = 100$ works better than $q = 10$ using the same initial guesses in the example.

The effect of regularization parameter becomes apparent in the noisy case. Since the problem is highly nonlinear, we seek the best regularization parameter empirically. We present the best regularization parameter with their relative errors, see Table 4. When the optimal regularization parameter is used, the algorithm ends at relatively better coefficients.

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