LIMIT CYCLES BIFURCATING FROM THE PERIOD ANNULUS OF A UNIFORM ISOCHRONOUS CENTER IN A QUARTIC POLYNOMIAL DIFFERENTIAL SYSTEM

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ABSTRACT. We study the number of limit cycles that bifurcate from the periodic solutions surrounding a uniform isochronous center located at the origin of the quartic polynomial differential system

$$\dot{x} = -y + xy(x^2 + y^2), \quad \dot{y} = x + y^2(x^2 + y^2),$$

when perturbed in the class of all quartic polynomial differential systems. Using the averaging theory of first order we show that at least 8 limit cycles bifurcate from the period annulus of the center. Recently this problem was studied by Peng and Feng [9], where the authors found 3 limit cycles.

1. INTRODUCTION AND STATEMENT OF MAIN RESULTS

One of the main open problems in the qualitative theory of polynomial differential systems in $\mathbb{R}^2$ is the determination of their limit cycles, see for instance [5]. A classical method to produce limit cycles is by perturbing a system which has a center. In this case the perturbed system displays limit cycles that bifurcate either from the center (having the so-called Hopf bifurcation); or from some of the periodic orbits around the center, (see [2, 10] and references therein); or from the graph in the boundary of the period annulus of the center.

Isochronous differential systems constitute a large class of polynomial systems with interesting properties. They arise in many applications. For recent studies on the bifurcation of limit cycles of planar polynomial differential systems having a uniform isochronous center see for instance [4, 6, 7]. In this paper we shall perturb the uniform isochronous center of the quartic polynomial differential system

$$\dot{x} = -y + xy(x^2 + y^2), \quad \dot{y} = x + y^2(x^2 + y^2), \tag{1.1}$$

inside the class of all quartic polynomial differential systems.

Peng and Feng [9] studied the differential system (1.1), showing that under any quartic homogeneous polynomial perturbations, at most 2 limit cycles bifurcate from the period annulus of such system using averaging theory of first order, and

that this upper bound can be reached. In addition these authors proved that for the family of perturbed quartic polynomial differential systems

$$
\begin{aligned}
\dot{x} = {}& -y + xy(x^2 + y^2) + \varepsilon(a_{10}x + a_{01}y + a_{11}xy + a_{21}x^2y + a_{03}y^3 \\
& + a_{40}x^4 + a_{31}x^3y + a_{22}x^2y^2 + a_{13}xy^3 + a_{04}y^4), \\
\dot{y} = {}& x + y^2(x^2 + y^2) + \varepsilon(b_{10}x + b_{01}y + b_{20}x^2 + b_{02}y^2 + b_{30}x^3 \\
& + b_{12}xy^2 + b_{40}x^4 + b_{31}x^3y + b_{22}x^2y^2 + b_{13}xy^3 + b_{04}y^4),
\end{aligned} \tag{1.2}
$$

there are at most 3 limit cycles bifurcating from the period annulus of (1.1) using averaging theory of first order, and that this upper bound is sharp. We remark that the perturbed system (1.2) studied by Peng and Feng do not cover all possible perturbed quartic polynomial differential systems because the authors do not consider the coefficients a_{00}, a_{20}, a_{02}, a_{30}, a_{12}, b_{00}, b_{11}, b_{21}, b_{03} in their analysis.

We study the limit cycles which bifurcate from the periodic solutions of the uniform isochronous center located at the origin of system (1.1) when it is perturbed inside the whole class of quartic polynomial differential systems. More precisely we consider the following differential systems

$$
\begin{aligned}
\dot{x} &= -y + xy(x^2 + y^2) + \varepsilon \sum_{i=0}^{4} p_i(x, y), \\
\dot{y} &= x + y^2(x^2 + y^2) + \varepsilon \sum_{i=0}^{4} q_i(x, y),
\end{aligned} \tag{1.3}
$$

where

$$
p_i = \sum_{j+k=i} a_{jk}x^j y^k, \quad q_i = \sum_{j+k=i} b_{jk}x^j y^k
$$

are homogeneous polynomials of degree i, and $a_{jk}, b_{jk} \in \mathbb{R}$. In what follows we state our main result.

Theorem 1.1. *For $|\varepsilon| \neq 0$ sufficiently small there are quartic polynomial differential systems (1.3) having at least 8 limit cycles bifurcating from the periodic orbits of the period annulus of the uniform isochronous center located at the origin of system (1.1).*

Note that Theorem 1.1 improves the result of Peng and Feng by providing five additional limit cycles. All our calculations were performed with the assistance of the software *Mathematica*.

2. Preliminary results

In this section we introduce some preliminary results on uniform isochronous centers and on averaging theory that we shall use in our study.

Let $p \in \mathbb{R}^2$ be a center of a differential polynomial system in $\mathbb{R}^2$. Without loss of generality we can assume that p is the origin of coordinates. We say that p is an *isochronous center* if it is a center having a neighborhood such that all the periodic orbits in this neighborhood have the same period. We say that p is a *uniform isochronous center* if the system, in polar coordinates (r, θ) where $x = r\cos\theta$, $y = r\sin\theta$, takes the form $\dot{r} = G(\theta, r)$, $\dot{\theta} = k$, $k \in \mathbb{R} \setminus \{0\}$, for more details see Conti [3]. The *period annulus* of a center is the biggest connected set of periodic

solutions surrounding a center and having in its inner boundary the center itself. The next result is well-known.

Proposition 2.1. *Assume that a planar differential polynomial system $\dot{x} = P(x, y)$, $\dot{y} = Q(x, y)$ of degree n has a center at the origin of coordinates. Then, this center is uniform isochronous if and only if through a linear change of variables and time rescaling it can be written under the form*

$$\dot{x} = -y + x\, f(x, y), \quad \dot{y} = x + y\, f(x, y), \tag{2.1}$$

where $f(x, y)$ is a polynomial in x and y of degree $n - 1$, and $f(0, 0) = 0$.

In 1994, Conti [3] proved the following theorem.

Theorem 2.2. *Let $f(x, y) = \sum_{i+j=n-1} p_{i,j} x^i y^j$ be a homogeneous polynomial of degree $n - 1$. Then system (2.1) has a uniform isochronous center at the origin if either n is even, or if n is odd and*

$$\sum_{\nu=0}^{n-1} \left[p_{n-1-\nu,\nu} \int_0^{2\pi} \cos^{n-1-\nu} \theta \sin^\nu \theta\, d\theta \right] = 0.$$

The next result is the first-order averaging theory developed for continuous differential systems. Consider the differential system

$$\dot{x} = \varepsilon F_1(t, x) + \varepsilon^2 F_2(t, x, \varepsilon), \quad x(0) = x_0 \tag{2.2}$$

with $x \in D$, where D is an open subset of $\mathbb{R}^n$, $t \geq 0$. Furthermore we suppose that the functions $F_1(t, x)$ and $F_2(t, x, \varepsilon)$ are $T-$periodic in t. We define in D the averaged differential system

$$\dot{y} = \varepsilon f_1(y), \quad y(0) = x_0, \tag{2.3}$$

where

$$f_1(y) = \frac{1}{T} \int_0^T F_1(t, y) dt.$$

As we shall see under convenient assumptions, the equilibria solutions of the averaged system will provide $T-$periodic solutions of system (2.2).

Theorem 2.3. *Consider the two initial value problems (2.2) and (2.3). Assume that*

(i) *the functions F_1, $\partial F_1/\partial x$, $\partial^2 F_1/\partial x^2$, F_2 and $\partial F_2/\partial x$ are defined, continuous and bounded by a constant independent of ε in $[0, \infty) \times D$ and $\varepsilon \in (0, \varepsilon_0]$;*

(ii) *the functions F_1 and F_2 are $T-$periodic in t (T independent of ε).*

Then the following statements hold.

(a) *If p is an equilibrium point of the averaged system (2.3) satisfying*

$$\det \left(\frac{\partial f_1}{\partial y} \right) \Big|_{y=p} \neq 0,$$

then there is a $T-$periodic solution $\varphi(t, \varepsilon)$ of system (2.2) such that $\varphi(0, \varepsilon) \to p$ as $\varepsilon \to 0$.

(b) *The kind of stability or instability of the periodic solution $\varphi(t, \varepsilon)$ coincides with the kind of stability or instability of the equilibrium point p of the averaged system (2.3). The equilibrium point p has the kind of stability behavior of the Poincaré map associated to the periodic solution $\varphi(t, \varepsilon)$.*

For a proof of the above, see [11, sec. 6.3, 11.8]. The next theorem provides a method to write a perturbed differential system under the form (2.2).

Theorem 2.4. *Consider the unperturbed system $\dot{x} = P(x, y)$, $\dot{y} = Q(x, y)$, where $P, Q : \mathbb{R}^2 \to \mathbb{R}$ are continuous functions, and assume that this system has a continuous family of period solutions $\{\Gamma_h\} \subset \{(x, y) : \mathcal{H}(x, y) = h, h_1 < h < h_2\}$, where $\mathcal{H}$ is a first integral of the system. For a given first integral H assume that $xQ(x, y) - yP(x, y) \neq 0$ for all (x, y) in the period annulus formed by the ovals $\{\Gamma_h\}$. Let $\rho : (\sqrt{h_1}, \sqrt{h_2}) \times [0, 2\pi) \to [0, \infty)$ be a continuous function such that*

$$H(\rho(R, \theta) \cos \theta, \rho(R, \theta) \sin \theta) = R^2$$

for all $R \in (\sqrt{h_1}, \sqrt{h_2})$ and all $\theta \in [0, 2\pi)$. Then the differential equation which describes the dependence between the square root of the energy $R = \sqrt{h}$ and the angle θ for the perturbed system $\dot{x} = P(x, y) + \varepsilon p(x, y)$, $\dot{y} = Q(x, y) + \varepsilon q(x, y)$, where $p, q : \mathbb{R}^2 \to \mathbb{R}$ are continuous functions is

$$\frac{dR}{d\theta} = \varepsilon \frac{\mu(x^2 + y^2)(Qp - Pq)}{2R(Qx - Py)} + \mathcal{O}(\varepsilon^2) \tag{2.4}$$

where $\mu = \mu(x, y)$ is the integrating factor corresponding to the first integral H of the unperturbed system and $x = \rho(R, \theta) \cos \theta$, $y = \rho(R, \theta) \sin \theta$.

For more details see [1]. We also need the next result, which can be found in [8, Prop. 1].

Proposition 2.5. *Let $f_0, \ldots, f_n$ be analytic functions defined on an open interval $I \subset \mathbb{R}$. If $f_0, \ldots, f_n$ are linearly independent then there exists $s_1, \ldots, s_n \in I$ and $\lambda_0, \ldots, \lambda_n \in \mathbb{R}$ such that for every $j \in \{1, \ldots, n\}$ we have $\sum_{i=0}^n \lambda_i f_i(s_j) = 0$.*

3. Proof of Theorem 1.1

By Theorem 2.2 it follows that system (1.1) has a uniform isochronous center at the origin. A first integral H and its corresponding integrating factor μ for system (1.1) are

$$H(x, y) = \frac{1}{3(x^2 + y^2)^{3/2}} - \frac{x}{(x^2 + y^2)^{1/2}}, \quad \mu(x, y) = \frac{1}{(x^2 + y^2)^{5/2}},$$

respectively. If $h \in (1, +\infty)$ then $H(x, y) = h$ are periodic solutions around the center $(0, 0)$. For proving Theorem 1.1 we shall use Theorem 2.4. We choose

$$\rho(R, \theta) = \frac{1}{(R^2 + 3\cos\theta)^{1/3}},$$

then $H(\rho \cos \theta, \rho \sin \theta) = R^2/3$ for all $R > \sqrt{3}$ and $\theta \in [0, 2\pi)$. Therefore all the hypotheses of Theorem 2.4 are satisfied for system (1.1). Using Theorem 2.4 we transform the perturbed differential system (1.3) into the form

$$\frac{dR}{d\theta} = \varepsilon \left(\frac{3}{2R} \frac{Qp - Pq}{\rho^5} \right) \Big|_{x = \rho \cos \theta, y = \rho \sin \theta} + O(\varepsilon^2), \tag{3.1}$$

where

$$Qp - Pq = A + B,$$

with

$$A = a_{00}x + b_{00}y + (a_{02} + b_{11})xy^2 + a_{20}x^3 + (a_{00} + b_{03})y^4$$
$$- b_{00}xy^3 + (a_{00} + a_{12} + b_{21})x^2y^2 - b_{00}x^3y + a_{30}x^4 + a_{02}y^6$$

$$+ (a_{02} + a_{20} - b_{11})x^2y^4 + (a_{20} - b_{11})x^4y^2 + (a_{12} - b_{03})xy^6$$
$$(a_{12} + a_{30} - b_{03} - b_{21})x^3y^4 + (a_{30} - b_{21})x^5y^2,$$

$$B = a_{10}x^2 + (a_{01} + b_{10})xy + b_{01}y^2 + (a_{11} + b_{20})x^2y + b_{02}y^3$$
$$+ (a_{21} + b_{30})x^3y + (a_{03} + b_{12})xy^3 + a_{40}x^5 + (a_{31} + b_{40} - b_{10})x^4y$$
$$+ (a_{22} + a_{10} + b_{31} - b_{01})x^3y^2 + (a_{13} + a_{01} + b_{22} - b_{10})x^2y^3$$
$$+ (a_{04} + a_{10} + b_{13} - b_{01})xy^4 + (a_{01} + b_{04})y^5 - b_{20}x^5y$$
$$+ (a_{11} - b_{20} - b_{02})x^3y^3 + (a_{11} - b_{02})xy^5 - b_{30}x^6y$$
$$+ (a_{21} - b_{30} - b_{12})x^4y^3 + (a_{21} + a_{03} - b_{12})x^2y^5 + a_{03}y^7$$
$$- b_{40}x^7y + (a_{40} - b_{31})x^6y^2 + (a_{31} - b_{40} - b_{22})x^5y^3$$
$$+ (a_{40} + a_{22} - b_{31} - b_{13})x^4y^4 + (a_{31} + a_{13} - b_{22} - b_{04})x^3y^5.$$

We remark that the coefficients $\{a_{ij}, b_{ij}\}_{i,j \in \{0,\ldots,4\}}$ which appear in A and B are different. The expression B corresponds to the perturbed system (1.2) studied in [9]. The authors of [9] obtained for this system the following averaging function

$$g_B(R) = \frac{3}{4R}\Big[\Big(M_4 - \frac{3M_1 + 4M_2 + 8M_3}{36}\Big)R^2 - \frac{M_1 + 2M_2}{82}R^6 - \frac{2M_1}{729}R^{10}$$
$$+ \Big(\frac{2M_1}{729}R^{12} + \frac{2M_2}{81}R^8 + \frac{2M_3}{9}R^4 - 2(M_1 + M_2 + M_3)\Big)\frac{1}{\sqrt{R^4 - 9}}\Big],$$
$$(3.2)$$

where

$$M_1 = a_{22} - a_{40} - a_{04} + b_{31} - b_{13},$$
$$M_2 = -2a_{22} + a_{40} + 3a_{04} - b_{31} + 2b_{13}, \qquad (3.3)$$
$$M_3 = a_{22} - 3a_{04} - b_{13}, \quad M_4 = a_{10} + b_{01}.$$

Peng and Feng proved that the function $g_B(R)$ has at most 3 zeros in $R \in (\sqrt{3}, +\infty)$, and using the averaging theory of first order they showed that the maximum number of limit cycles of system (1.2) emerging from the period annulus of the unperturbed system (1.1) is 3.

In this work we extend the results presented in [9] by calculating the part of the averaging function of system (1.3) which corresponds to the expression A. In order to do that, we perturb the center of system (1.1) inside the whole class of quartic polynomial differential systems. We note that (3.1) is continuous and bounded for $\theta \in (0, 2\pi)$ and $R \in (\sqrt{3}, +\infty)$ therefore the integral of (3.1) is the sum of the integrals of its parts A and B. Then from the expression (3.1) we have

$$\frac{dR}{d\theta} = \varepsilon\Big(\frac{3}{2R}\frac{A}{\rho^5}\Big)\Big|_{\substack{x=\rho\cos\theta \\ y=\rho\sin\theta}} + \varepsilon\Big(\frac{3}{2R}\frac{B}{\rho^5}\Big)\Big|_{\substack{x=\rho\cos\theta \\ y=\rho\sin\theta}} + O(\varepsilon^2).$$

We obtain the averaging function $f(R) = g_A(R) + g_B^*(R)$ where

$$g_A(R) = a_{00}g_0(R) + a_{02}g_1(R) + a_{12}g_2(R) + a_{20}g_3(R) + a_{30}g_4(R)$$
$$+ b_{03}g_5(R) + b_{11}g_6(R) + b_{21}g_7(R),$$

$$g_B^*(R) = \sum_{i=1}^{4} M_i\, g_{Mi}(R),$$

and $g_B^*(R)$ is the function (3.2) rearranged in a convenient way, with M_i, $i \in \{1, \ldots, 4\}$ given in (3.3). The expressions of $g_i(R)$, $i \in \{0, \ldots, 7\}$ are shown in the Appendix 4, and the functions $g_{Mi}(R)$, $i \in \{1, \ldots, 4\}$ are presented in the Appendix 5.

Out of the 12 functions $G_i = g_i : (\sqrt{3}, +\infty) \to \mathbb{R}$, $i \in \{0, \ldots, 7\}$, $G_{i+7} = g_{Mi} : (\sqrt{3}, +\infty) \to \mathbb{R}$, $i \in \{1, \ldots, 4\}$ we have that 9 are linearly independent. Indeed, using the software *Mathematica* to calculate the Taylor expansions for those 12 functions in the variable R until its 15th power around $R = 2$, which are too long and therefore they are not presented here, we construct a 12×16 matrix, where in the k row we place the 16 coefficients of $R^0, R^1, \ldots, R^{15}$ of the Taylor expansion of G_k, $k \in \{0, \ldots, 11\}$, and we conclude that the rank of such matrix is 9.

By Proposition 2.5 since there are 9 linearly independent functions among the 12 previously described, then there exists a linear combination of them with at least 8 zeros, because all the coefficients of these functions are linearly independent, as it is easy to check. Thus there exist $R_1, R_2, \ldots, R_8 \in (\sqrt{3}, +\infty)$ and coefficients $a_{ij}, b_{ij} \in \mathbb{R}$, $i, j \in \{0, \ldots, 4\}$ such that $f(R_k) = 0$, $k \in \{1, \ldots, 8\}$.

In summary, applying Theorem 2.3 we conclude that there are planar quartic polynomial differential systems (1.3) having at least 8 limit cycles bifurcating from the period orbits of the period annulus of the uniform isochronous center located at the origin of the unperturbed differential system (1.1).

4. Appendix: Averaging functions $g_i(R)$, $i \in \{0, \ldots, 7\}$

$$
\begin{aligned}
g_0 = -3\pi\Big(& (R^2+3)(-6R^{10}(R^2+3)^{2/3} + 59R^6(R^2+3)^{2/3} - 1440R^2(R^2+3)^{2/3} \\
& + 6R^8((R^4-9)^{2/3}\sqrt[3]{R^2-3} + 3(R^2+3)^{2/3}) - R^4\big(709(R^4-9)^{2/3}\sqrt[3]{R^2-3} \\
& + 177(R^2+3)^{2/3}\big) + 360(12(R^2+3)^{2/3} - 61\sqrt[3]{R^2-3}(R^4-9)^{2/3})) \\
& \times {}_2F_1(-\frac{1}{2}, \frac{2}{3}; 1; \frac{6}{R^2+3})\sqrt[3]{R^2-3} + \Big(7320(R^4-9)^{2/3}\sqrt[3]{R^2-3} \\
& - 1440(R^2+3)^{2/3} + R^2\Big(1346(R^4-9)^{2/3}\sqrt[3]{R^2-3} + 6R^8(R^2+3)^{2/3} \\
& + 618(R^2+3)^{2/3} - 6R^6\big((R^4-9)^{2/3}\sqrt[3]{R^2-3} + (R^2+3)^{2/3}\big) \\
& - R^4(12(R^4-9)^{2/3}\sqrt[3]{R^2-3} + 71(R^2+3)^{2/3}) + R^2(685(R^4-9)^{2/3}\sqrt[3]{R^2-3} \\
& + 59(R^2+3)^{2/3})\Big)\Big){}_2F_1(\frac{1}{2}, \frac{2}{3}; 1; \frac{6}{R^2+3})(R^2-3)^{4/3} \\
& + (R^2+3)^{4/3}\Big(\big(-1346(R^4-9)^{2/3}\sqrt[3]{R^2+3} + 618(R^2-3)^{2/3} \\
& + R^2\Big(685(R^4-9)^{2/3}\sqrt[3]{R^2+3} - 59(R^2-3)^{2/3} \\
& + R^2\Big(12(R^4-9)^{2/3}\sqrt[3]{R^2+3} - 71(R^2-3)^{2/3} + 6R^2\big(-(R^4-9)^{2/3}\sqrt[3]{R^2+3} \\
& + R^2(R^2-3)^{2/3} + (R^2-3)^{2/3}\big)\Big)\Big)\Big)R^2 + 120(61(R^4-9)^{2/3}\sqrt[3]{R^2+3} \\
& + 12(R^2-3)^{2/3})\Big){}_2F_1(\frac{1}{2}, \frac{2}{3}; 1; -\frac{6}{R^2-3}) - (R^2-3)\sqrt[3]{R^2+3} \\
& \times \Big(6R^{10}(R^2-3)^{2/3} - 59R^6(R^2-3)^{2/3} + 1440R^2(R^2-3)^{2/3}
\end{aligned}
$$

$$+ 360(61(R^4-9)^{2/3}\sqrt[3]{R^2+3}+12(R^2-3)^{2/3})$$
$$+ R^4(709\sqrt[3]{R^2+3}(R^4-9)^{2/3}-177(R^2-3)^{2/3})+6R^8(3(R^2-3)^{2/3}$$
$$- \sqrt[3]{R^2+3}(R^4-9)^{2/3})\Big)\,_2F_1(-\frac{1}{2},\frac{2}{3};1;-\frac{6}{R^2-3})\Big)$$
$$\div 14560R(R^2-3)^{2/3}(R^2+3)^{2/3}(R^4-9)^{2/3};$$

$$g_1 = \pi\Big(-(2R^4-39)(3R^2\sqrt[3]{R^2-3}+9\sqrt[3]{R^2-3}+2(R^2+3)^{2/3}\sqrt[3]{R^4-9})R^2$$
$$_2F_1(-\frac{2}{3},\frac{1}{2};1;\frac{6}{R^2+3})+2(R^4-12)((R^2+3)(3R^2\sqrt[3]{R^2+3}$$
$$-9\sqrt[3]{R^2+3}+2(R^2-3)^{2/3}\sqrt[3]{R^4-9})\,_2F_1(\frac{1}{3},\frac{1}{2};1;-\frac{6}{R^2-3})$$
$$+ (R^2-3)(3R^2\sqrt[3]{R^2-3}+9\sqrt[3]{R^2-3}+2(R^2+3)^{2/3}\sqrt[3]{R^4-9})$$
$$\times\,_2F_1(\frac{1}{3},\frac{1}{2};1;\frac{6}{R^2+3}))-R^2(2R^4-39)(3R^2\sqrt[3]{R^2+3}-9\sqrt[3]{R^2+3}$$
$$+ 2(R^2-3)^{2/3}\sqrt[3]{R^4-9})\,_2F_1(-\frac{2}{3},\frac{1}{2};1;-\frac{6}{R^2-3})\Big)$$
$$\div 880R\sqrt[3]{R^4-9};$$

$$g_2 = 3\pi\Big((R^2+3)\Big(72R^{14}(R^2+3)^{2/3}-840R^{10}(R^2+3)^{2/3}+391R^6(R^2+3)^{2/3}$$
$$+ 50688R^2(R^2+3)^{2/3}-72R^{12}((R^4-9)^{2/3}\sqrt[3]{R^2-3}$$
$$+ 3(R^2+3)^{2/3})+24R^8(73(R^4-9)^{2/3}\sqrt[3]{R^2-3}+105(R^2+3)^{2/3})$$
$$- 51R^4(261(R^4-9)^{2/3}\sqrt[3]{R^2-3}+23(R^2+3)^{2/3})$$
$$- 1728(88(R^2+3)^{2/3}-15\sqrt[3]{R^2-3}(R^4-9)^{2/3})\Big)$$
$$\times\,_2F_1(-\frac{1}{2},\frac{2}{3};1;\frac{6}{R^2+3})\sqrt[3]{R^2-3}+((12294(R^4-9)^{2/3}\sqrt[3]{R^2-3}$$
$$- 24702(R^2+3)^{2/3}+R^2(8535(R^4-9)^{2/3}\sqrt[3]{R^2-3}+391(R^2+3)^{2/3}$$
$$+ R^2(-2640(R^4-9)^{2/3}\sqrt[3]{R^2-3}-1711(R^2+3)^{2/3}$$
$$- 24R^2(61(R^4-9)^{2/3}\sqrt[3]{R^2-3}+3R^6(R^2+3)^{2/3}+35(R^2+3)^{2/3}$$
$$- 3R^4((R^4-9)^{2/3}\sqrt[3]{R^2-3}+(R^2+3)^{2/3})-R^2(6(R^4-9)^{2/3}\sqrt[3]{R^2-3}$$
$$+ 41(R^2+3)^{2/3})))))R^2+576(88(R^2+3)^{2/3}-15\sqrt[3]{R^2-3}(R^4-9)^{2/3}))$$
$$\times\,_2F_1(\frac{1}{2},\frac{2}{3};1;\frac{6}{R^2+3})(R^2-3)^{4/3}+\sqrt[3]{R^2+3}Big((50688(R^2-3)^{2/3}$$
$$+ R^2((391(R^2-3)^{2/3}+24R^2(73(R^4-9)^{2/3}\sqrt[3]{R^2+3}+3R^6(R^2-3)^{2/3}$$
$$- 35R^2(R^2-3)^{2/3}-105(R^2-3)^{2/3}+R^4(9(R^2-3)^{2/3}$$
$$- 3\sqrt[3]{R^2+3}(R^4-9)^{2/3})))R^2+51(23(R^2-3)^{2/3}$$
$$- 261\sqrt[3]{R^2+3}(R^4-9)^{2/3})))R^2+1728(15(R^4-9)^{2/3}\sqrt[3]{R^2+3}$$

$$+ 88(R^2 - 3)^{2/3}\big) \, _2F_1(-\frac{1}{2}, \frac{2}{3}; 1; -\frac{6}{R^2 - 3})(R^2 - 3)$$

$$- (R^2 + 3)^{4/3}\big(\big((-8535(R^4 - 9)^{2/3}\sqrt[3]{R^2 + 3} + 391(R^2 - 3)^{2/3}$$

$$+ R^2(-2640(R^4 - 9)^{2/3}\sqrt[3]{R^2 + 3} + 1711(R^2 - 3)^{2/3}$$

$$+ 24R^2(61(R^4 - 9)^{2/3}\sqrt[3]{R^2 + 3} - 35(R^2 - 3)^{2/3} + R^2(6(R^4 - 9)^{2/3}\sqrt[3]{R^2 + 3}$$

$$- 41(R^2 - 3)^{2/3} + 3R^2(-(R^4 - 9)^{2/3}\sqrt[3]{R^2 + 3} + R^2(R^2 - 3)^{2/3}$$

$$+ (R^2 - 3)^{2/3})))))R^2 + 6(2049(R^4 - 9)^{2/3}\sqrt[3]{R^2 + 3}$$

$$+ 4117(R^2 - 3)^{2/3}))R^2 + 576(15(R^4 - 9)^{2/3}\sqrt[3]{R^2 + 3}$$

$$+ 88(R^2 - 3)^{2/3})) \, _2F_1(\frac{1}{2}, \frac{2}{3}; 1; -\frac{6}{R^2 - 3})\big)$$

$$\div 442624R(R^2 - 3)^{2/3}(R^2 + 3)^{2/3}(R^4 - 9)^{2/3};$$

$$g_3 = \pi\big(\big(-29R^2\sqrt[3]{R^2 + 3} + 87\sqrt[3]{R^2 + 3} + 6R^6\sqrt[3]{R^2 + 3}$$

$$+ 98(R^2 - 3)^{2/3}\sqrt[3]{R^4 - 9} + R^4(4(R^2 - 3)^{2/3}\sqrt[3]{R^4 - 9}$$

$$- 18\sqrt[3]{R^2 + 3}))R^2 \, _2F_1(-\frac{2}{3}, \frac{1}{2}; 1; -\frac{6}{R^2 - 3})$$

$$+ (-29R^2\sqrt[3]{R^2 - 3} - 87\sqrt[3]{R^2 - 3} + 6R^6\sqrt[3]{R^2 - 3}$$

$$+ 98(R^2 + 3)^{2/3}\sqrt[3]{R^4 - 9} + 2R^4(9\sqrt[3]{R^2 - 3}$$

$$+ 2(R^2 + 3)^{2/3}\sqrt[3]{R^4 - 9}))R^2 \, _2F_1(-\frac{2}{3}, \frac{1}{2}; 1; \frac{6}{R^2 + 3})$$

$$- 2(R^2 + 3)(8R^2\sqrt[3]{R^2 + 3} - 24\sqrt[3]{R^2 + 3} + 3R^6\sqrt[3]{R^2 + 3}$$

$$+ 64(R^2 - 3)^{2/3}\sqrt[3]{R^4 - 9} + R^4(2(R^2 - 3)^{2/3}\sqrt[3]{R^4 - 9}$$

$$- 9\sqrt[3]{R^2 + 3})) \, _2F_1(\frac{1}{3}, \frac{1}{2}; 1; -\frac{6}{R^2 - 3}) - 2(R^2 - 3)(8R^2\sqrt[3]{R^2 - 3}$$

$$+ 3R^6\sqrt[3]{R^2 - 3} + R^4(9\sqrt[3]{R^2 - 3} + 2(R^2 + 3)^{2/3}\sqrt[3]{R^4 - 9})$$

$$+ 8(3\sqrt[3]{R^2 - 3} + 8(R^2 + 3)^{2/3}\sqrt[3]{R^4 - 9})) \, _2F_1(\frac{1}{3}, \frac{1}{2}; 1; \frac{6}{R^2 + 3})\big)$$

$$\div 880R\sqrt[3]{R^4 - 9};$$

$$g_4 = 3\pi\Big((R^2 + 3)(6R^8 - 7R^4 - 1056)(R^2 - 3)^{2/3} \, _2F_1(-\frac{1}{2}, \frac{2}{3}; 1; \frac{6}{R^2 + 3})$$

$$+ (R^2 + 3)^{2/3}(6R^8 - 7R^4 - 1056)(R^2 - 3) \, _2F_1(-\frac{1}{2}, \frac{2}{3}; 1; -\frac{6}{R^2 - 3})$$

$$- (6R^8 + 12R^6 + 17R^4 + 58R^2 - 352)(R^2 - 3)^{5/3} \, _2F_1(\frac{1}{2}, \frac{2}{3}; 1; \frac{6}{R^2 + 3})$$

$$+ (R^2 + 3)^{2/3}(-6R^{10} - 6R^8 + 19R^6 + 7R^4 + 526R^2 + 1056)$$

$$\times \, _2F_1(\frac{1}{2}, \frac{2}{3}; 1; -\frac{6}{R^2 - 3})\Big)\Big/ 2912R(R^4 - 9)^{2/3};$$

$$g_5 = -9\pi\bigg(-(R^2+3)(-120R^{14}(R^2+3)^{2/3}+1704R^{10}(R^2+3)^{2/3}$$
$$-3641R^6(R^2+3)^{2/3}-11520R^2(R^2+3)^{2/3}+120R^{12}((R^4-9)^{2/3}$$
$$\times\sqrt[3]{R^2-3}+3(R^2+3)^{2/3})+17280(7(R^4-9)^{2/3}\sqrt[3]{R^2-3}$$
$$+2(R^2+3)^{2/3})-8R^8(403(R^4-9)^{2/3}\sqrt[3]{R^2-3}+639(R^2+3)^{2/3})$$
$$+3R^4(10587(R^4-9)^{2/3}\sqrt[3]{R^2-3}+3641(R^2+3)^{2/3}))$$
$$\times{}_2F_1(-\frac{1}{2},\frac{2}{3};1;\frac{6}{R^2+3})\sqrt[3]{R^2-3}+\Big(35994(R^4-9)^{2/3}\sqrt[3]{R^2-3}$$
$$-9858(R^2+3)^{2/3}+R^2\Big(22585(R^4-9)^{2/3}\sqrt[3]{R^2-3}+3641(R^2+3)^{2/3}$$
$$+R^2\big(-5008(R^4-9)^{2/3}\sqrt[3]{R^2-3}-6449(R^2+3)^{2/3}$$
$$-8R^2(343(R^4-9)^{2/3}\sqrt[3]{R^2-3}+213(R^2+3)^{2/3}$$
$$+3R^2(-10(R^4-9)^{2/3}\sqrt[3]{R^2-3}+5R^4(R^2+3)^{2/3}-81(R^2+3)^{2/3}$$
$$\times\sqrt[3]{R^2-3}+(R^2+3)^{2/3})))\Big)\Big)R^2+5760(7(R^4-9)^{2/3}\sqrt[3]{R^2-3}$$
$$+2(R^2+3)^{2/3})){}_2F_1(\frac{1}{2},\frac{2}{3};1;\frac{6}{R^2+3})(R^2-3)^{4/3}$$
$$+\sqrt[3]{R^2+3}((11520(R^2-3)^{2/3}+R^2(-31761(R^4-9)^{2/3}\sqrt[3]{R^2+3}$$
$$+10923(R^2-3)^{2/3}+R^2(3641(R^2-3)^{2/3}+8R^2(403(R^4-9)^{2/3}$$
$$\times\sqrt[3]{R^2+3}+15R^6(R^2-3)^{2/3}-213R^2(R^2-3)^{2/3}-639(R^2-3)^{2/3}$$
$$+15R^4(3(R^2-3)^{2/3}-\sqrt[3]{R^2+3}(R^4-9)^{2/3})))))R^2$$
$$+17280(2(R^2-3)^{2/3}-7\sqrt[3]{R^2+3}(R^4-9)^{2/3}))$$
$${}_2F_1(-\frac{1}{2},\frac{2}{3};1;-\frac{6}{R^2-3})(R^2-3)-(R^2+3)^{4/3}((35994(R^4-9)^{2/3}$$
$$\times\sqrt[3]{R^2+3}+9858(R^2-3)^{2/3}+R^2(-22585(R^4-9)^{2/3}\sqrt[3]{R^2+3}$$
$$+3641(R^2-3)^{2/3}+R^2(-5008(R^4-9)^{2/3}\sqrt[3]{R^2+3}+6449(R^2-3)^{2/3}$$
$$+8R^2(343(R^4-9)^{2/3}\sqrt[3]{R^2+3}-213(R^2-3)^{2/3}$$
$$+3R^2(10(R^4-9)^{2/3}\sqrt[3]{R^2+3}-81(R^2-3)^{2/3}+5R^2(-(R^4-9)^{2/3}$$
$$\times\sqrt[3]{R^2+3}+R^2(R^2-3)^{2/3}+(R^2-3)^{2/3}))))))R^2+5760(2(R^2-3)^{2/3}$$
$$-7\sqrt[3]{R^2+3}(R^4-9)^{2/3})){}_2F_1(\frac{1}{2},\frac{2}{3};1;-\frac{6}{R^2-3})\bigg)$$
$$\div2213120R(R^2-3)^{2/3}(R^2+3)^{2/3}(R^4-9)^{2/3};$$

$$g_6=\pi\Big(2(29R^2\sqrt[3]{R^2+3}-87\sqrt[3]{R^2+3}-6R^6\sqrt[3]{R^2+3}+78(R^2-3)^{2/3}$$
$$\times\sqrt[3]{R^4-9}+2R^4(9\sqrt[3]{R^2+3}-2(R^2-3)^{2/3}\sqrt[3]{R^4-9}))R^2$$
$$\times{}_2F_1(-\frac{2}{3},\frac{1}{2};1;-\frac{6}{R^2-3})+2(29R^2\sqrt[3]{R^2-3}+87\sqrt[3]{R^2-3}$$

$$-6R^6\sqrt[3]{R^2-3} + 78(R^2+3)^{2/3}\sqrt[3]{R^4-9} - 2R^4(9\sqrt[3]{R^2-3}$$
$$+ 2(R^2+3)^{2/3}\sqrt[3]{R^4-9}))R^2\,{}_2F_1(-\frac{2}{3},\frac{1}{2};1;\frac{6}{R^2+3})$$
$$+ 4(R^2+3)(8R^2\sqrt[3]{R^2+3} + 3R^6\sqrt[3]{R^2+3} - 24(\sqrt[3]{R^2+3}$$
$$+ (R^2-3)^{2/3}\sqrt[3]{R^4-9}) + R^4(2(R^2-3)^{2/3}\sqrt[3]{R^4-9} - 9\sqrt[3]{R^2+3}))$$
$$\times\,{}_2F_1(\frac{1}{3},\frac{1}{2};1;-\frac{6}{R^2-3}) + 4(R^2-3)(8R^2\sqrt[3]{R^2-3}$$
$$+ 3R^6\sqrt[3]{R^2-3} + R^4(9\sqrt[3]{R^2-3} + 2(R^2+3)^{2/3}\sqrt[3]{R^4-9})$$
$$+ 24(\sqrt[3]{R^2-3} - (R^2+3)^{2/3}\sqrt[3]{R^4-9}))\,{}_2F_1(\frac{1}{3},\frac{1}{2};1;\frac{6}{R^2+3}))\Big)$$
$$\div 1760R\sqrt[3]{R^4-9};$$

$$g_7 = 3\pi\Big(3(R^2+3)(2R^8-11R^4+64)(R^2-3)^{2/3}\,{}_2F_1(-\frac{1}{2},\frac{2}{3};1;\frac{6}{R^2+3})$$
$$+ 3(R^2+3)^{2/3}(2R^8-11R^4+64)(R^2-3)\,{}_2F_1(-\frac{1}{2},\frac{2}{3};1;-\frac{6}{R^2-3})$$
$$- (6R^8+12R^6-9R^4+6R^2+64)(R^2-3)^{5/3}\,{}_2F_1(\frac{1}{2},\frac{2}{3};1;\frac{6}{R^2+3})$$
$$- (R^2+3)^{5/3}(6R^8-12R^6-9R^4-6R^2+64)\,{}_2F_1(\frac{1}{2},\frac{2}{3};1;-\frac{6}{R^2-3})\Big)$$
$$\div 2912R(R^4-9)^{2/3};$$

where $_2F_1(a,b,c,z)$ is the hypergeometric function which has the series expansion

$$\sum_{k=0}^{+\infty}\frac{(a)_k(b)_k}{(c)_k}\frac{z^k}{k!},$$

with

$$(a)_k = \begin{cases} 1 & \text{if } k=0; \\ a(a+1)(a+2)\cdots(a+k-1) & \text{if } k>0. \end{cases}$$

5. Averaging functions $g_{Mi}(R)$, $i\in\{1,\dots,4\}$

$$g_{M1} = -\frac{R}{16} - \frac{R^5}{108} - \frac{R^9}{486} - \frac{3}{2R\sqrt{R^4-9}} + \frac{R^{11}}{486\sqrt{R^4-9}};$$
$$g_{M2} = -\frac{R}{12} - \frac{R^5}{54} + \frac{\sqrt{R^4-9}}{6R} + \frac{1}{54}\sqrt{R^4-9}R^3;$$
$$g_{M3} = -\frac{R}{6} + \frac{\sqrt{R^4-9}}{6R}; \quad g_{M4} = \frac{3R}{4}.$$

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