Electronic Journal of Differential Equations, Vol. 2015 (2015), No. 250, pp. 1–9.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

REGULARITY FOR THE AXISYMMETRIC NAVIER-STOKES EQUATIONS

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Abstract. In this article, we establish a regularity criterion for the Navier-Stokes system with axisymmetric initial data. It is proved that if the local axisymmetric smooth solution u satisfies $\|u^\theta\|_{L^\alpha(0,T;L^\beta)} < \infty$, where $\frac{2}{\alpha}+\frac{3}{\beta} \le 1$, and $3 < \beta \le \infty$, then the strong solution keeps smoothness up to time T.

1. Introduction

We study the following classic 3D incompressible Navier-Stokes equations in the whole space,

$$\begin{gathered} \partial_t u + (u\cdot\nabla u)u + \nabla p = \nu\Delta u,\\ \nabla\cdot u = 0,\\ u(x,t=0) = u_0, \end{gathered} \tag{1.1}$$

where $u(x,t) \in \mathbb{R}^3$ and $p(x,t) \in \mathbb{R}$ denote the unknowns, velocity and pressure respectively, while ν denotes the viscous coefficient of the system.

A lot of works have been devoted to study the above system, but global well-posedness for (1.1) with arbitrary large initial data is still a challenging open problem, see [3, 5, 8, 12, 13, 14].

Here, we are concerned with (1.1) with axisymmetric initial data. If u_0 is axisymmetric in system (1.1), then the solution $u(x,t)$ of system (1.1)is also axisymmetric [10, 7]. So, it is convenient to write $u(x,t)$ as in the form

$$u(x,t) = u^r(r,z,t)e_r + u^\theta(r,z,t)e_\theta + u^z(r,z,t)e_z,$$

where e_r, e_θ and e_z are the standard orthonormal unit vectors in cylindrical coordinate system

$$\begin{gathered} e_r = (\frac{x_1}{r},\frac{x_2}{r},0) = (\cos\theta,\sin\theta,0),\\ e_\theta = (-\frac{x_2}{r},\frac{x_1}{r},0) = (-\sin\theta,\cos\theta,0),\\ e_z = (0,0,1), \end{gathered}$$

2010 *Mathematics Subject Classification.* 35Q30, 76D03.
Key words and phrases. Navier-Stokes equations; axi-symmetric flow; regularity criterion.

Submitted June 11, 2015. Published September 25, 2015.

with $r = (x_1^2 + x_2^2)^{1/2}$. By direct computations, it is easy to show the following relations.

$$\begin{gathered}\nabla = (\partial_{x_1}, \partial_{x_2}, \partial_z)^T = \partial_r e_r + \frac{\partial_\theta}{r} e_\theta + \partial_z e_z,\\ \Delta = \nabla\cdot\nabla = \frac{1}{r}\partial_r(r\partial_r) + \frac{1}{r^2}\frac{\partial^2}{\partial_\theta^2} + \frac{\partial^2}{\partial z^2},\\ \frac{\partial e_r}{\partial\theta} = e_\theta, \quad \frac{\partial e_\theta}{\partial\theta} = -e_r.\end{gathered}$$

Accordingly, the system (1.1) can be rewritten equivalently as

$$\begin{gathered}\frac{\tilde{D}}{Dt}u^r - \nu(\partial_r^2 + \partial_z^2 + \frac{1}{r}\partial_r - \frac{1}{r^2})u^r - \frac{(u^\theta)^2}{r} + \partial_r p = 0,\\ \frac{\tilde{D}}{Dt}u^\theta - \nu(\partial_r^2 + \partial_z^2 + \frac{1}{r}\partial_r - \frac{1}{r^2})u^\theta + \frac{u^r u^\theta}{r} = 0,\\ \frac{\tilde{D}}{Dt}u^z - \nu(\partial_r^2 + \partial_z^2 + \frac{1}{r}\partial_r)u^z + \partial_z p = 0,\\ u|_{t=0} = u_0^r\cdot e_r + u_0^\theta\cdot e_\theta + u_0^z\cdot e_z,\end{gathered} \tag{1.2}$$

where $\frac{\tilde{D}}{Dt}$ denotes the material derivative

$$\frac{\tilde{D}}{Dt} = \partial_t + u^r\partial_r + u^z\partial_z.$$

If $u^\theta = 0$ (so-called without swirl), Ukhovskii and Yudovich [10] (see also [7]) proved the existence of generalized solutions, uniqueness and regularity. When $u^\theta \neq 0$ (with swirl), it is much complicated and difficult. For recent progress, one can find results on regularity criteria or global existence with small initial data in [2, 6, 7, 11]. In particular, very recently in [11], the following regularity criterion was established:

$$\|u^\theta 1_{r\le\varsigma}\|_{L^\alpha((0,T);L^\beta)} < \infty, \quad \text{with } \frac{2}{\alpha} + \frac{3}{\beta} < 1, \quad \beta > 6, \quad \text{or } (\alpha,\beta) = (4,6), \tag{1.3}$$

where $\varsigma > 0$ is given.

The aim of this paper is to give a regularity criteria in terms of u^θ. More precisely, we have the following theorem.

Theorem 1.1. *Let $u_0 \in H^2$, and $u \in C([0,T); H^2(\mathbb{R}^3)) \cap L^2_{loc}([0,T); \dot{H}^3(\mathbb{R}^3))$ be the solution of* (1.1)*. If it satisfies*

$$\|u^\theta\|_{L^\alpha(0,T;L^\beta)} < \infty, \quad \text{where } \frac{2}{\alpha} + \frac{3}{\beta} = 1, \text{ and } 3 < \beta \le \infty, \tag{1.4}$$

then $u(x,t)$ can be continued beyond T.

In Section 2 some key lemmas are given. Then Section 3 is devoted to the proof of the main result.

2. Key lemmas

Before going to the details, let us introduce some notation. $L^{p,q}$ norm be defined by

$$\|u\|_{L^{p,q}} = \begin{cases} \big(\int_0^t \|u\|_{L^q}^p \mathrm{d}\tau\big)^{1/p} & \text{if } 1 \le p < \infty,\\ \operatorname{ess\,sup}_{0<\tau<t}\|u\|_{L^q} & \text{if } q = \infty.\end{cases} \tag{2.1}$$

And we define $\tilde{\nabla} \doteq (\partial_r, \partial_z)$.

Next, let us introduce the vorticity field and the corresponding equation,

$$\omega = \nabla \times u = (\frac{1}{r}\frac{\partial u^z}{\partial \theta} - \frac{\partial u^\theta}{\partial z})e_r + (\frac{\partial u^r}{\partial z} - \frac{\partial u^z}{\partial r})e_\theta + (\frac{1}{r}\frac{\partial}{\partial r}(u^\theta r) - \frac{1}{r}\frac{\partial u^r}{\partial \theta})e_z,$$

or equivalently,

$$\omega = \omega^r e_r + \omega^\theta e_\theta + \omega^z e_z = -\partial_z u^\theta e_r + (\partial_z u^r - \partial_r u^z)e_\theta + (\partial_r u^\theta + \frac{u^\theta}{r})e_z.$$

Then we have the vorticity equation

$$\begin{gathered}
\frac{\tilde{D}}{Dt}\omega^r - \nu(\partial_r^2 + \partial_z^2 + \frac{1}{r}\partial_r - \frac{1}{r^2})\omega^r - (\omega^r\partial_r + \omega^z\partial_z)u^r = 0, \\
\frac{\tilde{D}}{Dt}\omega^\theta - \nu(\partial_r^2 + \partial_z^2 + \frac{1}{r}\partial_r - \frac{1}{r^2})\omega^\theta - \frac{2u^\theta\partial_z u^\theta}{r} - \frac{u^r\omega^\theta}{r} = 0, \\
\frac{\tilde{D}}{Dt}\omega^z - \nu(\partial_r^2 + \partial_z^2 + \frac{1}{r}\partial_r)\omega^z - (\omega^r\partial_r + \omega^3\partial_3)u^3 = 0, \\
\omega|_{t=0} = \omega_0^r e_r + \omega_0^\theta e_\theta + \omega_0^z e_z.
\end{gathered} \tag{2.2}$$

If, we set $\tilde{u} \doteq u^r e_r + u^z e_z$, then

$$\nabla \cdot \tilde{u} = 0 \quad \text{and} \quad \nabla \times \tilde{u} = \omega^\theta e_\theta.$$

To proof Theorem 1.1, we need the following key lemmas.

Lemma 2.1 ([2, Lemma 2])**.** *Suppose that $u(x,t)$ is an axisymmetric vector field with $\operatorname{div} u = 0$, and $\omega = curl\ u$ vanishes sufficiently fast near infinity in $\mathbb{R}^3$, then ∇u and $\nabla(u^\theta e_\theta)$ can be represented as the singular integral form*

$$\begin{gathered}
\nabla\tilde{u}(x) = C\omega^\theta e_\theta(x) + [K * (\omega^\theta e_\theta)](x), \\
\nabla(u^\theta e_\theta(x)) = C\tilde{\omega}(x) + [H * (\tilde{\omega})](x),
\end{gathered}$$

where the kernels $K(x)$ and $H(x)$ are matrix valued functions homogeneous of degree -3, defining a singular integral operator by convolution, and $f * g(x) = \int_{\mathbb{R}^3} f(x-y)g(y)\mathrm{d}y$ denotes the standard convolution operator. The matrices C and $\tilde{C}$ are constant.

Lemma 2.2. *Based on the above Lemma 2.1 and the L^p boundness of Calderon-Zygmund singular integral operators with $1 < p < \infty$, we can deduce that*

$$\|\nabla\tilde{u}\|_{L^p} \lesssim \|\omega\|_{L^p}, \quad \|\nabla(u^\theta e_\theta)\|_{L^p} \lesssim \|\omega\|_{L^p}.$$

Lemma 2.3. *Let u be a sufficiently smooth vector field, then for all $1 < p < \infty$, we have*

$$\|\nabla u\|_{L^p} \le C(p)\|\omega\|_{L^p}.$$

Lemma 2.4 ([6, Lemma 3])**.** *Let u be a sufficiently smooth divergence-free axisymmetric vector field. Then there exist constants $C_1(p) > 0$ and $C_2 > 0$, independent of u, such that for $1 < p < \infty$, we have*

$$\begin{gathered}
\|\nabla u^r\|_{L^p} + \|\frac{u^r}{r}\|_{L^p} \le C_1(p)\|\omega^\theta\|_{L^p}, \\
\|\partial_r(\frac{u^\theta}{r})\|_{L^p} \le C_2\|\nabla^2 u\|_{L^p}.
\end{gathered}$$

Lemma 2.5 ([6, Lemma 4]). *Suppose that u is a sufficiently smooth axisymmetric vector field, then there exists a constant $C > 0$ that is independent of u, such that for all $1 \le p \le \infty$, we have*

$$\|\nabla u^\theta\|_{L^p} + \|\frac{u^\theta}{r}\|_{L^p} \le C\|\nabla u\|_{L^p},$$
$$\|\partial_r(\frac{u^\theta}{r})\|_{L^p} \le C\|\Delta u\|_{L^p}.$$

Lemma 2.6 ([6, Lemma 5]). *Let u be the sufficiently smooth and divergence-free axisymmetric vector field. Then there exist $C_1(p)$, C_2, independent of u, such that for $1 < p < \infty$*

$$\begin{aligned} C_1(p)\|\Delta u\|_{L^p} &\le \|\frac{\omega^r}{r}\|_{L^p} + \|\frac{\omega^\theta}{r}\|_{L^p} + \|\nabla\omega^r\|_{L^p}, + \|\nabla\omega^\theta\|_{L^p} + \|\nabla\omega^z\|_{L^P} \\ &\le C_2\|\Delta u\|_{L^p}. \end{aligned}$$

Lemma 2.7. *Let u be the unique local axisymmetric solution of (1.1), then we have*

$$\begin{aligned} \|\nabla^2 u\|_{L^2}^2 &= \|\nabla\partial_r u^r\|_{L^2}^2 + \|\nabla\frac{u^r}{r}\|_{L^2}^2 + \|\nabla\partial_z u^r\|_{L^2}^2 + \|\partial_z\frac{u^r}{r}\|_{L^2}^2 \\ &\quad + \|\nabla\partial_r u^\theta\|_{L^2}^2 + \|\nabla\frac{u^\theta}{r}\|_{L^2}^2 + \|\nabla\partial_z u^\theta\|_{L^2}^2 + \|\partial_z\frac{u^\theta}{r}\|_{L^2}^2 + \|\nabla^2 u^z\|_{L^2}^2 \\ &\quad + \int_{\mathbb{R}^3}\frac{2}{r^2}\big\{(\partial_r u^r)^2 + (\frac{u^r}{r})^2 - \partial_r u^r\frac{u^r}{r}\big\}\mathrm{d}x \\ &\quad + \int_{\mathbb{R}^3}\frac{2}{r^2}\big\{(\partial_r u^\theta)^2 + (\frac{u^\theta}{r})^2 - \partial_r u^\theta\frac{u^\theta}{r}\big\}\mathrm{d}x \end{aligned}$$

Lemma 2.8 (Proposition 2.5]MZ). *Let u be the sufficiently smooth and divergence-free axisymmetric vector field, and $\nabla \times u = \omega$, then one can obtain that*

$$\frac{u^r}{r} = \Delta^{-1}\partial_z(\frac{\omega^\theta}{r}) - 2\frac{\partial_r}{r}\Delta^{-2}\partial_z(\frac{\omega^\theta}{r}) \tag{2.3}$$

where

$$\frac{\partial_r}{r}f(r,z) = \frac{x_2^2}{r^2}R_{11}f + \frac{x_1^2}{r^2}R_{22}f - 2\frac{x_1x_2}{r^2}R_{12}f \tag{2.4}$$

here $R_{ij} = \Delta^{-1}\partial_i\partial_j$.

Lemma 2.9. *Based on Lemma 2.7, for $1 < p < \infty$, one can deduce easily the following results*

$$\|\hat\nabla\frac{u^r}{r}\|_{L^p} \le C(p)\|\frac{\omega^\theta}{r}\|_{L^p} \tag{2.5}$$

$$\|\hat\nabla\hat\nabla\frac{u^r}{r}\|_{L^p} \le C(p)\|\partial_z(\frac{\omega^\theta}{r})\|_{L^p} \tag{2.6}$$

The Lemma below is a general Sobolev-Hardy inequality, which was deduced by Hui chen et al [4, v]. About more Sobolev-Hardy inequality one can see [1, Theorem 2.1].

Lemma 2.10 ([4, Lemma 2.4]). *We assume that There exist a positive constant $C(s,q^*)$, $q^* \in [2, 2(2-s)]$ with $0 \le s < 2$ and $r = (x_1^2 + x_2^2)^{1/2}$ such that for all*

$u \in \mathscr{D}^{1,q}(\mathbb{R}^3)$, one can obtain that

$$\|\frac{u}{r^{\frac{s}{q^*}}}\|_{L^{q^*}} \le C(q^*,s)\|u\|_{L^2}^{\frac{3-s}{q^*}-\frac{1}{2}}\|\nabla u\|_{L^2}^{\frac{3}{2}-\frac{3-s}{q^*}}.$$

Lemma 2.11 ([4]). *Let u be the unique axisymmetric solution of (1.1), then we have*

$$\begin{aligned}
&\|\frac{\omega^\theta}{r}\|^2_{L^\infty(0,T;L^2)} + \|\frac{\omega^r}{r}\|^2_{L^\infty(0,T;L^2)} + \|\nabla\frac{\omega^\theta}{r}\|^2_{L^2(0,T;L^2)} + \|\nabla\frac{\omega^r}{r}\|^2_{L^2(0,T;L^2)}\\
&\le C\Big\{\|\frac{\omega_0^r}{r}\|_{L^2} + \|\frac{\omega_0^\theta}{r}\|_{L^2}\Big\}\exp\Big\{C\int_0^T \|u^\theta\|_{L^\beta}^{\frac{2}{1-\frac{3}{\beta}}}\,\mathrm{d}t\Big\}.
\end{aligned}$$

where (α,β) satisfies $\frac{2}{\alpha}+\frac{3}{\beta}\le 1$ with $3<\beta\le\infty$.

Proof. This proof can be found in [4]. For reader's convenience, we give it here. Multiplying the ω^r equation of (2.2) by $\frac{\omega^r}{r^2}$ and integrating the resulting equation over $\mathbb{R}^3$ leads to

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\frac{\omega^r}{r}\|^2_{L^2} + \nu\|\hat\nabla\frac{\omega^r}{r}\|^2_{L^2}\\
&= \int_{\mathbb{R}^3}(\omega^r\partial_r + \omega^z\partial_z)\frac{u^r}{r}\frac{\omega^r}{r}\cdot r\mathrm{d}x\\
&= -2\pi\int_{\mathbb{R}}\int_0^\infty \partial_z u^\theta\cdot\partial_r\frac{u^r}{r}\cdot\frac{\omega^r}{r}\mathrm{d}r\mathrm{d}z + 2\pi\int_{\mathbb{R}}\int_0^\infty\frac{\partial_r(ru^\theta)}{r}\cdot\partial_z\frac{u^r}{r}\cdot\frac{\omega^r}{r}\cdot r\mathrm{d}r\mathrm{d}z\\
&= \int_{\mathbb{R}^3} u^\theta(\partial_z\partial_r\frac{u^r}{r})\frac{\omega^r}{r} + u^\theta(\partial_r\frac{u^r}{r})(\partial_z\frac{\omega^r}{r})\mathrm{d}x\\
&\quad -\int_{\mathbb{R}^3} u^\theta\cdot(\partial_r\partial_z\frac{u^r}{r})(\frac{\omega^r}{r})\mathrm{d}x - \int_{\mathbb{R}^3} u^\theta\cdot(\partial_z\frac{u^r}{r})(\partial_r\frac{\omega^r}{r})\mathrm{d}x\\
&= \int_{\mathbb{R}^3} u^\theta\cdot(\partial_r\frac{u^r}{r})\cdot(\partial_z\frac{\omega^r}{r})\mathrm{d}x - \int_{\mathbb{R}^3} u^\theta\cdot(\partial_z\frac{u^r}{r})\cdot(\partial_r\frac{\omega^r}{r})\mathrm{d}x = H_1 + H_2
\end{aligned}$$

Form Lemma 2.9 we obtain

$$\begin{aligned}
|H_1| &\le \int_{\mathbb{R}^3}|u^\theta\cdot(\partial_r\frac{u^r}{r}\partial_z\frac{\omega^r}{r})|\mathrm{d}x\\
&\le \|u^\theta\|_{L^\beta}\|\partial_r\frac{u^r}{r}\|_{L^{\frac{2\beta}{\beta-2}}}\|\partial_z\frac{\omega^r}{r}\|_{L^2}\quad\text{(Hölder inequality)}\\
&\le C\|u^\theta\|_{L^\beta}\|\hat\nabla\partial_r\frac{u^r}{r}\|_{L^2}^{\frac{3}{\beta}}\|\partial_r\frac{u^r}{r}\|_{L^2}^{1-\frac{3}{\beta}}\|\partial_z\frac{\omega^r}{r}\|_{L^2}\\
&\quad\Big(\text{(Lemma 2.10 } s=0,\ q^*=\frac{2\beta}{\beta-2}\Big)\\
&\le C\|u^\theta\|_{L^\beta}\|\hat\nabla\frac{\omega^\theta}{r}\|_{L^2}^{\frac{3}{\beta}}\|\frac{\omega^\theta}{r}\|_{L^2}^{1-\frac{3}{\beta}}\|\hat\nabla\frac{\omega^r}{r}\|_{L^2}\quad\text{(Lemma 2.9)}\\
&\le C_\delta\|u^\theta\|_{L^\beta}^{\frac{2}{1-\frac{3}{\beta}}}\|\frac{\omega^\theta}{r}\|^2_{L^2} + \delta\|\hat\nabla\frac{\omega^r}{r}\|^2_{L^2} + \delta\|\hat\nabla\frac{\omega^\theta}{r}\|^2_{L^2}\quad\text{(Young inequality)}
\end{aligned}$$

The quantity H_2 can be estimated similarly as H_1:

$$|H_2| \le C_\delta\|u^\theta\|_{L^\beta}^{\frac{2}{1-\frac{3}{\beta}}}\|\frac{\omega^\theta}{r}\|^2_{L^2} + \delta\|\hat\nabla\frac{\omega^r}{r}\|^2_{L^2} + \delta\|\hat\nabla\frac{\omega^\theta}{r}\|^2_{L^2}.$$

Thus we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\frac{\omega^r}{r}\|_{L^2}^2+\nu\|\hat{\nabla}\frac{\omega^r}{r}\|_{L^2}^2 \\ &\le C_\delta\|u^\theta\|_{L^\beta}^{\frac{2}{1-\frac{3}{\beta}}}\|\frac{\omega^\theta}{r}\|_{L^2}^2+2\delta\|\hat{\nabla}\frac{\omega^r}{r}\|_{L^2}^2+2\delta\|\hat{\nabla}\frac{\omega^\theta}{r}\|_{L^2}^2 \end{aligned} \tag{2.7}$$

Multiplying ω^θ equation of (2.2) by $\frac{\omega^\theta}{r^2}$, and integrating over $\mathbb{R}^3$, after integrating by parts we obtain that

$$\frac{1}{2}\frac{d}{dt}\|\frac{\omega^\theta}{r}\|_{L^2}^2+\nu\|\hat{\nabla}\frac{\omega^\theta}{r}\|_{L^2}^2=2\int_{\mathbb{R}^3}\frac{u^\theta}{r}\frac{\omega^r}{r}\frac{\omega^\theta}{r}\mathrm{d}x:=H_3$$

$$\begin{aligned} |H_3| &\le \int_{\mathbb{R}^3}|u^\theta\cdot(r^{-\frac12}\frac{\omega^\theta}{r})(r^{-\frac12}\frac{\omega^r}{r})|\mathrm{d}x \\ &\le C\|u^\theta\|_{L^\beta}\|r^{-\frac12}\frac{\omega^\theta}{r}\|_{L^{\frac{2\beta}{\beta-1}}}\|r^{-\frac12}\frac{\omega^r}{r}\|_{L^{\frac{2\beta}{\beta-1}}} \quad \text{(Hölder inequality)} \\ &\le C\|u^\theta\|_{L^\beta}\|\frac{\omega^\theta}{r}\|_{L^2}^{\frac12-\frac{3}{2\beta}}\|\hat{\nabla}\frac{\omega^\theta}{r}\|_{L^2}^{\frac12+\frac{3}{2\beta}}\|\frac{\omega^r}{r}\|_{L^2}^{\frac12-\frac{3}{2\beta}}\|\hat{\nabla}\frac{\omega^r}{r}\|_{L^2}^{\frac12+\frac{3}{2\beta}} \\ &\quad \text{(where we used Lemma 2.10 } s=\frac{\beta}{\beta-1},\ q^*=\frac{2\beta}{\beta-1}) \\ &\le C_\delta\|u^\theta\|_{L^\beta}^{\frac{2}{1-\frac{3}{\beta}}}\|\frac{\omega^r}{r}\|_{L^2}\|\frac{\omega^\theta}{r}\|_{L^2}+\delta\|\hat{\nabla}\frac{\omega^\theta}{r}\|_{L^2}^2+\delta\|\hat{\nabla}\frac{\omega^r}{r}\|_{L^2}^2 \quad \text{(Young ineq.)} \\ &\le C_\delta\|u^\theta\|_{L^\beta}^{\frac{2}{1-\frac{3}{\beta}}}(\|\frac{\omega^r}{r}\|_{L^2}^2+\|\frac{\omega^\theta}{r}\|_{L^2}^2)+\delta\|\hat{\nabla}\frac{\omega^\theta}{r}\|_{L^2}^2+\delta\|\hat{\nabla}\frac{\omega^r}{r}\|_{L^2}^2 \end{aligned}$$

Then we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\frac{\omega^\theta}{r}\|_{L^2}^2+\nu\|\hat{\nabla}\frac{\omega^\theta}{r}\|_{L^2}^2 \\ &\le C_\delta\|u^\theta\|_{L^\beta}^{\frac{2}{1-\frac{3}{\beta}}}(\|\frac{\omega^r}{r}\|_{L^2}^2+\|\frac{\omega^\theta}{r}\|_{L^2}^2)+\delta\|\hat{\nabla}\frac{\omega^\theta}{r}\|_{L^2}^2+\delta\|\hat{\nabla}\frac{\omega^r}{r}\|_{L^2}^2 \end{aligned} \tag{2.8}$$

Combining (2.7) and (2.8) together and let δ be a small enough constant, we have then by Gronwall's inequality that

$$\begin{aligned} &\|\frac{\omega^r}{r}\|_{L^\infty(0,T;L^2)}^2+\|\hat{\nabla}\frac{\omega^r}{r}\|_{L^2(0,T;L^2)}^2+\|\frac{\omega^\theta}{r}\|_{L^\infty(0,T;L^2)}^2+\|\hat{\nabla}\frac{\omega^\theta}{r}\|_{L^2(0,T;L^2)}^2 \\ &\le C(\|\frac{\omega_0^r}{r}\|_{L^2}+\|\frac{\omega_0^\theta}{r}\|_{L^2})\exp\Big\{C\int_0^T\|u^\theta\|_{L^\beta}^{\frac{2}{1-\frac{3}{\beta}}}\mathrm{d}t\Big\}. \end{aligned}$$

□

3. Proof of regularity criteria

Proof. Let u be an axisymmetric smooth solution of the Navier-Stokes equations. Taking curl on the both sides of the Navier-Stokes equations, then we can obtain the equation

$$\partial_t\omega-\Delta\omega+(u\cdot\nabla)\omega=(\omega\cdot\nabla)u.$$

By multiplying ω on the both sides of the above equations, and integrating over $\mathbb{R}^3$, one obtain:

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}|\omega|^2\mathrm{d}x+\int_{\mathbb{R}^3}|\nabla\omega|^2\mathrm{d}x$$

$$
\begin{aligned}
&= \int_{\mathbb{R}^3} (\omega\cdot\nabla)u\cdot\omega \mathrm{d}x \\
&= \int_{\mathbb{R}^3} \omega^r\partial_r u^r\omega^r \mathrm{d}x - \int_{\mathbb{R}^3} \frac{\omega^\theta}{r}u^\theta\omega^r \mathrm{d}x + \int_{\mathbb{R}^3} \omega^z\partial_z u^r\omega^r \mathrm{d}x + \int_{\mathbb{R}^3} \omega^r\partial_r u^\theta\omega^\theta \mathrm{d}x \\
&\quad + \int_{\mathbb{R}^3} \frac{\omega^\theta}{r}u^r\omega^\theta \mathrm{d}x + \int_{\mathbb{R}^3} \omega^z\partial_z u^\theta\omega^\theta \mathrm{d}x + \int_{\mathbb{R}^3} \omega^r\partial_r u^z\omega^\theta \mathrm{d}x + \int_{\mathbb{R}^3} \omega^z\partial_z u^z\omega^z \mathrm{d}x \\
&:= I_1+I_2+I_3+I_4+I_5+I_6+I_7+I_8.
\end{aligned}
$$

We will estimate the terms one by one, for the term I_1, using integration by parts, we have

$$
\begin{aligned}
I_1 &= \int_{\mathbb{R}^3} \omega^r\partial_r u^r\omega^r \mathrm{d}x \\
&= -\int_{\mathbb{R}^3} \partial_z u^\theta\cdot\partial_r u^r\cdot\omega^r \mathrm{d}x \\
&= \int_{\mathbb{R}^3} (u^\theta\cdot\partial_z\partial_r u^r\cdot\omega^r + u^\theta\cdot\partial_r u^r\partial_z\omega^r)\mathrm{d}x
\end{aligned}
$$

then

$$
|I_1| \le \int_{\mathbb{R}^3} |u^\theta\cdot\partial_z\partial_r u^r\cdot\omega^r|\mathrm{d}x + \int_{\mathbb{R}^3} |u^\theta\cdot\partial_r u^r\partial_z\omega^r|\mathrm{d}x \doteq I_1^1 + I_1^2
$$

For the term I_1^1,

$$
\begin{aligned}
I_1^1 &\le \Big\{\int_{R^3} |u^\theta\cdot\omega^r|^2\mathrm{d}x\Big\}^{1/2}\cdot\|\partial_z\partial_r u^r\|_{L^2} \quad \text{(Hölder inequality)} \\
&\le C_\delta\int_{R^3} |u^\theta\cdot\omega^r|^2\mathrm{d}x + \delta\|\partial_z\partial_r u^r\|_{L^2}^2 \quad \text{(Young inequality)} \\
&\le C_\delta\int_{R^3} |u^\theta\cdot\omega^r|^2\mathrm{d}x + \delta\|\nabla\omega\|_{L^2}^2 \quad \text{(By Lemma 2.6)} \\
&\le C_\delta\|u^\theta\|_{L^\beta}^2\|\omega^r\|_{L^{\frac{2\beta}{\beta-2}}}^2 + \delta\|\nabla\omega\|_{L^2}^2 \quad \text{(Hölder inequality)} \\
&\le C_\delta\|u^\theta\|_{L^\beta}^2\left\{\|\omega\|_{L^2}^\theta\|\nabla\omega\|_{L^2}^{1-\theta}\right\}^2 + \delta\|\nabla\omega\|_{L^2}^2 \\
&\qquad \text{(Gagliardo-Nirenberg inequlity and } \theta = 1-\frac{3}{\beta}) \\
&\le C_\delta\|u^\theta\|_{L^\beta}^{\frac{2}{\theta}}\|\omega\|_{L^2}^2 + 2\delta\|\nabla\omega\|_{L^2}^2 \quad \text{(Young inequality)}
\end{aligned}
$$

For the term of I_1^2,

$$
\begin{aligned}
I_1^2 &= \int_{\mathbb{R}^3} |u^\theta\cdot\partial_r u^r\cdot\partial_z\omega^r|\mathrm{d}x \\
&\le \Big\{\int_{\mathbb{R}^3} |u^\theta\cdot\partial_r u^r|^2\mathrm{d}x\Big\}^{1/2}\cdot\|\partial_z\omega^r\|_{L^2} \quad \text{(Hölder inequality)} \\
&\le C_\delta\Big\{\int_{\mathbb{R}^3} |u^\theta\cdot\partial_r u^r|^2\mathrm{d}x\Big\} + \delta\|\partial_z\omega^r\|_{L^2}^2 \quad \text{(Young inequality)} \\
&\le C_\delta\|u^\theta\|_{L^\beta}^2\|\partial_r u^r\|_{L^{\frac{2\beta}{\beta-2}}}^2 + \delta\|\partial_z\omega^r\|_{L^2}^2 \quad \text{(Hölder inequality)} \\
&\le C_\delta\|u^\theta\|_{L^\beta}^2\Big\{\|\omega\|_{L^2}^\theta\|\nabla\omega\|_{L^2}^{1-\theta}\Big\}^{1/2} + \delta\|\partial_z\omega^r\|_{L^2}^2
\end{aligned}
$$

$$(\text{Gagliardo-Nirenberg inequlity and } \theta = 1 - \frac{3}{\beta})$$
$$\le C_\delta \|u^\theta\|_{L^\beta}^{\frac{2}{\theta}} \|\omega\|_{L^2}^2 + 2\delta \|\nabla\omega\|_{L^2}^2 \quad (\text{Young inequality and Lemma 2.6})$$

Then we obtain

$$|I_1| \le C_\delta \|u^\theta\|_{L^\beta}^{\frac{2}{1-\frac{3}{\beta}}} \|\omega\|_{L^2}^2 + 2\delta \|\nabla\omega\|_{L^2}^2$$

Similarly, one has

$$|I_2|, |I_3|, |I_4|, |I_6|, |I_7| \le C_\delta \|u^\theta\|_{L^\beta}^{\frac{2}{1-\frac{3}{\beta}}} \|\omega\|_{L^2}^2 + 2\delta \|\nabla\omega\|_{L^2}^2.$$

One can see that I_5 is a difficult term, but we can obtain a term that can be estimated by Lemma 2.10,

$$\begin{aligned}
|I_5| &\le \int_{\mathbb{R}^3} |\frac{\omega^\theta}{r}\omega^\theta u^r| \mathrm{d}x \le C\|u^r\|_{L^2} (\int_{\mathbb{R}^3} \frac{(\omega^\theta)^4}{r^2} \mathrm{d}x)^{1/2} \quad (\text{Hölder inequality}) \\
&\le C\|\frac{\omega^\theta}{r}\|_{L^6} \|\omega^\theta\|_{L^3} \le C\|\frac{\omega^\theta}{r}\|_{L^6} \|\omega^\theta\|_{L^2}^{1/2} \|\nabla\omega^\theta\|_{L^2}^{1/2} \quad (\text{Gagliardo-Nirenberg ineq.}) \\
&\le C_\delta \|\nabla \frac{\omega^\theta}{r}\|_{L^2}^{\frac{4}{3}} \|\omega^\theta\|_{L^2}^{\frac{2}{3}} + \delta \|\nabla\omega^\theta\|_{L^2}^2 \quad (\text{Young inequality }) \\
&\le \|\nabla \frac{\omega^\theta}{r}\|_{L^2}^2 + C_\delta \|\omega^\theta\|_{L^2}^2 + \delta \|\nabla\omega^\theta\|_{L^2}^2. \quad (\text{Young inequality }) \\
&\le \|\nabla \frac{\omega^\theta}{r}\|_{L^2}^2 + C_\delta \|\omega\|_{L^2}^2 + \delta \|\nabla\omega\|_{L^2}^2. \quad (\text{Lemma 2.6})
\end{aligned}$$

Using integration by parts, one has

$$\begin{aligned}
I_8 &= \int_{\mathbb{R}^3} \omega^z \partial_z u^z \omega^z \mathrm{d}x = \int_{\mathbb{R}^3} (\partial_r u^\theta + \frac{u^\theta}{r}) \partial_z u^z \omega^r \mathrm{d}x \\
&= \int_{\mathbb{R}^3} \partial_r u^\theta \partial_z \cdot u^z \cdot \omega^r \mathrm{d}x + \int_{\mathbb{R}^3} \frac{u^\theta}{r} \cdot \partial_z u^z \cdot \omega^r \mathrm{d}x \\
&= -\int_{\mathbb{R}^3} u^\theta \cdot \partial_r \partial_z u^z \cdot \omega^r \mathrm{d}x - \int_{\mathbb{R}^3} u^\theta \cdot \partial_z u^z \cdot \partial_r \omega^r \mathrm{d}x + \int_{\mathbb{R}^3} u^\theta \cdot \partial_z u^z \cdot \frac{\omega^r}{r} \mathrm{d}x \\
&:= I_8^1 + I_8^2 + I_8^3
\end{aligned}$$

For the term I_8^1,

$$\begin{aligned}
|I_8^1| &\le \int_{\mathbb{R}^3} |u^\theta \cdot \partial_r \partial_z \cdot \omega^r| \mathrm{d}x \le \Big\{ \int_{\mathbb{R}^3} |u^\theta \cdot \omega^r|^2 \mathrm{d}x \Big\}^{1/2} \|\partial_r \partial_z u^z\|_{L^2} \quad (\text{Hölder ineq.}) \\
&\le C_\delta \int_{\mathbb{R}^3} |u^\theta \cdot \omega^r|^2 \mathrm{d}x + \delta \|\partial_r \partial_z u^z\|_{L^2} \quad (\text{Young inequality }) \\
&\le C_\delta \|u^\theta\|_{L^\beta}^2 \|\omega\|_{\frac{2\beta}{\beta-2}}^2 + \delta \|\nabla\omega\|_{L^2}^2 \quad (\text{Hölder inequality and Lemma 2.7}) \\
&\le C_\delta \|u^\theta\|_{L^\beta}^2 \left\{ \|\omega\|_{L^2}^\theta \|\nabla\omega\|_{L^2}^{1-\theta} \right\}^2 + \delta \|\nabla\omega\|_{L^2}^2 \quad (\text{Gagliardo-Nirenberg inequlity }) \\
&\le C_\delta \|u^\theta\|_{L^\beta}^{\frac{2}{\theta}} \|\omega\|_{L^2}^2 + 2\delta \|\nabla\omega\|_{L^2}^2 \quad (\text{Young inequality })
\end{aligned}$$

The quantities $|I_8^2|$, $|I_8^3|$ can be estimated similarly to $|I_8^1|$. Putting together the above estimates, and taking δ small enough, then one have

$$\frac{d}{dt}\|\omega\|_{L^2}^2 + \|\nabla\omega\|_{L^2}^2 \le C\Big\{1 + \|u^\theta\|_{L^\beta}^{\frac{2}{1-\frac{3}{\beta}}}\Big\}\|\omega\|_{L^2}^2 + C\|\nabla\frac{\omega^\theta}{r}\|_{L^2}^2$$

Using Gronwall's inequality, we have

$$\begin{aligned}
&\|\omega\|_{L^\infty((0,T);L^2)}^2 + \|\nabla\omega\|_{L^2((0,T);L^2)}^2 \\
&\le C\big\{\|\omega_0\|_{L^2}^2 + \|\nabla\frac{\omega^\theta}{r}\|_{L^2(0,T;L^2)}^2\big\}\exp\Big\{C + C\int_0^T \|u^\theta\|_{L^\beta}^{\frac{2\beta}{\beta-3}}\mathrm{d}t\Big\}.
\end{aligned}$$

Therefore, combining with Lemma 2.11 then we completes the proof of Theorem 1.1. □

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