Electronic Journal of Differential Equations, Vol. 2015 (2015), No. 254, pp. 1–17.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

TRAVELING CURVED FRONTS OF BISTABLE REACTION-DIFFUSION EQUATIONS WITH DELAY

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Abstract. This article concerns the existence and stability of traveling curved fronts for bistable reaction-diffusion equations with delay in two-dimensional space. Using the comparison principle and establishing super- and sub- solutions, we prove the existence of traveling curved fronts. We also show that such traveling curved front is unique and stable.

1. Introduction

Traveling wave solutions of reaction-diffusion equations with delay have been widely studied because of its significant in physics and biology, for example, see [2, 3, 8, 9, 11, 12, 13, 14, 19, 21, 26, 27, 28, 30, 31, 32] and references therein. Recently, the nonplanar traveling wave solutions of reaction-diffusion equations in high dimensional spaces have attracted a lot of attention, see [1, 4, 5, 6, 7, 15, 16, 22, 23, 24] for the existence and stability results of nonplanar traveling wave solutions of autonomous reaction-diffusion equation and see [20, 29] for non-autonomous case. Moreover, we refer to [17, 25] for the existence of nonplanar traveling fronts to reaction diffusion system. Compared to the planar traveling wave, the profiles of nonplanar traveling wave solutions become more complicated and have various new types in multidimensional space. Since many practical problems from physics, chemistry and ecology are high-dimensional problems, the nonplanar traveling wave solutions have important applications to describe multi-dimensional chemical waves and ecology phenomena in multidimensional space, see [1, 6, 7, 15, 17, 22, 25] and so on. It is then natural to ask whether such traveling curved fronts of reaction-diffusion equations with delay exist and are stable. Resolving this issue is the main contribution of our current paper.

More precisely, in this paper, we are interested in two-dimensional V-shaped traveling fronts of the following bistable reaction diffusion equations with time delayed

$$\frac{\partial u}{\partial t} = u_{xx} + u_{yy} + f(u(x,y,t), u(x,y,t-\tau)), \quad (x,y) \in \mathbb{R}^2,\ t > 0, \tag{1.1}$$

2010 *Mathematics Subject Classification.* 35C07, 35R10, 35K57.
Key words and phrases. Existence; stability; traveling curved fronts; bistable; reaction-diffusion equation; delay.

Submitted July 14, 2015. Published September 29, 2015.

where $\tau > 0$ is a given constant and $f(u(x,y,t), u(x,y,t-\tau))$ satisfies the following structure hypotheses:

- (H1) $f \in C^1(I^2, \mathbb{R})$ for some open interval $I \subset \mathbb{R}$ with $[0,1] \subset I$ and $\partial_2 f(u,v) \geq 0$ for $(u,v) \in I^2$;
- (H2) $f(0,0) = f(1,1) = 0$, $\partial_1 f(0,0) + \partial_2 f(0,0) < 0$ and $\partial_1 f(1,1) + \partial_2 f(1,1) < 0$.
- (H3) There exists a monotone traveling wave solution $U(xe_1 + ye_2 + ct)$ of (1.1) connecting 0 and 1, that is $U'(\xi) > 0$, $U(-\infty) = 0$, $U(\infty) = 1$ and $U(\xi)$ satisfy $U'' - cU' + f(U(\xi), U(\xi - c\tau)) = 0$. Furthermore, assume that wave speed $c > 0$.

Note that the assumption (H1) and (H2) are standard. Assumption (H3) implies that a traveling wave solution of (1.1) with the form $U(xe_1 + ye_2 + ct)$ is a planar traveling wave solution in $\mathbb{R}^2$. A typical example of f satisfying (H1), (H2) and (H3) is the typical Huxley nonlinearity

$$f(u,v) = \begin{cases} u(1-u)(v-a), & \text{for } 0 \leq u \leq 1, v \in \mathbb{R}, \\ u(1-u)(u-a), & u \in (-\infty, 0) \cup (1, +\infty) \end{cases}$$

with $0 < a < 1$. Let $\widehat{f} : I^2 \to \mathbb{R}$ be a smooth extension of $f : [0,1]^2 \to \mathbb{R}$, then $\widehat{f}$ satisfies (H1) and (H2), see Smith and Zhao [21, Remark 3.1]. Following from Schaaf [19], for Huxley nonlinearity, there exists a unique function $U(\xi) : \mathbb{R} \to \mathbb{R}$ and a unique constant $c \in \mathbb{R}$ such that

$$\begin{gathered} U'' - cU' + f(U(\xi), U(\xi - c\tau)) = 0, \quad \forall \xi \in \mathbb{R}, \\ U(+\infty) = 1, \quad U(-\infty) = 0, \\ U'(\xi) > 0 \quad \text{in } \mathbb{R}. \end{gathered} \tag{1.2}$$

As usual c is called the wave speed and U is the wave profile of front. In particular, equation (1.1) with the typical Huxley nonlinearity $f(u,v)$ has an increasing traveling wave solution for wave speed $c > 0$ if $a \in (0, \frac{1}{2})$. Smith and Zhao [21] proved the global asymptotic stability, Lyapunov stability and uniqueness of traveling wave solutions of (1.2) under the assumption (H1), (H2) and (H3). It is known from Wang et al.[28] (see also Schaaf[19]) that when (H1)-(H3) hold, there exist positive constant β_1 and C_1 such that

$$\max\{U(-\xi), |U(\xi) - 1|, |U'(\pm\xi)|, |U''(\pm\xi)|\} \leq C_1 e^{-\beta_1 \xi}, \quad \forall \xi \geq 0. \tag{1.3}$$

Generally the curvature effect is excepted to accelerate the speed. Assume $c > 0$. Fix $s > c$ and we try to find two-dimensional V-shaped traveling fronts with wave speed s to (1.1). Without loss of generality, we assume that the solutions travel towards the y-direction. Take

$$u(x,y,t) = w(x,z,t), \quad z = y + st,$$

we have

$$\begin{gathered} \frac{\partial w}{\partial t} = w_{xx} + w_{zz} - s\frac{\partial w}{\partial z} + f(w(x,z,t), w(x, z - s\tau, t - \tau)), \\ w(x,z,r) = \phi(x,z,r), \quad (x,z) \in \mathbb{R}^2, \ r \in [-\tau, 0]. \end{gathered} \tag{1.4}$$

We denote the solution of (1.4) with $w(x,z,r) = \phi(x,z,r)$ by $w(x,z,t;\phi)$.

The purpose of the current paper is to seek for $V(x,z)$ with

$$\mathcal{L}[V] := -V_{xx} - V_{zz} + s\frac{\partial V}{\partial z} - f(V(x,z), V(x, z - s\tau)) = 0 \quad \text{for } (x,z) \in \mathbb{R}^2. \tag{1.5}$$

Let $m_* = \sqrt{s^2 - c^2}/c$, we know that $U\big(\frac{c}{s}(z - m_* x)\big)$ and $U\big(\frac{c}{s}(z + m_* x)\big)$ are two planar traveling waves of (1.1). Then the function

$$v^-(x,z) := \max\big\{U\big(\frac{c}{s}(z - m_* x)\big), U\big(\frac{c}{s}(z + m_* x)\big)\big\} = U\big(\frac{c}{s}(z + m_*|x|)\big)$$

is a subsolution of (1.5). In particular, $v_z^-(x,z) > 0$.

The following theorem is the main result of this article.

Theorem 1.1. *Assume that* (H1)-(H3) *hold. Then for each $s > c$, there exists a solution $u(x,y,t) = V(x, y + st)$ of* (1.1) *with $V(x,z) > v^-(x,z)$ and*

$$\lim_{R\to\infty} \sup_{x^2+z^2\ge R^2} |V(x,z) - v^-(x,z)| = 0.$$

Let $\phi(x,z,r)$ with $\phi(x,z,r) \ge v^-(x,z)$ for $(x,z) \in \mathbb{R}^2$ and $r \in [-\tau, 0]$ and

$$\lim_{R\to\infty} \sup_{x^2+z^2\ge R^2, r\in[-\tau,0]} |\phi(x,z,r) - v^-(x,z)| = 0.$$

Then the solution $w(x,z,t;\phi)$ of (1.4) *satisfies*

$$\lim_{t\to\infty} \|w(x,z,t;\phi) - V(x,z)\|_{L^\infty(\mathbb{R}^2)} = 0. \tag{1.6}$$

In the following, we call $V(x, y + st)$ a traveling curved front of (1.1). Note that for every $\tau \ge 0$ there is exactly one c and one unique (up to translation) planar traveling wave solution of (1.2) and the sign of the wave speed c can be obtained by the sign of $\int_0^1 f_0(r)dr$, where $f_0(r) = f(r,r)$, see [19, Theorem 3.13 and 3.16]. Because of the curvature effect, the speed of traveling curved front must be greater than c. Following Theorem 1.1, for every given $s > c$, there exists a traveling curved front $V(x,z)$ of (1.1) and it is unique and stable for the initial value $\phi(x,z,r)$ with $\phi(x,z,r) \ge v^-(x,z)$ in $\mathbb{R}^2$ and $r \in [-\tau, 0]$. However, as the effect of time delay τ, it is difficult to prove that the traveling curved front is stable for initial value $\phi(x,z,r) \le v^-(x,z)$ in $\mathbb{R}^2$ and $r \in [-\tau, 0]$ as that in [16, 29]. It remains as an interested open problem. Furthermore, for (1.1) with nonlocal decay

$$\frac{\partial u}{\partial t} = u_{xx} + u_{yy} + g\Big(u(x,y,t), \big(h * S(u)\big)(x,y,t)\Big), \quad \text{for } (x,y) \in \mathbb{R}^2,\ t > 0, \tag{1.7}$$

Wang et al [27] established the existence of traveling wave solution for (1.7). There are some difficulties to construct the supersolution of (1.7) and the existence of traveling curved fronts of (1.7) remains as an open problem. It is worth to point out that, for some special kernels, the traveling curved fronts of (1.7) can be obtained, see [25, Example 2] for the details proof of existence of traveling curved fronts to a Lotka-Volterra competition-diffusion system with spatio-temporal delays.

This article is organized as follows. In Section 2, we make some preparations which is needed in the sequel. In Section 3, we establish a super-solution and prove the existence of traveling curved fronts of (1.1). In Section 4, we prove that the traveling curved front is unique and stable.

2. Preliminaries

In this section, we establish a comparison theorem for equation (1.1) on $\mathbb{R}^2$. Let $f_0(\cdot,\cdot) : I \to \mathbb{R}$ be defined by $f_0(u) = f(u,u)$, $u \in I$. By the continuity of f_0 and assumption (H2), it then easily follows that there exist δ_0, $a^-, a^+ \in (0,1)$ with $[-\delta_0, 1+\delta_0] \subset I$ and $a^- \le a^+$ such that $f_0(\cdot) : [-\delta_0, 1+\delta_0] \to \mathbb{R}$ satisfies

$$f_0(0) = f_0(a^-) = f_0(a^+) = f_0(1) = 0,$$

$$f_0(u) > 0 \quad \text{for} \quad u \in (-\delta_0, 0) \cup (a^+, 1),$$
$$f_0(u) < 0 \quad \text{for} \quad u \in (0, a^-) \cup (1, 1+\delta_0).$$

Let $X = BUC(\mathbb{R}^2, \mathbb{R})$ be the Banach space of all bounded and uniformly continuous functions from $\mathbb{R}^2$ to $\mathbb{R}$ with the usual supremum norm. Let $X^+ = \{\phi \in X : \phi(x,y) \geq 0,\ (x,y) \in \mathbb{R}^2\}$. Let $\mathcal{C} = \mathcal{C}([-\tau, 0], X)$ be the Banach space of continuous functions from $[-\tau, 0]$ into X with the supremum norm and $\mathcal{C}^+ = \{\phi \in \mathcal{C} : \phi(s) \in X^+, s \in [-\tau, 0]\}$. Then $\mathcal{C}^+$ is a positive cone of $\mathcal{C}$. We identify an element $\phi \in \mathcal{C}$ as a function from $\mathbb{R}^2 \times [-\tau, 0]$ into $\mathbb{R}$ defined by $\phi(x,y,s) = \phi(s)(x,y)$. For any continuous function $\omega(\cdot) : [-\tau, b) \to X$, $b > 0$, we define $\omega_t \in \mathcal{C}$, $t \in [0,b)$, by $\omega_t(s) = \omega(t+s)$, $s \in [-\tau, 0]$. For any $\phi \in [-\delta_0, 1+\delta_0]_{\mathcal{C}} = \big\{\phi \in \mathcal{C} : \phi(x,y,s) \in [0,1], (x,y) \in \mathbb{R}^2, s \in [-\tau, 0]\big\}$, define

$$F(\phi)(x,y) = f(\phi(x,y,0), \phi(x,y,-\tau)), \quad (x,y) \in \mathbb{R}^2.$$

By the global Lipschitz continuity of $f(\cdot,\cdot)$ on $[-\delta_0, 1+\delta_0]^2$, we can verify that $F(\phi) \in X$ and $F : [-\delta_0, 1+\delta_0]_{\mathcal{C}} \to X$ is globally Lipschitz continuous. In addition, it follows from assumption (H1) that F is quasi-monotone on $[-\delta_0, 1+\delta_0]_{\mathcal{C}}$ in the sense that

$$\lim_{h \to 0^+} \frac{1}{h} \operatorname{dist}\big(\psi(0) - \phi(0) + h[F(\psi) - F(\phi)]; X^+\big) = 0$$

for all $\psi, \phi \in [-\delta_0, 1+\delta_0]_{\mathcal{C}}$ with $\psi \geq \phi$. Let

$$T(t)\phi(x,y) = \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}^2} \exp\Big(-\frac{(x-x_1)^2 + (y-y_1)^2}{4t}\Big)\phi(x_1, y_1)\,dx_1\,dy_1,$$

for $(x,y) \in \mathbb{R}^2$, $t > 0$, and $\phi(\cdot,\cdot) \in X$.

Definition 2.1. A continuous function $v : [-\tau, b) \to X$, $b > 0$, is called a mild supersolution (subsolution) of (1.4) on $[0,b)$ if

$$v(t) \geq (\leq) T(t-s)v(s) + \int_s^t T(t-r)F(v_r)dr \tag{2.1}$$

for all $0 \leq s \leq t < b$. If v is both supersolution and subsolution on $[0,b)$, then we call it a mild solution of (1.4).

Remark 2.2. *Assume that there is a $v(x,z,t) \in \mathrm{BUC}(\mathbb{R}^2 \times [-\tau, b), \mathbb{R})$, $b > 0$, such that $v(x,z,t)$ is C^2 in $(x,z) \in \mathbb{R}^2$ and C^1 in $t \in [0,b]$, and*

$$\frac{\partial v}{\partial t} \geq (\leq) v_{xx} + v_{zz} - s\frac{\partial v}{\partial z} + f(v(x,z,t), v(x,z-s\tau, t-\tau)), \quad (x,z) \in \mathbb{R}^2,\ t \in (0,b).$$

Then by the positivity of the linear semigroup $T(t) : X \to X$ implies (2.1) holds. In this case v is called a smooth supersolution (subsolution) of (1.4) on $(0,b)$.

Similar to [21, Theorem 2.2] and [27, Theorem 2.3], we have the following existence and comparison theorem, here we omit the details of the proof.

Theorem 2.3. *Assume that (H1) and (H2) hold. Then for any $\phi \in [-\delta_0, 1+\delta_0]_{\mathcal{C}}$, (1.4) has a unique mild solution $w(x,z,t;\phi)$ on $[0,\infty)$ and $w(x,z,t;\phi)$ is a classical solution of (1.4) for $(x,z,t) \in \mathbb{R}^2 \times [\tau, +\infty)$. Moreover, suppose that $w^+(x,z,t)$ and $w^-(x,z,t)$ are supersolution and subsolution of (1.4) on $\mathbb{R}^2 \times \mathbb{R}^+$, respectively, and satisfy $w^\pm(x,z,t) \in [-\delta_0, 1+\delta_0]$ for $(x,z) \in \mathbb{R}^2$, $t \in [-\tau, +\infty)$ and $w^-(x,z,s) \leq w^+(x,z,s)$ for any $(x,z) \in \mathbb{R}^2$ and $s \in [-\tau, 0]$. Then one has $w^-(x,z,t) \leq w^+(x,z,t)$ for $(x,z) \in \mathbb{R}^2$ and $t \geq 0$.*

3. Existence of traveling curved fronts

In this section, we establish the existence of traveling curved fronts of (1.1) by constructing a suitable supersolution $v^+(x,z)$ with $v^+(x,z) \geq v^-(x,z)$ in $\mathbb{R}^2$.

For any $s > c$, it follows from [15] and the reference therein that there exists a unique function $\varphi(x,s)$ with asymptotic lines $y = m_*|x|$ satisfying

$$s = \frac{\varphi_{xx}}{1+\varphi_x^2} + c\sqrt{1+\varphi_x^2}. \tag{3.1}$$

About the shape of the function φ, the readers can refer to [15, Fig. 3]. Following from [15, Lemma 2.1], there exist positive constants β_2, C_i $(i = 2,3,4)$ and $\mu_\pm$ such that

$$\max\{|\varphi''(x)|, |\varphi'''(x)|\} \leq C_2 \operatorname{sech}(\beta_2 x), \tag{3.2}$$

$$C_3 \operatorname{sech}(\beta_2 x) \leq \frac{s}{\sqrt{1+\varphi_x^2}} - c \leq C_4 \operatorname{sech}(\beta_2 x), \tag{3.3}$$

$$m_*|x| \leq \varphi(x), \tag{3.4}$$

$$\mu_- \leq \mu(x) \leq \mu_+ \tag{3.5}$$

for all $x \in \mathbb{R}$, where

$$\mu(x) = \frac{s(\varphi(x) - m_*|x|)}{s - c\sqrt{1+\varphi_x^2}}, \quad \beta_2 = \frac{s\sqrt{s^2-c^2}}{c}.$$

The following lemma constructs a supersolution of (1.5).

Lemma 3.1. *There exist a positive constant ε_0^+ and a positive function $\alpha_0^+(\varepsilon)$ such that, for $0 < \varepsilon < \varepsilon_0^+$ and $0 < \alpha \leq \alpha_0^+(\varepsilon)$,*

$$v^+(x,z;\varepsilon,\alpha) = U\Big(\frac{z + \varphi(\alpha x)/\alpha}{\sqrt{1+\varphi'^2(\alpha x)}}\Big) + \varepsilon \operatorname{sech}(\beta_2 \alpha x)$$

is a supersolution of (1.5) with

$$\lim_{R\to\infty} \sup_{x^2+z^2>R^2} |v^+(x,z;\varepsilon,\alpha) - v^-(x,z)| \leq 2\varepsilon, \tag{3.6}$$

$$v^-(x,z) < v^+(x,z;\varepsilon,\alpha), \quad \textit{for } (x,z) \in \mathbb{R}^2, \tag{3.7}$$

$$v_z^+(x,z;\varepsilon,\alpha) > 0, \quad \textit{for } (x,z) \in \mathbb{R}^2. \tag{3.8}$$

Proof. Let

$$\xi = \alpha x, \quad \zeta = \frac{z + \varphi(\alpha x)/\alpha}{\sqrt{1+\varphi'^2(\alpha x)}}, \quad \sigma(\xi) = \varepsilon \operatorname{sech}(\beta_2 \xi).$$

Then a direct calculation yields (see [15])

$$\zeta_x = -\frac{\alpha\varphi'\varphi''}{1+\varphi'^2}\zeta + \frac{\varphi'}{\sqrt{1+\varphi'^2}},$$

$$\zeta_{xx} = -\frac{\alpha\varphi''^2 + \alpha^2\varphi'\varphi'''}{1+\varphi'^2}\zeta + \frac{3\alpha^2\varphi'^2\varphi''^2}{(1+\varphi'^2)^2}\zeta - \frac{\alpha(\varphi'^2-1)\varphi''}{(1+\varphi'^2)^{3/2}}.$$

Then $v^+(x,z;\varepsilon,\alpha) = U(\zeta) + \sigma(\xi)$. Since $U(\zeta)$ is a solution of (1.2), we have

$$U_{\zeta\zeta} - cU_\zeta + f\left(U(\zeta), U(\zeta - c\tau)\right) = 0 \quad \text{for } \zeta \in \mathbb{R}.$$

By assumptions (H1), (H3) and (3.3), we have that

$$f\left(U(\zeta)+\sigma(\xi),U(\zeta-c\tau)+\sigma(\xi)\right)\geq f\Big(U(\zeta)+\sigma(\xi),U\Big(\zeta-\frac{s\tau}{\sqrt{1+\varphi'^2}}\Big)+\sigma(\xi)\Big).$$

Direct calculations yield

$$\begin{aligned}
\mathcal{L}[v^+]&=-v^+_{xx}-v^+_{zz}+s\frac{\partial v^+}{\partial z}-f\left(v^+(x,z),v^+(x,z-s\tau)\right)\\
&=U_{\zeta\zeta}\Big(-\zeta_x^2-\frac{1}{1+\varphi'^2(\xi)}\Big)-U_\zeta\zeta_{xx}+\frac{sU_\zeta}{\sqrt{1+\varphi'^2(\xi)}}-\alpha^2\sigma''(\xi)\\
&\quad-f\Big(U(\zeta)+\sigma(\xi),U\Big(\zeta-\frac{s\tau}{\sqrt{1+\varphi'^2(\xi)}}\Big)+\sigma(\xi)\Big)\\
&\quad+f\left(U(\zeta),U\left(\zeta-c\tau\right)\right)+U_{\zeta\zeta}-cU_\zeta\\
&\geq U_{\zeta\zeta}\Big(1-\zeta_x^2-\frac{1}{1+\varphi'^2(\xi)}\Big)-U_\zeta\zeta_{xx}+\Big(\frac{s}{\sqrt{1+\varphi'^2(\xi)}}-c\Big)U_\zeta\\
&\quad-f\left(U(\zeta)+\sigma(\xi),U(\zeta-c\tau)+\sigma(\xi)\right)\\
&\quad+f\left(U(\zeta),U(\zeta-c\tau)\right)-\alpha^2\sigma''(\xi)\\
&=I_1+I_2+I_3+I_4.
\end{aligned}$$

Let

$$\begin{gathered}
I_1:=U_{\zeta\zeta}\Big(1-\zeta_x^2-\frac{1}{1+\varphi'^2(\xi)}\Big)=-\alpha U_{\zeta\zeta}\Big[\Big(\frac{\varphi'\varphi''}{1+\varphi'^2}\Big)^2\alpha\zeta^2-\frac{2\varphi'^2\varphi'}{(1+\varphi'^2)^{3/2}}\zeta\Big],\\
I_2:=-U_\zeta\zeta_{xx}=\alpha\Big[\frac{\varphi''^2+\varphi'\varphi''}{1+\varphi'^2}\alpha\zeta-\frac{3\varphi'^2\varphi''^2}{(1+\varphi'^2)^3}\alpha\zeta+\frac{(\varphi'^2-1)\varphi''}{(1+\varphi'^2)^{3/2}}\zeta\Big]U_\zeta,\\
I_3:=\Big(\frac{s}{\sqrt{1+\varphi'^2(\xi)}}-c\Big)U_\zeta
\end{gathered}$$

and

$$\begin{aligned}
I_4:&=f\left(U(\zeta),U(\zeta-c\tau)\right)-f\left(U(\zeta)+\sigma(\xi),U(\zeta-c\tau)+\sigma(\xi)\right)-\alpha^2\sigma''(\xi)\\
&=-\Big\{\int_0^1[\partial_1 f\left(U(\zeta)+\eta\sigma(\xi),U(\zeta-c\tau)+\eta\sigma(\xi)\right)\\
&\quad+\partial_2 f\left(U(\zeta)+\eta\sigma(\xi),U(\zeta-c\tau)+\eta\sigma(\xi)\right)]d\eta\Big\}\sigma(\xi)-\alpha^2\sigma''(\xi).
\end{aligned}$$

Then by (1.3) and (3.2)-(3.5), we have

$$|I_1|\leq C_5\alpha\operatorname{sech}(\beta_2\xi),\quad |I_2|\leq C_6\alpha\operatorname{sech}(\beta_2\xi),\quad I_3\geq C_3U_\zeta\operatorname{sech}(\beta_2\xi)>0$$

for $0<\alpha\leq 1$ and $0<\varepsilon<1$, where C_3,C_5 and C_6 are positive constants independent of α and ε. By assumption (H2), there are

$$\begin{aligned}
\lim_{(u,v,\beta)\to(0,0,0)}(\partial_1 f(u,v)+\partial_2 f(u,v)+\beta)=\partial_1 f(0,0)+\partial_2 f(0,0)<0,\\
\lim_{(u,v,\beta)\to(1,1,0)}(\partial_1 f(u,v)+\partial_2 f(u,v)+\beta)=\partial_1 f(1,1)+\partial_2 f(1,1)<0.
\end{aligned}$$

Then we can fix $\beta_0>0$ and choose $\delta^*\in(0,\delta_0)$ such that $[-\delta^*,1+\delta^*]\subset I$ and

$$\partial_1 f(u,v)+\partial_2 f(u,v)<-\beta_0,\quad \text{for } (u,v)\in[-\delta^*,\delta^*]^2\cup[1-\delta^*,1+\delta^*]^2. \tag{3.9}$$

Since $\lim_{\xi\to\infty} U(\xi) = 0$ and $\lim_{\xi\to-\infty} U(\xi) = 1$, there exists $M_0 = M_0(U, \beta_0, \delta^*) > 0$ such that

$$U(\zeta) \geq 1 - \frac{\delta^*}{2} \quad \text{for } \zeta \geq M_0 - c\tau, \tag{3.10}$$

$$U(\zeta) \leq \frac{\delta^*}{2} \quad \text{for } \zeta \leq -M_0 + c\tau. \tag{3.11}$$

Take

$$\begin{aligned} c_1 = \max\Big\{|\partial_1 f(u,v)| : (u,v) \in [-\delta^*, 1+\delta^*]^2\Big\} \\ + \max\Big\{|\partial_2 f(u,v)| : (u,v) \in [-\delta^*, 1+\delta^*]^2\Big\}. \end{aligned}$$

Next, we distinguish among three cases to consider I_4 and prove $\mathcal{L}[v^+] \geq 0$.

Case (i): $|\zeta| \leq M_0$. For $0 < \varepsilon \leq \delta^*/2$, we have $\sigma(\xi) = \varepsilon \operatorname{sech}(\beta_2\xi) \leq \frac{\delta^*}{2}$. By the choice of M_0 and c_1,

$$\begin{aligned} &\Big| -\Big\{ \int_0^1 \partial_1 f\left(U(\zeta) + \eta\sigma(\xi), U(\zeta - c\tau) + \eta\sigma(\xi)\right) \\ &+ \partial_2 f\left(U(\zeta) + \eta\sigma(\xi), U(\zeta - c\tau) + \eta\sigma(\xi)\right) d\eta \Big\} \Big| \leq c_1. \end{aligned}$$

Note that there exists a constant $C_7 > 0$ such that $\sigma''(\xi) \leq C_7\sigma(\xi)$ for $\xi \in \mathbb{R}$. Then

$$|I_4| \leq \left(c_1 + \alpha^2 C_7\right) \varepsilon \operatorname{sech}(\beta_2\xi).$$

Consequently, letting $m_0 = m_0(U, \beta_0, \delta^*) = \min\{U'(\zeta) : |\zeta| \leq M_0\} > 0$, we have

$$\begin{aligned} \mathcal{L}[v^+] &= I_1 + I_2 + I_3 + I_4 \\ &\geq [-C_5\alpha - C_6\alpha + C_3 m_0 - c_1\varepsilon - \alpha C_7] \operatorname{sech}(\beta_2\xi) \geq 0 \end{aligned}$$

provided that ε and α satisfy

$$0 < \varepsilon \leq \min\big\{1, \frac{\delta^*}{2}, \frac{C_3 m_0}{2c_1}\big\} \quad \text{and} \quad \alpha \leq \min\big\{1, \frac{C_3 m_0}{2(C_5 + C_6 + C_7)}\big\}.$$

Case (ii): $\zeta \geq M_0$. Clearly by (3.10), $1 \geq U(\zeta) \geq 1 - \frac{\delta^*}{2}$ and $1 \geq U\left(\zeta - c\tau\right) \geq 1 - \frac{\delta^*}{2}$. For $0 < \varepsilon \leq \frac{\delta^*}{2}$ and any $\eta \in (0,1)$, we have

$$1 - \frac{\delta^*}{2} \leq U(\zeta) + \eta\sigma(\xi) < \delta^* + 1,$$

$$1 - \frac{\delta^*}{2} \leq U(\zeta - c\tau) + \eta\sigma(\xi) < \delta^* + 1.$$

By (3.9), we have $I_4 \geq \beta_0\sigma(\xi) - \alpha^2\sigma''(\xi) \geq (\beta_0 - C_7)\sigma(\xi)$. Then

$$\mathcal{L}[v^+] = I_1 + I_2 + I_3 + I_4 \tag{3.12}$$

$$\geq (-C_5\alpha - C_6\alpha + (\beta_0 - C_7)\varepsilon) \operatorname{sech}(\beta_2\xi) \geq 0 \tag{3.13}$$

provided that $0 < \alpha \leq \min\Big\{1, \frac{\beta_0\varepsilon}{C_5 + C_6 + C_7}\Big\}$.

Case (iii): $\zeta \leq -M_0$. By (3.11), we have $0 \leq U(\zeta), U\left(\zeta - c\tau\right) \leq \frac{\delta^*}{2}$ and hence for any $\eta \in (0,1)$,

$$0 \leq U(\zeta) + \eta\sigma(\xi) \leq \delta^*,$$

$$0 \leq U\left(\zeta - c\tau\right) + \eta\sigma(\xi) \leq \delta^*$$

for $0 < \varepsilon \le \delta^*/2$. Similarly, we have (3.12) holds for $\zeta \le -M_0$. Thus, combining above three cases, we have $v^+(x,z;\varepsilon,\alpha)$ is a supersolution of (1.5). Furthermore, if we take $\alpha < \frac{\varepsilon e^2 c^2 \beta_1^2 \mu_-}{4C_1C_4 s}$, where e is the exponential, we can prove (3.6) and (3.7) by an argument similar to that in [15] and we omit it. In addition, (3.8) directly follows from the definition of $v^+(x,z;\varepsilon,\alpha)$.

Finally, we take

$$\varepsilon_0^+ = \min\{1, \frac{\delta^*}{2}, \frac{C_3 m_0}{2c_1}\},$$

$$\alpha_0^+(\varepsilon) = \min\{1, \frac{C_3 m_0}{2(C_5+C_6+C_7)}, \frac{\beta_0\varepsilon}{C_5+C_6+C_7}, \frac{\varepsilon e^2 c^2 \beta_1^2 \mu_-}{4C_1C_4 s}\}.$$

This completes the proof. □

Now, we are searching for a traveling curved front towards y-direction. From Lemma 3.1, $v^+(x,z;\varepsilon,\alpha)$ is also a supersolution of (1.4). Let $w(x,z,t;v^-)$ be the solution of (1.4) with initial value $\phi(x,z,r) = v^-(x,z)$. By Theorem 2.3, we have

$$v^-(x,z) \le w(x,z,t;v^-) \le v^+(x,z), \quad \text{for } (x,z) \in \mathbb{R}^2,\ t > 0.$$

Since $v^-(x,z)$ is a subsolution, we have $w(x,z,t;v^-)$ is monotone increasing in t. Similar to Wang et al [28, Proposition 4.3], we have that

$$\|w(x,z,y;v^-)\|_{C^{2,1}(\mathbb{R}^2\times[2(\tau+1),\infty),\mathbb{R})} < \infty.$$

Applying [10, Theorem 5.1.4 and 5.1.8], we have that

$$\|w(x,z,y;v^-)\|_{C^{2+\theta,1+\theta/2}(\mathbb{R}^2\times[2(\tau+1),\infty),\mathbb{R})} < \infty \quad \text{for some } \theta \in (0,1).$$

Then there exists a function $V(x,z) \in C^2(\mathbb{R}^2)$ such that

$$w(x,z,t;v^-) \to V(x,z) \quad \text{in} \quad C^2_{\rm loc}(\mathbb{R}^2) \text{ as } t \to \infty. \tag{3.14}$$

Furthermore,

$$\mathcal{L}[V] = 0 \quad \text{on } \mathbb{R}^2,$$

$$\frac{\partial}{\partial z}V(x,z) \ge 0, \quad v^-(x,z) \le V(x,z) \le v^+(x,z;\varepsilon,\alpha), \quad \forall (x,z) \in \mathbb{R}^2.$$

By (3.6) and the arbitrariness of ε, we have

$$\lim_{\mathbb{R}\to\infty} \sup_{x^2+y^2\ge\mathbb{R}^2} |V(x,z) - v^-(x,z)| = 0.$$

Theorem 2.3 and the maximum principle implies that $v^-(x,z) < V(x,z) < 1$ and $V(x,z) \le v^+(x,z;\varepsilon,\alpha)$ for all $(x,z) \in \mathbb{R}^2$. Moreover, one has $\frac{\partial}{\partial z}V(x,z) > 0$ for all $(x,z) \in \mathbb{R}^2$. Thus, the function $u(x,y,t) = V(x, y+st)$ is just the expected traveling curved fronts of (1.1).

4. Uniqueness and stability of traveling curved fronts

In this section we develop the arguments in [15] and [29] to establish the asymptotic stability and uniqueness of traveling curved fronts $V(x,z)$ obtained in Section 3. We prove (1.6) for $\phi(x,z,r) \ge v^-(x,z)$ in $\mathbb{R}^2$ and $r \in [-\tau, 0]$.

Let $w^i(x,z,t)$ be the solution of

$$\frac{\partial w^i}{\partial t} = w^i_{xx} + w^i_{zz} - s\frac{\partial w^i}{\partial z} + f(w^i(x,z,t), w^i(x,z-s\tau,t-\tau))\ \forall (x,z) \in \mathbb{R}^2,\ t > 0,$$
$$w^i(x,z,r) = \phi^i(x,z,r), \quad \forall (x,z) \in \mathbb{R}^2,\ r \in [-\tau, 0],$$

for $i = 1, 2$.

Lemma 4.1. *Let $\phi^i(x,z,r) \in [-\delta_0, 1+\delta_0]_{\mathcal{C}}$ for $r \in [-\tau, 0]$ and $(x,z) \in \mathbb{R}^2$, $i = 1, 2$. Then*

$$\sup_{(x,z)\in\mathbb{R}^2} |w^2(x,z,t) - w^1(x,z,t)| \le A^{(t+1)} \sup_{(x,z)\in\mathbb{R}^2, r\in[-\tau,0]} |\phi^2(x,z,r) - \phi^1(x,z,r)|,$$

for $t > 0$, where A is a positive constant independent of ϕ^1 and ϕ^2.

Proof. By the abstract setting in [14], $w^i(t) := w^i(x,z,t)$ $(i = 1,2)$ is a solution to its associated integral equation

$$\begin{gathered} w^i(t) = T(t)w^i(0) + \int_0^t T(t-s)F(w_s^i)ds, \quad t > 0, \\ w^i(0) = \phi^i \in [-\delta_0, 1+\delta_0]_{\mathcal{C}}. \end{gathered}$$

As aforementioned in Section 2, $F : [-\delta_0, 1+\delta_0]_{\mathcal{C}} \to X$ is globally Lipschitz continuous. Let $\widehat{w}(t) = e^{Mt}T(t)\widehat{w}(0)$ for $t \ge 0$, where

$$\widehat{w}(0) := \sup_{(x,z)\in\mathbb{R}^2, r\in[-\tau,0]} |\phi^2(x,z,r) - \phi^1(x,z,r)|$$

and $M = \max\{|F'(w)| : w \in [-\delta_0, 1+\delta_0]\}$. Then $\widehat{w}(t)$ satisfies

$$\widehat{w}(t) = T(t)\widehat{w}(0) + M\int_0^t T(t-s)\widehat{w}(s)ds, \quad t \ge 0.$$

Define $\widetilde{w}(t) := w^2(t) - w^1(t)$. Note that $\widetilde{w}(t)$ satisfies

$$\begin{aligned} \widetilde{w}(t) &= T(t)\widetilde{w}(0) + \int_0^t T(t-s)(F(w_s^2) - F(w_s^1))ds \\ &\le T(t)\widetilde{w}(0) + M\int_0^t T(t-s)\widetilde{w}(s)ds. \end{aligned}$$

By [14, Proposition 3] with $v^- = -\infty$, $v^+ = \widehat{w}(t)$, $S(t,s) = S^+(t,s) = T(t,s) = T(t-s)$, $t \ge s \ge 0$ and $B(t,\psi) = B^+(t,\psi) = M\psi(0)$, we have $\widehat{w}(t) \ge \widetilde{w}(t)$ for all $t \ge 0$. Thus it follows that $w^2(t) - w^1(t) \le e^{Mt}T(t)\widehat{w}(0)$ for $t \ge 0$. Similarly, we have $w^1(t) - w^2(t) \le e^{Mt}T(t)\widehat{w}(0)$ for $t \ge 0$. Note that there exist positive constants A_1 and A_2 such that

$$|T(t)| \le \frac{A_1}{t}\exp\{-A_2\frac{(x-x_1)^2 + (z-z_1)^2}{t}\}$$

for $(x,z) \in \mathbb{R}^2$ and $t \in (0,1]$. Then there exists a constant $A > 1$ such that $\widehat{w}(t) \le A\widehat{w}(0)$ for $t \in (0,1]$. Note that $\widehat{w}(t+s) = \widehat{w}(t)\widehat{w}(s)$. For any $t > 0$, we have

$$\widehat{w}(t) = e^{Mt}T(t-[t])\widehat{w}([t]),$$

where $[t] = \max\{n \in \mathbb{Z}, n \le t\}$. By induction, we obtain

$$\widehat{w}(t) \le A^{[t]+1}\widehat{w}(0) \le A^{t+1}\widehat{w}(0), \quad \forall t > 0.$$

This implies

$$\sup_{(x,z)\in\mathbb{R}^2} |w^2(x,z,t) - w^1(x,z,t)| \le A^{(t+1)} \sup_{(x,z)\in\mathbb{R}^2, r\in[-\tau,0]} |\phi^2(x,z,r) - \phi^1(x,z,r)|.$$

$\square$

Lemma 4.2. *There exists a positive constant β_3 satisfying*

$$\frac{\partial V}{\partial z}(x,z) \geq \beta_3, \quad \text{when } \delta^* \leq V(x,z) \leq 1-\delta^*,$$
$$\frac{\partial v^+}{\partial z}(x,z) \geq \beta_3, \quad \text{when } \delta^* \leq v^+(x,z) \leq 1-\delta^*.$$

This lemma is used to prove the uniqueness of traveling curved front and the proof is completely similar to [15, Lemma 4.3], so we omit it here.

Lemma 4.3. *Let $\bar{v}$ be a supersolution to (1.5) with*

$$\bar{v}_z(x,z) > 0, \quad -\frac{\delta^*}{2} < \bar{v}(x,z) < 1+\frac{\delta^*}{2}, \quad \text{for all } (x,z)\in\mathbb{R}^2,$$
$$\bar{v}_z(x,z) \geq \beta_3, \quad \frac{\delta^*}{2} < \bar{v}(x,z) < 1-\frac{\delta^*}{2}, \quad \text{for all } (x,z)\in\mathbb{R}^2.$$

Let $\underline{v}$ be a subsolution to (1.5) with

$$\underline{v}_z(x,z) > 0, \quad -\frac{\delta^*}{2} < \underline{v}(x,z) < 1+\frac{\delta^*}{2}, \quad \text{for all } (x,z)\in\mathbb{R}^2,$$
$$\underline{v}_z(x,z) \geq \beta_3, \quad \frac{\delta^*}{2} < \underline{v}(x,z) < 1-\frac{\delta^*}{2}, \quad \text{for all } (x,z)\in\mathbb{R}^2.$$

Then there exist a positive constant ρ sufficiently large and a positive constant β small enough such that for any $0<\delta<\frac{\delta^*}{2}e^{-\beta\tau}$, the functions w^+ and w^- defined by

$$w^+(x,z,t;\bar{v}) = \bar{v}(x,z+\rho\delta(1-e^{-\beta t})) + \delta e^{-\beta t},$$
$$w^-(x,z,t;\underline{v}) = \underline{v}(x,z-\rho\delta(1-e^{-\beta t})) - \delta e^{-\beta t}$$

are super- and subsolution of (1.4), respectively.

Proof. Since $\bar{v}(x,z)$ be a supersolution of (1.5), we have

$$-\bar{v}_{xx} - \bar{v}_{zz} + s\bar{v}_z \geq f\left(\bar{v}(x,z), \bar{v}(x,z-s\tau)\right), \quad \forall (x,z)\in\mathbb{R}^2.$$

For $\beta>0$ and any $t\in\mathbb{R}$, $\rho\delta(1-e^{-\beta(t-\tau)}) < \rho\delta(1-e^{-\beta t})$. Then by the assumption of $\bar{v}(x,z)$,

$$\bar{v}\Big(x, z+\rho\delta(1-e^{-\beta(t-\tau)}) - s\tau\Big) < \bar{v}\left(x, z+\rho\delta(1-e^{-\beta t}) - s\tau\right).$$

Let $\xi = z+\rho\delta(1-e^{-\beta t})$. Direct calculations yield that

$$\begin{aligned}
&\mathcal{N}[w^+]\\
&:= \frac{\partial w^+}{\partial t} - w^+_{zz} - w^+_{xx} + s\frac{\partial w^+}{\partial z} - f(w^+(x,z,t), w^+(x,z-s\tau,t-\tau))\\
&\geq \delta\beta e^{-\beta t}\Big(\rho\frac{\partial \bar{v}}{\partial z} - 1\Big) + f\left(\bar{v}(x,\xi), \bar{v}(x,\xi-s\tau)\right)\\
&- f\Big(\bar{v}(x,\xi)+\delta e^{-\beta t}, \bar{v}(x, z+\rho\delta(1-e^{-\beta(t-\tau)})) + \delta e^{-\beta(t-\tau)}\Big)\\
&\geq \delta\beta e^{-\beta t}\Big(\rho\frac{\partial \bar{v}}{\partial z} - 1\Big) + f\left(\bar{v}(x,\xi), \bar{v}(x,\xi-s\tau)\right)\\
&\quad - f\Big(\bar{v}(x,\xi)+\delta e^{-\beta t}, \bar{v}(x,\xi-s\tau) + \delta e^{-\beta(t-\tau)}\Big)\\
&\geq \delta e^{-\beta t}\Big\{\beta\rho\frac{\partial \bar{v}}{\partial z} - \beta - \Big[\int_0^1 \partial_1 f\Big(\bar{v}(x,\xi)+\theta\delta e^{-\beta t}, \bar{v}(x,\xi-s\tau)+\theta\delta e^{-\beta(t-\tau)}\Big)
\end{aligned}$$

$$+ e^{\beta\tau}\partial_1 f\left(\bar{v}(x,\xi) + \theta\delta e^{-\beta t}, \bar{v}(x,\xi - s\tau) + \theta\delta e^{-\beta(t-\tau)}\right) d\theta\Big]\Big\},$$

where we have used assumption (H1). By the assumption (H2), there exists a positive constant β such that

$$\lim_{(u,v,\beta)\to(0,0,0)} \left(\partial_1 f(u,v) + e^{\beta\tau}\partial_2 f(u,v)\right) = \partial_1 f(0,0) + \partial_2 f(0,0) < 0,$$
$$\lim_{(u,v,\beta)\to(1,1,0)} \left(\partial_1 f(u,v) + e^{\beta\tau}\partial_2 f(u,v)\right) = \partial_1 f(1,1) + \partial_2 f(1,1) < 0.$$

Then for $\beta > 0$ small enough, there exist $\eta \in (0,\beta]$ and $\delta^* \in (0,\delta_0)$ such that $[-\delta^*, 1+\delta^*] \subset I$ and

$$\partial_1 f(u,v) + e^{\beta\tau}\partial_2 f(u,v) < -\eta \quad \text{for } (u,v) \in [-\delta^*,\delta^*]^2 \cup [1-\delta^*, 1+\delta^*]^2. \tag{4.1}$$

Take

$$\begin{aligned} c_1' &= c_1'(\beta,\delta^*) \\ &= \max\left\{|\partial_1 f(u,v)| : (u,v) \in [-\delta^*, 1+\delta^*]^2\right\} \\ &\quad + e^{\beta\tau} \max\left\{|\partial_2 f(u,v)| : (u,v) \in [-\delta^*, 1+\delta^*]^2\right\}. \end{aligned}$$

Case (i): $\frac{\delta^*}{2} \le \bar{v} \le 1 - \frac{\delta^*}{2}$. For $\delta \in (0, \frac{1}{2}\delta^* e^{-\beta\tau})$ and any $\theta \in (0,1)$, we have

$$-\delta^* < \bar{v}(x,\xi) + \theta\delta e^{-\beta t},\ \bar{v}(x,\xi - s\tau) + \theta\delta e^{-\beta(t-\tau)} \le 1 + \delta^*.$$

Then we have

$$\mathcal{N}[w^+] \ge \delta e^{-\beta t}\left[\beta\rho\beta_3 - \beta - c_1'\right] \ge 0$$

provided that $\rho > \frac{\beta + c_1'}{\beta\beta_3}$.

Case (ii): $1 + \frac{\delta^*}{2} \ge \bar{v} \ge 1 - \frac{\delta^*}{2}$. In this case, for $\delta \in (0, \frac{1}{2}\delta^* e^{-\beta\tau})$ and any $\theta \in (0,1)$,

$$1 - \frac{\delta^*}{2} \le \bar{v}(x,\xi) + \theta\delta e^{-\beta t}, \bar{v}(x,\xi - s\tau) + \theta\delta e^{-\beta(t-\tau)} \le 1 + \delta^*.$$

By (4.1), we have

$$\mathcal{N}[w^+] \ge \delta e^{-\beta t}\Big(\beta\rho\frac{\partial\bar{v}}{\partial z} - \beta + \eta\Big) \ge 0$$

for ρ large enough.

Case (iii): $0 \le \bar{v} \le \frac{\delta^*}{2}$. Similarly, we have $0 \le \bar{v}(x,\xi) + \theta\delta e^{-\beta t}, \bar{v}(x,\xi - s\tau) + \theta\delta e^{-\beta(t-\tau)} \le \delta^*$ and $\mathcal{N}[w^+] \ge 0$.

Consequently, we have $\mathcal{N}[w^+] \ge 0$ for $(x,z) \in \mathbb{R}^2$ and $t \ge 0$. It then follows that $w^+(x,z,t;\bar{v})$ is a supersolution of (1.4). Similarly, we can prove $\mathcal{N}[w^-] \le 0$ and $w^-(x,z,t;\underline{v})$ is a subsolution of (1.4). This completes the proof. $\square$

Lemma 4.4. *Let $w(x,z,t)$ be the solution of (1.4) with*

$$\lim_{R\to\infty} \sup_{x^2+z^2 \ge R^2, r\in[-\tau,0]} |\phi(x,z,r) - v^-(x,z)| = 0.$$

Then

$$\lim_{R\to\infty} \sup_{x^2+z^2 \ge R^2} |w(x,z,T) - v^-(x,z)| = 0$$

holds for any fixed $T > \tau$.

Proof. For the sake of convenience, we define $w(x,z,t;\phi)$ by $w(x,z,t)$. Define $W(x,z) = U(\frac{z+\varphi(x)}{\sqrt{1+\varphi'^2(x)}})$. (3.2)-(3.5) implies that

$$\lim_{R\to\infty} \sup_{x^2+z^2>R^2} |W(x,z) - v^-(x,z)| = 0.$$

It then follows that

$$\lim_{R\to\infty} \sup_{x^2+z^2>R^2, r\in[-\tau,0]} |W(x,z) - \phi(x,z,r)| = 0.$$

Define $v(x,z,t) = w(x,z,t) - W(x,z)$. Consider the equation

$$\begin{gathered} \frac{\partial \widetilde{v}}{\partial t} - \widetilde{v}_{xx} - \widetilde{v}_{zz} + s\frac{\partial \widetilde{v}}{\partial z} - \partial_1 f(v+\theta W, W(x,z-s\tau))\widetilde{v} = h(x,z), \\ \widetilde{v}(x,z,r) = w(x,z,r) - W(x,z), \quad r\in[-\tau,0] \end{gathered} \tag{4.2}$$

for $(x,z)\in\mathbb{R}^2$, $t>0$ and some $\theta\in(0,1)$, where $h(x,z) = -\mathcal{L}[W]$ satisfies $\lim_{R\to\infty}\sup_{x^2+z^2>R^2}|h(x,z)| = 0$. In view of

$$\begin{aligned} &- f(v+W, v(x,z-s\tau,t-\tau) + W(x,z-s\tau)) + f(W, W(x,z-s\tau)) \\ &\le -f(v+W, W(x,z-s\tau)) + f(W, W(x,z-s\tau)) \\ &= -\partial_1 f(v+\theta W, W(x,z-s\tau))v, \end{aligned}$$

we have that $v(x,z,t)$ is a subsolution of (4.2). It follows that $v(x,z,t)\le\widetilde{v}(x,z,t)$ for $(x,z)\in\mathbb{R}^2$ and $t>0$. Define $\overline{v}(x,z,t)$ by

$$\begin{gathered} \frac{\partial \overline{v}}{\partial t} - \overline{v}_{xx} - \overline{v}_{zz} + s\frac{\partial \overline{v}}{\partial z} - M\overline{v} = |h(x,z)|, \\ \overline{v}(x,z,r) = |\widetilde{v}(x,z,r)|, \quad r\in[-\tau,0] \end{gathered} \tag{4.3}$$

for $(x,z)\in\mathbb{R}^2$, $t>0$ and $M = \max\{|\partial_1 f(u,v)| : u,v\in I\}$. The maximum principle (see [18]) yields $\overline{v}(x,z,t)\ge 0$ for $(x,z)\in\mathbb{R}^2$ and $t>0$. Then we have

$$\begin{aligned} &(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial z^2} + s\frac{\partial}{\partial z})(\overline{v}-\widetilde{v}) - \partial_1 f(v+\theta W, W(x,z-s\tau))(\overline{v}-\widetilde{v}) \\ &= (M - \partial_1 f(v+\theta W, W(x,z-s\tau)))\overline{v} + (|h(x,z)| - h(x,z)) \ge 0 \end{aligned}$$

for $(x,z)\in\mathbb{R}^2$ and $t>0$ with the initial value $(\overline{v}-\widetilde{v})|_{t=r\in[-\tau,0]}\ge 0$. The maximum principle implies $\overline{v}(x,z,t)\ge\widetilde{v}(x,z,t)$ for $(x,z)\in\mathbb{R}^2$ and $t>0$. It follows that $\overline{v}(x,z,t)\ge v(x,z,t)$ for $(x,z)\in\mathbb{R}^2$ and $t>0$. Apply the similar arguments to $(-v)$, we obtain $\overline{v}(x,z,t)\ge|v(x,z,t)|$ for $(x,z)\in\mathbb{R}^2$ and $t>0$.

To complete the proof, it suffices to prove that $\lim_{R\to\infty}\sup_{x^2+z^2>R^2}|\overline{v}(x,z,t)|\to 0$ for each $T>\tau$. Because $\overline{v}(x,z,t)$ is the solution of (4.3) then there exists a solution function $|\Gamma(x,z,t;x_1,z_1,s)|\le\frac{B_1}{t-s}\exp\{-B_2\frac{(x-x_1)^2+(z-z_1)^2}{t-s}\}$ and

$$\begin{aligned} \overline{v}(x,z,t) = &\int_{\mathbb{R}^2}\Gamma(x,z,t;x_1,z_1,0)\overline{v}(x_1,z_1,0)dx_1dz_1 \\ &+ \int_0^t\int_{\mathbb{R}^2}\Gamma(x,z,t;x_1,z_1,s)|h(x_1,z_1)|dx_1dz_1, \end{aligned}$$

where $0\le s\le t<T$, B_1 and B_2 are positive constants dependent of T. Thus, the remainder proofs follows a similar arguments as that in [15, Lemma 4.5] and we omit it here. This completes the proof. $\square$

In the following, we prove the uniqueness and stability of traveling curved fronts of (1.1). Take $0 < \varepsilon \le \varepsilon_0^+$. Let

$$V^*(x,z) := \lim_{t\to\infty} w(x,z,t;v^+) \tag{4.4}$$

for any $(x,z) \in \mathbb{R}^2$ and $t \in \mathbb{R}$. Since $v^+(x,z;\varepsilon,\alpha)$ is also a supersolution of (1.5), we have $w(x,z,t;v^+)$ is nonincreasing in t and converges to some function $V^*(x,z)$ under the norm $\|\cdot\|_{C^{2+\theta,1+\theta/2}(\mathbb{R}^2\times[2(\tau+1),\infty),\mathbb{R})}$ as $t \to \infty$. Furthermore, $V^*(x,z)$ satisfies (1.5) and

$$V(x,z) \le V^*(x,z) \le v^+(x,z;\varepsilon,\alpha), \quad \forall (x,z) \in \mathbb{R}^2. \tag{4.5}$$

Lemma 4.5. *Let $V^*(x,z)$ and $V(x,z)$ be as* (4.4) *and* (3.14)*. Then*

$$V_*(x,z) \equiv V(x,z) \quad \textit{for all } (x,z) \in \mathbb{R}^2.$$

Proof. Assume that $V_*(x,z) \not\equiv V(x,z)$ by contradiction. Then the maximum principle and (4.5) imply that

$$V(x,z) < V^*(x,z) \quad \text{for all } (x,z) \in \mathbb{R}^2.$$

Take β and ρ as in Lemma 4.3. For any $0 < \delta < \frac{\delta^*}{2} e^{-\beta\tau}$, using (3.6) we make $\lambda > 0$ large enough such that

$$V^*(x,z) \le V(x,z+\lambda) + \delta \quad \text{for all } (x,z) \in \mathbb{R}^2.$$

By Lemma 4.3,

$$w^+(x,z+\lambda,t;V) = V\left(x, z+\lambda+\rho\delta(1-e^{-\beta t})\right) + \delta e^{-\beta t}$$

is a supersolution of (1.5). By Theorem 2.3, $V^*(x,z) \le w^+(x,z+\lambda,t;V)$ for any $(x,z) \in \mathbb{R}^2$ and $t > 0$. Then let $t \to \infty$, we have

$$V^*(x,z) \le V(x,z+\lambda+\rho\delta), \quad \forall (x,z) \in \mathbb{R}^2.$$

Define

$$\Lambda := \inf\left\{\lambda : V^*(x,z) \le V(x,z+\lambda) \text{ for all } (x,z) \in \mathbb{R}^2\right\}.$$

Then we have that $\Lambda \ge 0$ and

$$V^*(x,z) \le V(x,z+\Lambda) \quad \text{for all } (x,z) \in \mathbb{R}^2.$$

It is still needed to show $\Lambda = 0$ by contradiction and then obtain $V^*(x,z) \equiv V(x,z)$ holds in $\mathbb{R}^2$. Assume $\Lambda > 0$. By the strong maximum principle of elliptic equation (see [18]), we also have that either

$$V^*(x,z) = V(x,z+\Lambda) \quad \text{for all } (x,z) \in \mathbb{R}^2$$

or

$$V^*(x,z) < V(x,z+\Lambda) \quad \text{for all } (x,z) \in \mathbb{R}^2.$$

We show that the former is impossible. In fact, it is easy to see that

$$\lim_{x\to\pm\infty} V^*(x,-m_*x) = U(0) \quad \text{and} \quad \lim_{x\to\pm\infty} V^*(x,-m_*x+\Lambda) = U(\frac{c}{s}\Lambda),$$

which is a contraction if $V^*(x,z) = V(x,z+\Lambda)$. Next, we assume that $V^*(x,z) < V(x,z+\Lambda)$ for any $(x,z) \in \mathbb{R}^2$. Since $\lim_{R\to\infty} \sup_{x^2+z^2\ge R^2} |V(x,z) - v^-(x,z)| = 0$ and

$$\lim_{R\to\infty} \sup_{|z+m_*|x||\ge R} |v_z^-(x,z)| = 0,$$

by the interpolation $\|\cdot\|_{C^1} \le 2\sqrt{\|\cdot\|_{C^0}\|\cdot\|_2}$, we have

$$\lim_{R\to\infty} \sup_{|z+m_*|x||\ge R} |V_z(x,z+\Lambda)| = 0.$$

Take $R_* > 0$ such that

$$2\rho \sup_{|z+m_*|x||\ge R_*-\rho\delta^*} |\frac{\partial}{\partial z} V(x,z+\Lambda)| < 1.$$

Define $D := \{(x,z)\in\mathbb{R}^2 ||z+m_*|x|| \le R_*\}$. Since $V^*(x,z) < V(x,z+\Lambda)$ in D, we can choose a small positive constant h with

$$0 < h < \min\{\frac{\delta^*}{2}, \frac{\Lambda}{2\rho}\}, \quad V^*(x,z) \le V(x,z+\Lambda-2\rho h) \quad \text{in D.} \tag{4.6}$$

In $(x,z)\in\mathbb{R}^2\setminus D$, we have

$$V(x,z+\Lambda-2\rho h) - V(x,z+\Lambda) = -2\rho h\int_0^1 V_z(x,z+\Lambda-2\theta\rho h)d\theta \ge -h. \tag{4.7}$$

Then by (4.6) and (4.7),

$$V(x,z+\Lambda) \le V(x,z+\Lambda-2\rho h) + h \quad \text{in } (x,z)\in\mathbb{R}^2.$$

By Lemma 4.3,

$$w^+(x,z+\Lambda-2\rho h,t;h) = V(x,z+\Lambda-2\rho h+\rho\delta(1-e^{-\beta t})) + he^{-\beta t}$$

is a supersolution of (1.4). Theorem 2.3 implies that

$$V^*(x,z) \le w^+(x,z+\Lambda-2\rho h,t;h) \quad \forall (x,z)\in\mathbb{R}^2, t>0.$$

Let $t\to\infty$ yields that $V^*(x,z) \le V(x,z+\Lambda-2\rho h)$ for $(x,z)\in\mathbb{R}^2$, which contradicts the definition of Λ. Thus $\Lambda = 0$. This proof is completed. □

Next, we establish asymptotic stability of $V(x,z)$ and prove (1.6) for $\phi(x,z,r) \ge v^-(x,z)$ in $\mathbb{R}^2$ and $r\in[-\tau,0]$.

Proof. For simplicity, we denote $w(x,z,t;\phi)$ as $w(x,z,t)$. For any $\varepsilon_0 > 0$, we show that there exists $T_* > 0$ such that

$$\sup_{(x,z)\in\mathbb{R}^2} |w(x,z,t) - V(x,z)| \le \epsilon \quad \text{for } t > T_*.$$

Choose δ small enough such that

$$V^*(x,z+\rho\delta) \le V^*(x,z) + \frac{\epsilon}{3}, \quad 0 < \delta < \varepsilon_0^+, \tag{4.8}$$

where ε_0^+ and ρ are given in Lemma 3.1 and Lemma 4.3, respectively. By Theorem 2.3, there exists a positive constant $T_\delta > \tau$ such that

$$w(x,z,t;v^-) \le w(x,z,t) < 1 \quad \text{for } (x,z)\in\mathbb{R}^2,\ t\ge T_\delta. \tag{4.9}$$

Lemma 4.4 shows that for some $R > 0$,

$$w(x,z,T_\delta) \le v^-(x,z) + \frac{\delta}{2} \quad \text{for } x^2+z^2 \ge R^2. \tag{4.10}$$

If α is small enough, then we have

$$U(\zeta) = U\Big(\frac{z+\varphi(\xi)/\alpha}{\sqrt{1+\varphi'^2(\xi)}}\Big) \ge U\Big(\frac{c}{s}\Big(-R+\frac{\varphi(0)}{\alpha}\Big)\Big) \ge 1-\frac{\delta}{2}$$

for $x^2+z^2 \leq R^2$. Choose α small to satisfy $0<\alpha<\min\left\{\alpha_0^+(\varepsilon), \frac{\varphi(0)}{\frac{s}{c}[U^{-1}(\cdot)(1-\frac{\delta}{2})+R]}\right\}$,

$$v^+(x,z) \geq 1-\frac{\delta}{2} \quad \text{for } x^2+z^2 \leq R^2. \tag{4.11}$$

Combining (4.9)-(4.11), we obtain

$$w(x,z,T_\delta) < v^+(x,z)+\delta \leq w^+(x,z,0;\delta) \quad \text{for } (x,z) \in \mathbb{R}^2.$$

Theorem 2.3 implies

$$w(x,z,t+T_\delta;v^-) \leq w(x,z,t+T_\delta) \leq w^+(x,z,t;v^+) \tag{4.12}$$

for $(x,z) \in \mathbb{R}^2$ and $t \geq 0$. By Theorem 2.3 again, we have

$$w(x,z,t+s+T_\delta,v^-) \leq w(x,z,t+s+T_\delta) \leq w(x,z,s;u^t) \tag{4.13}$$

for $t \geq 0$ and $s \geq 0$, where $u^t = w^+(x,z,t;v^+)$. Since $w(x,z,t;v^+)$ monotonically converges to $V(x,z)$ as $t \to \infty$, there exists a positive constant $s_1 \geq \tau$ with

$$\sup_{(x,z)\in\mathbb{R}^2} |w(x,z,s_1;v^{+,\delta}) - V(x,z+\rho\delta)| \leq \frac{\epsilon}{3},$$

where $v^{+,\delta}(x,z) = v^+(x,z+\rho\delta)$. By Lemma 4.1, for any $\phi(x,z,r) \in [-\delta_0, 1+\delta_0]_{\mathcal{C}}$,

$$\begin{aligned} &\sup_{(x,z)\in\mathbb{R}^2} |w(x,z,s_1;\phi) - w(x,z,s_1;v^{+,\delta})| \\ &\leq A^{(s_1+1)} \sup_{(x,z)\in\mathbb{R}^2, r\in[-\tau,0]} |\phi(x,z,r) - v^{+,\delta}(x,z)|. \end{aligned} \tag{4.14}$$

Since $w^+(x,z,t;v^+) \to v^+(x,z+\rho\delta)$ as $t \to \infty$ uniformly in $(x,z) \in \mathbb{R}^2$, then there exists $T_1 > \tau$ large enough such that satisfy

$$A^{(s_1+1)} \sup_{(x,z)\in\mathbb{R}^2} |w^+(x,z,t;v^+) - v^+(x,z+\rho\delta)| \leq \frac{\epsilon}{3} \tag{4.15}$$

for $t \geq T_1$ and $(x,z) \in \mathbb{R}^2$. Then, by (4.14) and (4.15) we have

$$\begin{aligned} &|w(x,z,s_1;u^t) - V^*(x,z+\rho\delta)| \\ &\leq |w(x,z,s_1;u^t) - w(x,z,s_1;v^{+,\delta})| + |w(x,z,s_1;v^{+,\delta}) - V^*(x,z+\rho\delta)| \\ &\leq \frac{2}{3}\epsilon \end{aligned}$$

for any $t \geq T_1$ and $(x,z) \in \mathbb{R}^2$. By (4.13)

$$w(x,z,t+s_1+T_\delta,v^-) \leq w(x,z,s_1;u^t) \leq V^*(x,z+\rho\delta) + \frac{2}{3}\epsilon \tag{4.16}$$

holds for $t \geq T_1$ and $(x,z) \in \mathbb{R}^2$. Combining (4.8), (4.13), (4.16) and Lemma 4.5, we have

$$w(x,z,t;v^-) \leq w(x,z,t) \leq V(x,z) + \epsilon$$

for $(x,z) \in \mathbb{R}^2$ and $t \geq s_1 + T_1 + T_\delta$. Since $V(x,z) = \lim_{t\to\infty} w(x,z,t;v^-)$, we obtain

$$\lim_{t\to\infty} \|w(x,z,t) - V(x,z)\|_{L^\infty(\mathbb{R}^2)} = 0.$$

This completes the proof. $\square$

Acknowledgments. This work was partially supported by NNSF of China (no. 11371179), the Fundamental Research Funds for the Central Universities (lzujbky-2014-229).

References

[1] A. Bonnet, F. Hamel; Existence of nonplanar solutions of a simple model of premixed Bunsen flames, *SIAM J. Math. Anal.* **31** (1999), 80-118.

[2] S. A. Gourley, S. Ruan; Convergence and travelling fronts in functional differential equations with nonlocal terms: a competition model, *SIAM J. Math. Anal.* **35** (2003), 806-822.

[3] S. A. Gourley, J. H. W. So, J. Wu; Non-locality of reaction-diffusion equations induced by delay: biological modeling and nonlinear dynamics, *J. Math. Sci.* **124** (2004), 5119-5153.

[4] F. Hamel, N. Nadirashvili; Travelling fronts and entire solutions of the Fisher-KPP equation in $\mathbb{R}^N$, *Arch. Ration. Mech. Anal.* **157** (2001), 91-163.

[5] F. Hamel, R. Monneau, J.-M. Roquejoffre; Stability of travelling waves in a model for conical flames in two space dimensions, *Ann. Sci. École Norm. Sup.* **37** (2004), 469-506.

[6] F. Hamel, R. Monneau, J.-M. Roquejoffre; Existence and qualitative properties of multidimensional conical bistable fronts, *Discrete Contin. Dyn. Syst.* **13** (2005), 1069-1096.

[7] Y. Kurokawa, M. Taniguchi; Multi-dimensional pyramidal travelling fronts in the Allen-Cahn equations, *Proc. Roy. Soc. Edinburgh Sect. A* **141** (2011), 1031-1054.

[8] G. Lin, W.-T. Li; Bistable wavefronts in a diffusive and competitive Lotka-Volterra type system with nonlocal delays, *J. Differential Equations* **244** (2008), 487-513.

[9] W.-T. Li, G. Lin, S. Ruan; Existence of travelling wave solutions in delayed reaction-diffusion systems with applications to diffusion-competition systems, *Nonlinearity* **19** (2006), 1253-1273.

[10] A. Lunardi; Analytic Semigroups and Optimal Regularity in Parabolic Problems, Birkhäuser, Boston, 1995.

[11] S. Ma; Traveling wavefronts for delayed reaction-diffusion systems via a fixed point theorem, *J. Differential Equations* **171** (2001), 294-314.

[12] S. Ma, J. Wu, Existence; Uniqueness and asymptotic stability of traveling wavefronts in a non-local delayed diffusion equation, *J. Dynam. Differential Equations* **19** (2007), 391-436.

[13] S. Ma, X. Zou; Existence, uniqueness and stability of travelling waves in a discrete reaction-diffusion monostable equation with delay, *J. Differential Equations* **217** (2005), 54-87.

[14] R. H. Martin, H. L. Simth; Abstract functional-differential equations and reaction-diffusion systems, *Trans. Amer. Math. Soc.* **321** (1990), 1-44.

[15] H. Ninomiya, M. Taniguchi; Existence and global stability of traveling curved fronts in the Allen-Cahn equations, *J. Differential Equations* **213** (2005), 204-233.

[16] H. Ninomiya, M. Taniguchi; Global stability of traveling curved fronts in the Allen-Cahn equations, *Discrete Contin. Dyn. Syst.* **15** (2006), 819-832.

[17] W.-M. Ni, M. Taniguchi; Traveling fronts of pyramidal shapes in competition-diffusion systems, *Netw. Heterog. Media* **8** (2013), 379-395.

[18] M. H. Protter, H. F. Weinberger; Maximum Principles in Differential Equations, Prentice-Hall, Inc., Englewood Cliffs, N.J., 1967.

[19] K. W. Schaaf; Asymptotic behavior and traveling wave solutions for parabolic functional-differential equations, *Trans. Amer. Math. Soc.* **302** (1987), 587-615.

[20] W. J. Sheng, W.-T. Li, Z.-C. Wang; Periodic pyramidal traveling fronts of bistable reaction-diffusion equations with time-periodic nonlinearity, *J. Differential Equations* **252** (2012), 2388-2424.

[21] H. L. Smith, X-.Q. Zhao; Global asymptotic stability of traveling waves in delayed reaction-diffusion equations, *SIAM J. Math. Anal.* **31** (2000), 514-534.

[22] M. Taniguchi; Traveling fronts of pyramidal shapes in the Allen-Cahn equations, *SIAM J. Math. Anal.* **39** (2007), 319-344.

[23] M. Taniguchi; The uniqueness and asymptotic stability of pyramidal traveling fronts in the Allen-Cahn equations, *J. Differential Equations* **246** (2009), 2103-2130.

[24] M. Taniguchi; Multi-dimensional traveling fronts in bistable reaction-diffusion equations, *Discrete Contin. Dyn. Syst.* **32** (2012), 1011-1046.

[25] Z.-C. Wang; Traveling curved fronts in monotone bistable systems, *Discrete Contin. Dyn. Syst.* **32** (2012), 2339-2374.

[26] Z.-C. Wang, W.-T. Li, S. Ruan, Travelling wave fronts in reaction-diffusion systems with spatio-temporal delays, *J. Differential Equations* **222** (2006), 185-232.

[27] Z.-C. Wang, W.-T. Li, S. Ruan; Existence and stability of traveling wave fronts in reaction advection diffusion equations with nonlocal delay, *J. Differential Equations* **238** (2007), 153-200.

[28] Z.-C. Wang, W.-T. Li, S. Ruan; Entire solutions in bistable reaction-diffusion equations with nonlocal delayed nonlinearity, *Trans. Amer. Math. Soc.* **361** (2009), 2047-2084.

[29] Z.-C. Wang, J. Wu; Periodic traveling curved fronts in reaction-diffusion equation with bistable time-periodic nonlinearity, *J. Differential Equations* **250** (2011), 3196-3229.

[30] J. Wu; Theory and Applications of Partial Functional-Differential Equations, Applied Mathematical Sciences, 119. Springer-Verlag, New York. (1996).

[31] J. Wu, X. Zou; Traveling wave fronts of reaction-diffusion systems with delay, *J. Dynam. Differential Equations* **13** (2001), 651-687.

[32] X. Zou; Delay induced traveling wave fronts in reaction diffusion equations of KPP-Fisher type, *J. Comput. Appl. Math.* **146** (2002), 309-321.

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