Electronic Journal of Differential Equations, Vol. 2015 (2015), No. 255, pp. 1–18.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

UNIFICATION OF INTEGRABLE q-DIFFERENCE EQUATIONS

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Abstract. This article presents a unifying framework for q-discrete equations. We introduce a generalized q-difference equation in Hirota bilinear form and develop the associated three-q-soliton solutions which are described in polynomials of power functions by utilizing Hirota direct method. Furthermore, we present that the generalized q-difference soliton equation reduces to q-analogues of Toda, KdV and sine-Gordon equations equipped with their three-q-soliton solutions by appropriate transformations.

1. Introduction

The concept of integrability possesses a key position in the field of theoretical and mathematical physics. In the landmark article [12], Hirota introduced a very essential method, the so-called *Hirota direct method* which allows not only to construct multi-soliton solutions or some special type of solutions, but also to investigate the integrability criteria of a given nonlinear evolution equation [6, 8, 9, 10, 11]. Another important hallmark of Hirota's method over other methods; such as inverse scattering transform [5], or Bäcklund transformation [27], is the fact that it is algebraic rather than analytic. The intrinsic feature of the method is to convert a nonlinear partial differential or difference equation to *Hirota bilinear form* which is expressed by means of a polynomial in *Hirota-D derivative operator*. In the literature, it is conjectured that all integrable nonlinear evolution equations can be revealed in Hirota bilinear forms while the converse is not true. As an aside, notice that the equations in Hirota bilinear form equipped with three soliton solutions are defined to be *Hirota integrable* and they are widely considered to be integrable. [21, 22]. In the present paper, we stick to Hirota integrability definition.

Błaszak et al. [2] accomplished that all discrete systems that are generated by distinct vector fields are not globally equivalent. Besides, it is concluded that q-difference systems on $\mathbb{R}^-$, are not isomorphic to lattice systems on $\mathbb{R}$. This inequivalence beget a deeper analysis on the q-discretization of Toda lattice system of equations. In the literature, Hirota direct method was applied to a vast variety of differential or difference type of equations. In [26], the method was also shown to be applicable to q-difference equations such as q-difference-q-difference

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2010 *Mathematics Subject Classification.* 37K10, 37K40, 39A13, 39A14.
Key words and phrases. Integrability; q-soliton solutions; q-difference KdV equation; q-difference-q-difference Toda equation; q-difference sine-Gordon equation; Hirota direct method.

Submitted September 3, 2015. Published October 2, 2015.

and differential-q-difference Toda equations to obtain multi-soliton solutions. We observed that the constructed solutions obey classical soliton attitudes as well as they have power counterparts for q-discrete variables. We defined such solutions as *q-soliton solutions*.

In this same vein, one can present q-analogues of several soliton equations. Instead, it is of great interest to intensify on a single soliton equation that gathers various q-discrete type of equations under one roof. The aim of this paper is to create a unifying framework for q-discrete equations and analyze the applicability of Hirota direct method to develop their multi-soliton solutions. This formalism is based on the q-discretization of equations determined by q-forward jump operator. The fundamental feature of the framework is to introduce appropriate q-deformed Hirota bilinear forms in a way that they recover continuous Hirota bilinear forms. Significantly, q-deformed Hirota bilinear forms enjoy a key position as they provide not only q-analogues of corresponding equations but also their multi-soliton solutions. For this purpose, we present a generalized q-difference soliton equation which comprises q-analogues of various soliton equations such as Toda, KdV and sine-Gordon equations. We develop its three soliton solutions by the use of Hirota direct method and we stress that the constructed solutions appear to be q-solitons. Unlike the discrete generalized Toda equation [15] whose solutions are of exponential type, this generalized q-difference equation admits soliton solutions that are expressed in terms of a polynomial in power functions. This is a consequence of non-commutativity between q-forward shift and exponential transformation.

It is possible to present q-discretization of a continuous equation in several approaches. They can be derived by the frame of q-derivative operator, or analogously as in the present article, they can be constructed by the use of q-forward jump operator. In [26], a counter-example to Hirota's theorem [14] is revealed by examining q-differential-q-difference version of Toda equation which is expressed in terms of q-derivative operator. Although it can be presented in Hirota bilinear form and satisfies the sufficient conditions to admit at least two-soliton solutions, it possesses only solitary wave like a solution determined by q-exponential function. The nonexistence of further q-exponential type of soliton solutions is a consequence of the lack of additive property of q-exponentials [25] and lack of time-independency on interaction terms. Even though Hirota direct method is applicable to q-difference equations, it fails to produce q-exponential type of multi-soliton solutions for q-differential equations governed by q-derivative operator. Accordingly, in Section 4, we introduce Δ-Hirota D-operator to analyze Δ-differential equations on arbitrary time scales. In this more general case, classical Hirota perturbation does not provide multi-soliton solutions for any difference equation studied on discrete intervals with non-constant *graininess* (e.g. q-differential equations) or on such time scales. To be more precise, here we conjecture that other than the unifying framework proposed in the present article, it is not possible to acquire another unifying approach via classical Hirota perturbation for integrable Δ-differential equations on time scales with non-constant graininess.

The current article is organized as follows: In Section 2, we present the preliminary notions regarding Hirota D-operator, q-forward jump operator and q-exponential identity which allows to convert q-discrete equations into Hirota bilinear forms. Section 3 is devoted to derive three-q-soliton solutions of the generalized q-difference soliton equation as existence of three-soliton solutions is a benchmark for integrability. In Section 4, we introduce the proper reductions on this generalized equation yielding to the q-difference-q-difference Toda, q-difference-q-difference KdV and q-difference sine-Gordon equations. We intimately demonstrate the notion of continuous limit arising in association between q-deformed and classical Hirota bilinear forms. Furthermore, we present q-soliton solutions of the considered equations explicitly, resulting from the reductions on the findings of Section 3.

2. Preliminaries

In this section, to declare the source of Hirota bilinear forms of q-discrete equations, we intend to review *q-exponential identity*, stated in [26]. For this purpose, we first present Hirota D-derivative (operator)$D : S \times S \to S$

$$\begin{aligned}&[D_x^{m_1} D_t^{m_2} \dots]\{f \cdot g\}\\ &= [(\partial_x - \partial_{x'})^{m_1} (\partial_t - \partial_{t'})^{m_2} \dots] f(x,t,\dots) \cdot g(x',t',\dots)|_{x'=x, t'=t,\dots},\end{aligned} \tag{2.1}$$

where S is a space of differentiable functions $f : \mathbb{C}^n \to \mathbb{C}$, $x, t, \dots$ are independent variables and $m_i \in \mathbb{Z}^+$, for $i \geq 1$. Indeed, such differential operator (2.1) represents a novel calculus obeying the following properties:

Proposition 2.1 ([17]). *Let $f(x,t,\dots)$ and $g(x,t,\dots)$ be differentiable functions and $P(D)$ be any polynomial in D, then*

(i) $P(D)\{f \cdot 1\} = P(\partial) f$, $P(D)\{1 \cdot f\} = P(-\partial) f$;
(ii) $P(D)\{f \cdot g\} = P(-D)\{g \cdot f\}$,

hold, where ∂ denotes the ordinary differential operator.

Notice that, Hirota D-derivative (2.1) can also be introduced by the frame of the exponential identity

$$\exp(hD_x) f(x) g(x) = f(x+h) g(x-h), \tag{2.2}$$

which is very beneficial in deriving Hirota bilinear forms of differential-difference type of equations and Bäcklund transformations [17]. Here f, g are smooth functions of x and h is a parameter.

Suppose that we have continuous one-parameter group of diffeomorphisms $\mathbb{R} \ni h \mapsto \sigma_h$, acting as forward jump operators. There is one-to-one correspondence between one-parameter group of transformations and their *infinitesimal generators*. Moreover, such diffeomorphisms are determined by exponentiation of the infinitesimal generator as they appear to be the solutions of the ordinary differential equations [23]. Accordingly, for such σ_h we have [2]

$$\sigma_h(x) = e^{h\chi(x)\partial_x} x \tag{2.3}$$

if and only if

$$e^{h\chi(x)\partial_x} f(x) = f(e^{h\chi(x)\partial_x}.x) = f(\sigma_h(x)). \tag{2.4}$$

Here h is a positive deformation parameter, $f(x)$ is a smooth function and the vector field $\chi(x)\partial_x$ is infinitesimal generator.

It is possible to suggest the infinitesimal generators of the form $\chi(x)\partial_x = x^{1-n}\partial_x$ on $\mathbb{R}$. The choice $n = 1$ yields the *forward jump operator* of lattice type

$$\sigma_h(x) = e^{h\partial_x} x = x + h \quad \Leftrightarrow \quad e^{h\partial_x} f(x) = f(x+h). \tag{2.5}$$

In addition, the choice $n = 0$ gives rise to the *q-forward jump operator*

$$\sigma_h(x) = e^{hx\partial_x}x = e^h x = qx \quad (q \equiv e^h) \quad \Leftrightarrow \quad e^{hx\partial_x}f(x) = f(qx). \tag{2.6}$$

Definition 2.2. We define the q-forward jump operator E_q acting on any smooth function $f(x)$ as

$$E_q(f(x)) := e^{hx\partial_x}f(x) = f(qx), \tag{2.7}$$

where $x \in \mathbb{R}$ and h is a deformation parameter. Similarly, q-backward jump operator is introduced to act as

$$E_q^{-1}f(x) := e^{-hx\partial_x}f(x) = f(\frac{x}{q}). \tag{2.8}$$

Proposition 2.3. *The continuous limit of q-forward jump operator E_q is*

$$\lim_{q\to 1} E_q(x) = \lim_{h\to 0}\sigma_h(x) = x. \tag{2.9}$$

Proof. The limit process can be presented by the expansion into Taylor series with respect to h near zero. That is

$$\lim_{q\to 1} E_q(x) = \lim_{h\to 0}[x + hx\partial_x(x) + \frac{h^2}{2}(x\partial_x)^2(x) + O(h^3)] = x.$$

As well as

$$\lim_{q\to 1} E_q(x) = \lim_{q\to 1} qx = x.$$

□

We emphasize that all discrete systems generated by infinitesimal generators $\chi(x)\partial_x$ are not equivalent. To be more precise, if we consider $\chi(x) = x^{1-n}$(discrete case) where $n \neq 0$ is odd, and $\chi'(x') = 1$, it is possible to find a local transformation $x' = \frac{1}{n}x^n$ which is a bijection on $\mathbb{R} - \{0\}$. Thus, the discrete systems given by $\chi(x) = x^{1-n}$ with odd n, turns out to be Toda lattice type of equations. On the other hand, if we consider $\chi(x) = x$ (q-difference case) and $\chi'(x') = 1$, we have the transformation $x = e^{x'}$ and it is not a bijection if $x \in \mathbb{R}^-$.

Therefore, there does not exist an isomorphism between lattice systems on $\mathbb{R}$ and q-difference systems on $\mathbb{R}^-$ obtained by q-forward jump operators E_q given by (2.7).

Inspired by this in-equivalence, throughout this work we present q-difference equations that are determined by q-forward jump operators E_q. In order to construct Hirota bilinear forms of q-difference type of equations, it is worthwhile to state the q-analogue of exponential identity which is given in terms of q-forward and q-backward jump operators.

Theorem 2.4 ([26]). *Let $f(x)$, $g(x)$ be continuously differentiable functions, then the q-exponential identity*

$$e^{hxD_x}f(x)g(x) = f(qx)g(\frac{x}{q}) = E_q f(x)E_q^{-1}g(x), \quad x \in \mathbb{R}, \tag{2.10}$$

holds where h and q are quantum parameters related as $q = e^h$.

For the proof of the above theorem we refer [26].

3. q-SOLITON SOLUTIONS

In this section, we propose a generalized q-difference soliton equation, namely a q-discrete analogue of Hirota-Miwa equation

$$P(D_1, D_2, D_3)\{f \cdot f\} = \sum_{i=1}^{3} \lambda_i \cosh(D_i)\{f \cdot f\} = 0, \tag{3.1}$$

where λ_i' s are arbitrary parameters, D_i' s are linear combinations of the operators xD_x, yD_y, tD_t, i.e.,

$$D_i = a_i tD_t + b_i xD_x + c_i yD_y, \quad a_i, b_i, c_i \in \mathbb{R}, \quad i = 1, 2, 3. \tag{3.2}$$

Note that (3.1) reproduces various q-discretized soliton equations, by utilizing proper identifications and reductions of parameters. The associated transformations from (3.1) into several q-difference equations are intimately demonstrated in Section 4.

Corollary 3.1. *The q-exponential identity in three variables $t, x, y \in \mathbb{R}$*

$$\exp(a_i tD_t + b_i xD_x + c_i yD_y)f(t, x, y)g(t, x, y) = f(q_i t, p_i x, r_i y)g(\frac{t}{q_i}, \frac{x}{p_i}, \frac{y}{r_i}) \tag{3.3}$$

holds, for any continuously differentiable functions f and g, equipped with the relations between quantum parameters $e^{a_i} = q_i$, $e^{b_i} = p_i$ and $e^{c_i} = r_i$, for all $i = 1, 2, 3$, respectively.

To acquire multi-q-soliton solutions of the equation (3.1), we utilize the so-called Hirota perturbation. Upon substituting the finite perturbation expansions of the dependent variable $f(t, x, y)$ around a formal perturbation parameter ε

$$f(t, x, y) = 1 + \varepsilon f^{(1)}(t, x, y) + \varepsilon^2 f^{(2)}(t, x, y) + \dots. \tag{3.4}$$

into the Hirota bilinear form (3.1), we derive

$$\begin{aligned} &P(D_1, D_2, D_3)\{f(t, x, y) \cdot f(t, x, y)\} \\ &= P(D_1, D_2, D_3)[\{1.1\} + \varepsilon\{1 \cdot f^{(1)} + f^{(1)}.1\} + \varepsilon^2\{1 \cdot f^{(2)} + f^{(2)}.1 + f^{(1)} \cdot f^{(1)}\} \\ &\quad + \varepsilon^3\{1 \cdot f^{(3)} + f^{(3)} \cdot 1 + f^{(1)} \cdot f^{(2)} + f^{(2)} \cdot f^{(1)}\} \\ &\quad + \varepsilon^4\{1 \cdot f^{(4)} + f^{(4)} \cdot 1 + f^{(1)} \cdot f^{(3)} + f^{(3)} \cdot f^{(1)} + f^{(2)} \cdot f^{(2)}\} + \dots]. \end{aligned} \tag{3.5}$$

The last step towards the method is to analyze the conditions on the coefficients of ε^i, for all $i \geq 0$ for multi-q-soliton solutions. From the coefficient of the first term ε^0, we have

$$P(D_1, D_2, D_3)\{1 \cdot 1\} = \lambda_1 + \lambda_2 + \lambda_3.$$

Theorem 3.2 ([14])**.** *Any equation in Hirota bilinear form $P(D_t, D_x, D_y)f \cdot f = 0$, satisfying the sufficient conditions*

$$P(0, 0, 0) = 0, \tag{3.6}$$

$$P(D_t, D_x, D_y) = P(-D_t, -D_x, -D_y), \tag{3.7}$$

admits at least two-soliton solutions.

To satisfy the conditions of Theorem (3.2), hereafter we need to have the constraint

$$P(0, 0, 0) = \lambda_1 + \lambda_2 + \lambda_3 = 0. \tag{3.8}$$

The coefficient of ε^1 implies

$$\begin{aligned} &P(D_1, D_2, D_3)\{1 \cdot f^{(1)} + f^{(1)}.1\} \\ &= 2P(\partial_1, \partial_2, \partial_3) f^{(1)} \\ &= 2[\sum_{i=1}^{3} \frac{\lambda_i}{2} (\exp{(a_i t \partial_t + b_i x \partial_x + c_i y \partial_y)} \exp{(-a_i t \partial_t - b_i x \partial_x - c_i y \partial_y)})] f^{(1)} \\ &= 0. \end{aligned} \tag{3.9}$$

In the literature, soliton solutions of both differential [12] or difference [15] type of equations tend to be of the exponential form. However, q-difference equations have an exclusive nature. We remark that the q-difference equation (3.9) does not admit exponential type of solutions and its solution needs to include power counterparts for the q-discrete variables, which is indeed a consequence of change of variables. Therefore, as all variables are q-discrete, the q-difference equation (3.9) admits a starting solution of the power form

$$f^{(1)}(t, x, y) = \eta t^{\alpha} x^{\beta} y^{\gamma}, \tag{3.10}$$

where α, η, β are arbitrary constants. The solutions of the form (3.10) provide *q-soliton* solutions. The notion of q-solitons are introduced in [26] as follows.

Definition 3.3. A solution possessing usual soliton behaviors and having power counterparts for q-discrete variables are called as q-soliton solution.

Substituting such solution (3.10) in (3.9), we obtain the so-called dispersion relation which determines the relation among the parameters as

$$P(v) = \sum_{i=1}^{3} \frac{\lambda_i}{2} (q_i^{\alpha} p_i^{\beta} r_i^{\gamma} + q_i^{-\alpha} p_i^{-\beta} r_i^{-\gamma}) = 0, \tag{3.11}$$

where we denote the vector $v = (\alpha, \beta, \gamma)$. The coefficient of ε^2 resulted from (3.5) can be written as

$$P(D_1, D_2, D_3)\{f^{(1)} \cdot f^{(1)}\} = -2P(\partial_1, \partial_2, \partial_3) f^{(2)}. \tag{3.12}$$

Substituting $f^{(1)}$ given in (3.10) on the left hand side of (3.12), yields as

$$P(D_1, D_2, D_3)\{f^{(1)} \cdot f^{(1)}\} = (\lambda_1 + \lambda_2 + \lambda_3) \eta^2 t^{2\alpha} x^{2\beta} y^{2\gamma},$$

which vanishes by the ansatz (3.8). Therefore, for all $j \geq 2$, we assume that $f^{(j)} = 0$. As a generalization, for i-q-soliton solution, we assume $f^{(j)} = 0$ for all $j \geq i + 1$. Setting $\varepsilon = 1$, we express the solution describing one-q-soliton as

$$f(t, x, y) = 1 + \eta t^{\alpha} x^{\beta} y^{\gamma}. \tag{3.13}$$

To obtain two-q-soliton solutions, we start with the following solution of (3.9)

$$f^{(1)} = \sum_{i=1}^{2} \eta_i t^{\alpha_i} x^{\beta_i} y^{\gamma_i},$$

where $\eta_i, \alpha_i, \beta_i$' s are constants for all $i = 1, 2$. By the constraint (3.8), the coefficient of ε^0 vanishes and the coefficient of ε^1 implies the dispersion relation

$$P(v_j) = P(\alpha_j, \beta_j, \gamma_j) = \sum_{i=1}^{3} \frac{\lambda_i}{2} (q_i^{\alpha_j} p_i^{\beta_j} r_i^{\gamma_j} + q_i^{-\alpha_j} p_i^{-\beta_j} r_i^{-\gamma_j}) = 0, \quad \forall j = 1, 2. \tag{3.14}$$

Subsequently, from the coefficient of ε^2 we derive

$$\begin{aligned} -P(\partial)f^{(2)} = \eta_1\eta_2 t^{\alpha_1+\alpha_2}x^{\beta_1+\beta_2}y^{\gamma_1+\gamma_2}\sum_{i=1}^{3}\frac{\lambda_i}{2}[q_i^{\alpha_1-\alpha_2}p_i^{\beta_1-\beta_2}r_i^{\gamma_1-\gamma_2} \\ + q_i^{\alpha_2-\alpha_1}p_i^{\beta_2-\beta_1}r_i^{\gamma_2-\gamma_1}], \end{aligned} \tag{3.15}$$

which implies that $f^{(2)}$ is of the form

$$f^{(2)} = A(1,2)\eta_1\eta_2 t^{\alpha_1+\alpha_2}x^{\beta_1+\beta_2}y^{\gamma_1+\gamma_2}. \tag{3.16}$$

Substituting such $f^{(2)}$, given in (3.16) into (3.15), we find the phase shift among two-q-solitons as

$$A(1,2) = -\frac{P(v_1-v_2)}{P(v_1+v_2)},$$

where the vector notation stands for

$$P(v_1 \pm v_2) = \sum_{i=1}^{3}\frac{\lambda_i}{2}[q_i^{\alpha_1\pm\alpha_2}p_i^{\beta_1\pm\beta_2}r_i^{\gamma_1\pm\gamma_2} + q_i^{-(\alpha_1\pm\alpha_2)}p_i^{-(\beta_1\pm\beta_2)}r_i^{-(\gamma_1\pm\gamma_2)}].$$

Because of the fact that all higher order terms of $f^{(i)}$, $i \geq 3$ vanishes and the dispersion relation (3.14) holds, the coefficient of $\varepsilon^j = 0$ for all $j \geq 3$. Therefore, two-q-solitons are given by

$$f(t,x,y) = 1+\eta_1 t^{\alpha_1}x^{\beta_1}y^{\gamma_1}+\eta_2 t^{\alpha_2}x^{\beta_2}y^{\gamma_2}+A(1,2)\eta_1\eta_2 t^{\alpha_1+\alpha_2}x^{\beta_1+\beta_2}y^{\gamma_1+\gamma_2}. \tag{3.17}$$

For three-q-soliton solutions, we begin with

$$f^{(1)} = \sum_{i=1}^{3}\eta_i t^{\alpha_i}x^{\beta_i}y^{\gamma_i},$$

where $\alpha_i, \eta_i, \beta_i$ are constants for $i = 1,2,3$. The coefficient of ε^1 enables to have a similar dispersion relation

$$P(v_j) = \sum_{i=1}^{3}\frac{\lambda_i}{2}(q_i^{\alpha_j}p_i^{\beta_j}r_i^{\gamma_j} + q_i^{-\alpha_j}p_i^{-\beta_j}r_i^{-\gamma_j}) = 0, \quad \forall j = 1,2,3. \tag{3.18}$$

Further, $f^{(2)}$ follows

$$f^{(2)} = \sum_{i<j}^{(3)} A(i,j)\eta_i\eta_j t^{\alpha_i+\alpha_j}x^{\beta_i+\beta_j}y^{\eta_i+\eta_j},$$

from the coefficient of ε^2

$$\begin{aligned} -P(\partial)f^{(2)} = \sum_{i<j}^{(3)}\eta_i\eta_j t^{\alpha_i+\alpha_j}x^{\beta_i+\beta_j}y^{\gamma_i+\gamma_j}\Big[\sum_{k=1}^{3}\frac{\lambda_k}{2}(q_k^{\alpha_i-\alpha_j}p_k^{\beta_i-\beta_j}r_k^{\gamma_i-\gamma_j} \\ + q_k^{\alpha_j-\alpha_i}p_k^{\beta_j-\beta_i}r_k^{\gamma_j-\gamma_i})\Big], \end{aligned}$$

where $\sum_{i<j}^{(3)}$ denotes the summation over all elements such that $i < j$ and $i,j = 1,2,3$. Here one can derive the associated interaction terms

$$A(j,k) = -\frac{P(v_j-v_k)}{P(v_j+v_k)}, \quad j<k, \ j,k = 1,2,3, \tag{3.19}$$

among three-q-soliton solutions. The coefficient of ε^3 yields $f^{(3)}$ as

$$f^{(3)} = A(1,2,3)\eta_1\eta_2\eta_3 t^{\alpha_1+\alpha_2+\alpha_3} x^{\beta_1+\beta_2+\beta_3} y^{\gamma_1+\gamma_2+\gamma_3},$$

where

$$\begin{aligned} A(1,2,3) = -\Big(&A(1,2)P(v_3 - v_1 - v_2) + A(1,3)P(v_2 - v_1 - v_3) \\ &+ A(2,3)P(v_1 - v_2 - v_3)\Big)/P(v_1 + v_2 + v_3). \end{aligned} \tag{3.20}$$

If the coefficient of ε^4 is under consideration, having $f^{(4)} = 0$, we encounter another expression for $A(1,2,3)$

$$A(1,2,3) = A(1,2)A(1,3)A(2,3). \tag{3.21}$$

Both expressions (3.20) and (3.21) for $A(1,2,3)$ are equivalent provided that the *three-soliton solution condition*

$$\sum_{\sigma_i=\pm 1} P\Big(\sum_{i=1}^{3} \sigma_i v_i\Big) \prod_{i<j}^{(3)} P(\sigma_i v_i - \sigma_j v_j) = 0, \quad i,j = 1,2,3, \tag{3.22}$$

is satisfied (see [17]). To sum up, to guarantee the existence of three-soliton solutions, we end up with the condition (3.22) arising as a constraint on P. Thus, we present the three-q-soliton solutions

$$\begin{aligned} f(x,t) = 1 &+ \sum_{i=1}^{3} \eta_i t^{\alpha_i} x^{\beta_i} y^{\gamma_i} + \sum_{i<j}^{(3)} A(i,j)\eta_i\eta_j t^{\alpha_i+\alpha_j} x^{\beta_i+\beta_j} y^{\gamma_i+\gamma_j} \\ &+ A(1,2)A(1,3)A(2,3)\eta_1\eta_2\eta_3 t^{\alpha_1+\alpha_2+\alpha_3} x^{\beta_1+\beta_2+\beta_3} y^{\gamma_1+\gamma_2+\gamma_3}, \end{aligned} \tag{3.23}$$

which are expressed in terms of a polynomial in power functions.

4. Special cases

In the literature, the concept of a dependent variable transformation that converts a given nonlinear partial differential or difference equation into the bilinear form is one of the significant tools. The transformed new variables, that are expressed as Wronski or Casorati type of determinants, are said to be τ-*functions* and they solve equations in bilinear forms [24, 16]. In [19], two-dimensional q-discrete Toda lattice equation is presented and the τ functions are determined by means of Wronski determinant. Two different q-deformations of KdV hierarchies are introduced in [4] and in [20] on which the notion of integrability is discussed via bi-Hamiltonian structures. In [1], it is stated that any KdV τ-function provides a q-KdV τ-function. However, to our knowledge no research has been addressed to q-discretization of sine-Gordon equation. In this section, we intend to present other q-discretized versions of Toda, KdV equations and introduce q-difference sine-Gordon equation resulted by proper reductions of parameters on the generalized q-difference soliton equation (3.1). Such reductions on the three q-soliton solutions, presented in Section 3, provide three q-soliton solutions of the considered equations in one move.

Before embarking to the details we explain the source of q-discretization as follows.

There are several approaches to q-discretize continuous equations. The q-deformed equations can be constructed by the frame of q-forward jump operator E_q given by (2.7) or they can be obtained by means of q-derivative operator ∂_q [18],

$$\partial_{q,x} f(x) = \frac{f(qx) - f(x)}{qx - x}. \tag{4.1}$$

Here f is a q-differentiable function. However, classical Hirota method fails to produce q-exponential type of multi-soliton solutions for q-differential equations with respect to ∂_q. In [26], q-analogue of Hirota D-operator acting on q-differentiable functions f, g

$$D^m_{q,x}\{f.g\} := (\partial_{q,x} - \partial_{q,x'})^m f(x) g(x^{'})|_{x'=x}, \quad m \in \mathbb{Z}^+, \tag{4.2}$$

was introduced and developed where q-deformed Hirota bilinear forms were expressed in terms of such q-Hirota D-operator. q-differential-q-difference version of Toda equation was analyzed. We note that even though it is in q-deformed Hirota bilinear form

$$\begin{aligned} &P(D_{q,t}, D_x)\{f(x,t) \cdot f(x,t)\} \\ &= [D^2_{q,t} - (\exp(hxD_x) + \exp(-hxD_x) - 2)]\{f(x,t) \cdot f(x,t)\} = 0, \end{aligned}$$

and satisfies the sufficient conditions

$$P(D_{q,t}, D_x) = P(-D_{q,t}, -D_x), \quad P(0,0) = 0,$$

it admits only solitary wave like a solution determined by q-exponential function as $f^{(1)}(x,t) = \eta x^\alpha e_q^{\beta t}$. Indeed, classical Hirota perturbation (3.4) does not provide further q-exponential type of solutions. The restrictive condition

$$wz = qzw, \quad z, w \in q^{\mathbb{Z}},$$

to satisfy additive property of q-exponentials

$$e_q^z e_q^w = e_q^{z+w}, \quad z, w \in q^{\mathbb{Z}},$$

causes to detract one of the summands in two-q-soliton solutions and the solution reduces to one-q-soliton solution. Rather than the choice $f^{(2)} = A(1,2)x^{\alpha_1+\alpha_2} e_q^{\beta_1 t + \beta_2 t}$, one can analyze the function $f^{(2)} = A(1,2)x^{\alpha_1+\alpha_2} e_q^{\beta_1 t} e_q^{\beta_2 t}$. But in this case such a choice yields to have time dependency on the interaction constant $A(1,2)$. Note that, all other admissible choices for the function $f^{(2)}$ lead similar dead-ends.

To be more general, we can analyze Hirota approach on an arbitrary time scale. For this purpose, we introduce the notion of Δ-Hirota D-operator.

Definition 4.1. Let $\mathbb{T}$ be an arbitrary time scale. Let $\mathbb{T}^\kappa$ denote Hilger's above truncated set consisting of $\mathbb{T}$ except for a possible left scattered maximal point. Let $f(x), g(x) : \mathbb{T} \mapsto \mathbb{R}$ be arbitrary functions and $x \in \mathbb{T}^\kappa$. We introduce Δ-Hirota D-operator as

$$\mathcal{D}^m_x\{f.g\} := (\Delta_x - \Delta_{x'})^m f(x).g(x^{'})|_{x'=x}, \quad m \in \mathbb{Z}^+. \tag{4.3}$$

Here delta derivative of f is defined in [3], to act as

$$\Delta_x f(x) = \lim_{s \to x} \frac{f(\sigma(x)) - f(s)}{\sigma(x) - s}, \quad x \in \mathbb{T}^\kappa, s \in \mathbb{T}, \tag{4.4}$$

where $\sigma : \mathbb{T} \mapsto \mathbb{T}$ is forward jump operator given by

$$\sigma(x) = \inf\{y \in \mathbb{T} : y > x\}, \quad x \in \mathbb{T}. \tag{4.5}$$

We note that when $\mathbb{T}=\mathbb{R}$, then $\sigma(x)=x$, $\Delta_x=\frac{d}{dx}$ and Δ-Hirota D-operator (4.3) turns out to be usual Hirota D-operator (2.1). If $\mathbb{T}=\mathbb{K}_q:=q^{\mathbb{Z}}\bigcup\{0\}$, $q\neq 1$ then $\sigma(x)=qx$, $\Delta_x=\partial_{q,x}$ and Δ-Hirota D-operator (4.3) reduces to q-analogue of Hirota D-operator (4.2).

One can make use of the Δ-Hirota D-operator (4.3) to write Δ-differential equations [7] on arbitrary time scales in Hirota bilinear form. By utilizing classical Hirota pertubation (3.4) developed on such Δ-Hirota D-operator, one can produce multi soliton solutions only when the *graininess function* $\mu:\mathbb{T}\to[0,\infty)$

$$\mu(x):=\sigma(x)-x,\quad x\in\mathbb{T}, \tag{4.6}$$

is constant (e.g. difference equations on $h\mathbb{Z}$ or q-difference equations). To be more precise, in order to derive multi soliton solutions on arbitrary time scales with arbitrary non-constant graininess μ, one should construct a deformed perturbation different than the classical Hirota perturbation (3.4). However, such an approach will be no more classical Hirota direct method.

To sum up, we conjecture that other than the unifying framework proposed in the present article, it is not possible to create another unifying approach via classical Hirota perturbation (3.4) for integrable equations on discrete intervals with non-constant graininess μ, or on arbitrary such time scales.

In this work, we present q-discretization of equations that are expressed by q-forward jump operator E_q as it recovers the continuous case. Our framework is based on the inverse procedure. The core of the procedure is to introduce appropriate q-deformed Hirota bilinear forms (resulted from proper reductions on (3.1)) in a way that they reduce to continuous Hirota bilinear forms as $q\to 1$ (equivalently in the small limit of h). In addition, from such q-deformed Hirota bilinear forms, we develop not only q-analogues of corresponding equations but also their multi-q-soliton solutions.

4.1. **The q-difference-q-difference Toda equation.** It is proposed in [26] that q-difference-q-difference Toda equation admits the Hirota bilinear form

$$\begin{aligned}&[h^{-1}(\exp(h\tau D_\tau)+\exp(-h\tau D_\tau)-2)-(\exp(\bar{h}yD_y)\\&+\exp(-\bar{h}yD_y)-2)]\{f(\tau,y)\cdot f(\tau,y)\}=0.\end{aligned} \tag{4.7}$$

In the present work, we determine (4.7) by the reductions

$$D_1=h\tau D_\tau,\quad D_2=\bar{h}yD_y,\quad D_3=0,\quad \lambda_1=2h^{-1},\quad \lambda_2=-2,\quad \lambda_3=2-2h^{-1} \tag{4.8}$$

on the generalized q-difference equation (3.1). Note that $\lambda_1+\lambda_2+\lambda_3=0$ to satisfy the condition (3.8). In order to construct the standard form of the q-difference-q-difference Toda equation, we need to introduce the following operator.

Definition 4.2. [26] *The central q-difference operator Λ_τ^2 acting on a function $f(\tau)$, is defined as*

$$\Lambda_\tau^2 f(\tau)=f(q\tau)+f(\frac{\tau}{q})-2f(\tau),\quad q\neq 1,\ \tau\in\mathbb{R}. \tag{4.9}$$

Using the inverse procedure, the standard form of the q-difference-q-difference Toda equation is proposed written by the language of (4.9)

$$\Lambda_\tau^2\log(1+V(\tau,y))=\Lambda_y^2\log(1+hV(\tau,y)), \tag{4.10}$$

which follows from the transformation

$$V(\tau,y):=h^{-1}\Big[\frac{f(q\tau,y)f(\frac{\tau}{q},y)}{f^2(\tau,y)}-1\Big]=\frac{f(\tau,\bar{q}y)f(\tau,\frac{y}{\bar{q}})}{f^2(\tau,y)}-1, \tag{4.11}$$

on the bilinear form (4.7). Here $e^h=q$, $e^{\bar{h}}=\bar{q}$.

Proposition 4.3. *One-q-soliton solution of q-difference-q-difference Toda equation* (4.10) *is*

$$V(\tau,y)=\frac{\eta\tau^\alpha y^\beta[\bar{q}^\beta+\bar{q}^{-\beta}-2]}{(1+\eta\tau^\alpha y^\beta)^2}, \tag{4.12}$$

provided that the dispersion relation

$$h^{-1}(q^\alpha+q^{-\alpha}-2)=\bar{q}^\beta+\bar{q}^{-\beta}-2, \tag{4.13}$$

is satisfied.

Proof. The identifications

$$t=\tau,\quad \gamma=0,\quad a_1=h,\quad b_2=\bar{h},\quad a_2=a_3=b_1=b_3=c_1=c_2=c_3=0 \tag{4.14}$$

resulting from the reductions (4.8) allows to obtain $f^{(1)}=\eta\tau^\alpha y^\beta$. One-$q$-soliton solution follows from straightforward calculation of using $f=1+\eta\tau^\alpha y^\beta$ on the dependent variable transformation (4.11). The identifications (4.14) on the dispersion relation (3.11) lead to derive the dispersion relation (4.13) which determines the relation among the parameters in (4.12). □

If τ, $y\in q^{\mathbb{Z}}$, namely $\tau=q^n$ and $y=\bar{q}^m$, $n,m\in\mathbb{Z}$, then q-difference-q-difference Toda equation (4.10) can be rewritten

$$\begin{aligned}&\frac{(1+V(q^{n+1},(\bar{q})^m))(1+V(q^{n-1},(\bar{q})^m))}{(1+V(q^n,(\bar{q})^m))^2}\\&=\frac{(1+hV(q^n,(\bar{q})^{m+1}))(1+hV(q^n,(\bar{q})^{m-1}))}{(1+hV(q^n,(\bar{q})^m))^2},\end{aligned} \tag{4.15}$$

whose one-q soliton solution is explicitly deduced as

$$V(\tau,y)=\frac{\eta q^{n\alpha}(\bar{q})^{m\beta}[(\bar{q})^\beta+(\bar{q})^{-\beta}-2]}{(1+\eta q^{n\alpha}(\bar{q})^{m\beta})^2}. \tag{4.16}$$

Subsequently, it is possible to rewrite two-q-soliton (3.17) and three-q-soliton solutions (3.23) using the above reductions and identifications. Therefore, by the frame of the reductions (4.8) on (3.1) we recover the results found in the Ref.[26] for q-difference-q-difference Toda equation. Moreover, it is observed that the plotted waves associated to two-q-soliton and three-q-soliton solutions of q-difference-q-difference Toda equation obey the classical soliton behaviors. Additionally, since solutions are presented by means of power functions, the lengths of waves increase as space variable increases (We refer [26] for the graphs). Therefore the solutions presented in Section 3, appear to satisfy q-soliton conditions.

Proposition 4.4. *Hirota bilinear form of the q-difference-q-difference Toda equation* (4.7) *reduces to Hirota bilinear form of the differential-q-difference Toda equation*

$$[{D_t}^2-(\exp(\bar{h}yD_y)+\exp(-\bar{h}yD_y)-2)]\{f(t,y)\cdot f(t,y)\}=0, \tag{4.17}$$

in the small limit of h.

Indeed, if we replace h by h^2 and let $\tau = \exp(ht)$, then the expression on (4.7)

$$4h^{-2}\sinh^2(\frac{h^2\tau D_\tau}{2}) = h^{-2}(\exp(hD_t) + \exp(-hD_t) - 2), \tag{4.18}$$

tends to D_t^2 as $h \to 0$. To be more precise, Hirota bilinear form (4.7), is a generalization of Hirota bilinear form (4.17), from which we establish the standard form of the q-discretized Toda equation (4.10).

4.2. **The q-difference-q-difference KdV equation.** We propose the q-difference-q-difference KdV equation in Hirota bilinear form

$$\sinh(\frac{h^2\tau D_\tau + \bar{h}^2 yD_y}{2})[h^{-1}\sinh(h^2\tau D_\tau) + 2\sinh(\bar{h}^2 yD_y)]\{f(\tau,y)\cdot f(\tau,y)\} = 0, \tag{4.19}$$

that can be identified by the below reductions

$$D_1 = \frac{1}{2}(3h^2\tau D_\tau + \bar{h}^2 yD_y), \quad D_2 = \frac{1}{2}(h^2\tau D_\tau + 3\bar{h}^2 yD_y), D_3 = \frac{1}{2}(h^2\tau D_\tau - \bar{h}^2 yD_y), \tag{4.20}$$

$$\lambda_1 = 1, \quad \lambda_2 = 2h, \quad \lambda_3 = -1 - 2h, \tag{4.21}$$

on (3.1).

Proposition 4.5. *Hirota bilinear form of q-difference-q-difference KdV equation* (4.19) *reduces to Hirota bilinear form of the continuous KdV equation* [12]

$$[D_x(D_t + D_x^3)]\{f(t,x)\cdot f(t,x)\} = 0, \tag{4.22}$$

as h, $\bar{h}$ tends to zero.

Proof. Setting $\tau = \exp(ht)$, Hirota bilinear form of q-difference-q-difference KdV equation (4.19) turns out to be

$$\sinh(\frac{hD_t + \bar{h}^2 yD_y}{2})[h^{-1}\sinh(hD_t) + 2\sinh(\bar{h}^2 yD_y)]\{f\cdot f\} = 0. \tag{4.23}$$

One can verify that as $h \to 0$, the equation (4.23) reduces to

$$\sinh(\frac{\bar{h}^2 yD_y}{2})[D_t + 2\sinh(\bar{h}^2 yD_y)]\{f(t,y)\cdot f(t,y)\} = 0. \tag{4.24}$$

By a similar fashion, we set $y = \exp(\bar{h}x)$ which implies (4.24) as

$$\sinh(\frac{\bar{h}D_x}{2})[D_t + 2\sinh(\bar{h}D_x)]\{f(t,x)\cdot f(t,x)\} = 0. \tag{4.25}$$

As a final step, we fix $D_t = \frac{\bar{h}^3 D_t}{3} - 2\bar{h}D_x$ in (4.25) and we derive

$$\sinh(\frac{\bar{h}D_x}{2})[\frac{\bar{h}^3 D_t}{3} - 2\bar{h}D_x + 2\sinh(\bar{h}D_x)]\{f\cdot f\} = 0. \tag{4.26}$$

Dividing (4.26) with $\bar{h}^4$ and taking the small limit of $\bar{h}$, we end up with the continuous KdV equation in Hirota bilinear form (4.22). □

To be more precise, q-deformed Hirota bilinear form (4.19) is a generalization of Hirota bilinear form (4.22) of usual KdV equation.

4.3. **The standard form of q-difference-q-difference KdV equation.** In this section, we intend to introduce the standard form of the q-difference-q-difference KdV equation. Before embarking to the details we list some useful identities for hyperbolic functions.

Properties: [17] For continuously differentiable functions f, g, the following identities hold

$$\begin{aligned} &\sinh\frac{1}{2}(D_2-D_3)\{\sinh\frac{1}{2}(D_2+D_3)\cdot 2\sinh D_1 f\cdot f\} \\ &.\{\cosh(\frac{1}{2}(D_2+D_3)-D_1)f\cdot f\} \\ &=\sinh D_1\{\cosh D_2 f\cdot f\}\cdot\{\cosh D_3 f\cdot f\}, \end{aligned} \tag{4.27}$$

$$\cosh D_1\{\cosh D_2 f\cdot f\}\cdot\{\cosh D_2 f\cdot f\}=\cosh D_2\{\cosh D_1 f\cdot f\}.\{\cosh D_1 f\cdot f\}, \tag{4.28}$$

$$\exp(\alpha\partial_1)(\frac{f}{g})=exp(\alpha D_1)\{f\}.\{\frac{g}{\cosh(\alpha D_1)g.g}\}, \tag{4.29}$$

where the operators D_i are of the form (3.2) and α is an arbitrary constant.

Definition 4.6. The q-difference operator δ_τ, operating on an arbitrary function $u(\tau)$, is defined as follows

$$\delta_\tau u(\tau):=u(q\tau)-u(\frac{\tau}{q}),\quad \tau\in\mathbb{R},\quad q\neq 1. \tag{4.30}$$

We stress that the operators Λ^2, defined by (4.9) and δ^2 and are equivalent.

Proposition 4.7. *The standard form of the q-difference-q-difference KdV equation is given as*

$$\delta_\tau(\frac{1}{V(\tau,y)})=-2h^{1/2}\delta_y V(\tau,y), \tag{4.31}$$

provided that the dependent variable transformation

$$V(\tau,y):=-\frac{f(\tau,\bar{q}y)f(\tau,\frac{y}{\bar{q}})}{f(q\tau,y)f(\frac{\tau}{q},y)}, \tag{4.32}$$

holds, where $e^h=q$ *and* $e^{\bar{h}}=\bar{q}$.

Proof. We first adopt an equivalent version of Hirota bilinear form (4.19) of q-difference-q-difference KdV equation

$$\begin{aligned} &\sinh(\frac{h\tau D_\tau-\bar{h}yD_y}{2})\{\sinh(\frac{h\tau D_\tau+\bar{h}yD_y}{2})[h^{-1/2}\sinh(h\tau D_\tau) \\ &+2\sinh(\bar{h}yD_y)]f\cdot f\}\cdot\{\cosh(\frac{h\tau D_\tau-\bar{h}yD_y}{2})f\cdot f\}=0, \end{aligned} \tag{4.33}$$

where for convenience we interchange h^2 and $\bar{h}^2$ by h and $\bar{h}$, respectively. Using the identity (4.27), the bilinear form (4.33) can be rewritten in the form

$$\begin{aligned} &h^{-1/2}\sinh(h\tau D_\tau)\{\cosh(h\tau D_\tau)f\cdot f\}.\{\cosh(\bar{h}yD_y)f\cdot f\} \\ &=-2\sinh(\bar{h}yD_y)\{\cosh(h\tau D_\tau)f\cdot f\}.\{\cosh(\bar{h}yD_y)f\cdot f\}. \end{aligned} \tag{4.34}$$

Upon dividing the form (4.34) by (4.28) equipped with $D_1 = h\tau D_\tau$, $D_2 = \bar{h}yD_y$, we obtain

$$\begin{aligned}&\frac{h^{-1/2}\sinh(h\tau D_\tau)(\cosh(h\tau D_\tau)f\cdot f).(\cosh(\bar{h}yD_y)f\cdot f)}{\cosh(h\tau D_\tau)(\cosh(\bar{h}yD_y)f\cdot f)\cdot(\cosh(\bar{h}yD_y)f\cdot f)}\\&=\frac{-2\sinh(\bar{h}yD_y)(\cosh(h\tau D_\tau)f\cdot f).(\cosh(\bar{h}yD_y)f\cdot f)}{\cosh(\bar{h}yD_y)(\cosh(h\tau D_\tau f\cdot f))\cdot(\cosh(h\tau D_\tau)f\cdot f)},\end{aligned}$$

which takes the form

$$h^{-1/2}\sinh(h\tau\partial_\tau)(\frac{\cosh(h\tau D_\tau)f\cdot f}{\cosh(\bar{h}yD_y)f\cdot f}) = -2\sinh(\bar{h}y\partial_y)(\frac{\cosh(\bar{h}yD_y)f\cdot f}{\cosh(h\tau D_\tau)f\cdot f}), \quad (4.35)$$

Here we make use of the identity (4.29). Armed with the q-exponential identity (3.3), by the frame of $e^h = q$ and $e^{\bar{h}} = \bar{q}$, then (4.35) boils down to

$$\begin{aligned}&[\frac{f(q^2\tau,y)f(\tau,y)}{f(q\tau,\bar{q}y)\cdot f(q\tau,\frac{y}{\bar{q}})} - \frac{f(\tau,y)f(\frac{\tau}{q^2},y)}{f(\frac{\tau}{q},\bar{q}y)\cdot f(\frac{\tau}{q},\frac{y}{\bar{q}})}]\\&= 2h^{1/2}[\frac{f(\tau,y)f(\tau,\frac{y}{\bar{q}^2})}{f(q\tau,\frac{y}{\bar{q}})\cdot f(\frac{\tau}{q},\frac{y}{\bar{q}})} - \frac{f(\tau,\bar{q}^2y)f(\tau,y)}{f(q\tau,\bar{q}y)\cdot f(\frac{\tau}{q},\bar{q}y)}].\end{aligned} \quad (4.36)$$

Utilizing the dependent variable transformation (4.32), we rewrite the equation (4.36) as (4.31) in the language of q-difference operator (4.30). □

Proposition 4.8. *The q-difference-q-difference KdV equation* (4.31) *admits one-q-soliton solution*

$$V(\tau,y) = -\frac{[1+\eta\tau^\alpha(\bar{q}y)^\beta + \eta\tau^\alpha(\bar{q})^{-\beta}y^\beta + \eta^2\tau^{2\alpha}y^{2\beta}]}{(1+\eta(q\tau)^\alpha y^\beta)(1+\eta\tau^\alpha(q)^{-\alpha}y^\beta)}, \quad (4.37)$$

where the dispersion relation is

$$(\bar{q})^{\frac{\beta}{2}}[q^{\frac{3\alpha}{2}}-q^{-\frac{\alpha}{2}}]+(\bar{q})^{\frac{-\beta}{2}}[q^{\frac{-3\alpha}{2}}-q^{\frac{\alpha}{2}}]+2h\{q^{\frac{\alpha}{2}}[(\bar{q})^{\frac{3\beta}{2}}-(\bar{q})^{-\frac{\beta}{2}}]+q^{\frac{-\alpha}{2}}[(\bar{q})^{\frac{-3\beta}{2}}-(\bar{q})^{\frac{\beta}{2}}]\} = 0. \quad (4.38)$$

Proof. Using the identifications

$$t=\tau, \quad \gamma=0, \quad a_1=\frac{3h}{2}, \quad b_1=\frac{\bar{h}}{2}, \quad (4.39)$$

$$a_2=a_3=\frac{h}{2}, \quad b_2=\frac{3\bar{h}}{2}, \quad b_3=-\frac{\bar{h}}{2}, \quad c_1=c_2=c_3=0, \quad (4.40)$$

resulting from the reductions (4.20), (4.21) we obtain $f^{(1)} = \eta\tau^\alpha y^\beta$. One can find one-$q$-soliton solution (4.37) using $f = 1+\eta\tau^\alpha y^\beta$ on the transformation (4.32). Similarly, using (4.39), (4.40) on the dispersion relation (3.11) yields the dispersion relation (4.38). □

If τ, $y \in q^{\mathbb{Z}}$, i.e., $\tau = q^n$ and $y = \bar{q}^m$, $n, m \in \mathbb{Z}$, then one-q-soliton solution (4.37) turns out to be

$$V = -\frac{[1+\eta q^{n\alpha}(\bar{q})^{\beta(m+1)} + \eta q^{n\alpha}(\bar{q})^{\beta(m-1)} + \eta^2\tau^{2n\alpha}(\bar{q})^{2m\beta}]}{(1+\eta(q)^{\alpha(n+1)}(\bar{q})^{m\beta})(1+\eta q^{\alpha(n-1)}(\bar{q})^{m\beta})}. \quad (4.41)$$

Furthermore, one can explicitly present two and three-q-soliton solutions using the reductions on (3.17) and (3.23), respectively.

4.4. **The q-difference sine-Gordon Equation.** We propose Hirota bilinear form of q-difference sine-Gordon equation

$$\begin{aligned}&[2\sinh(\bar{h}^2 yD_y)\sinh(h^2\tau D_\tau)h\bar{h}\cosh(h^2\tau D_\tau+\bar{h}^2 yD_y+kzDz)\\&-h\bar{h}\cosh(h^2\tau D_\tau-\bar{h}^2 yD_y)]\{f(\tau,y,z)\cdot f(\tau,y,z)\}=0,\end{aligned}\tag{4.42}$$

which follows from the reductions below on the equation (3.1),

$$\begin{gathered}D_1=h^2\tau D_\tau+\bar{h}^2 yD_y,\\D_2=h^2\tau D_\tau+\bar{h}^2 yD_y+kzDz,\\D_3=h^2\tau D_\tau-\bar{h}^2 yD_y,\end{gathered}\tag{4.43}$$

$$\lambda_1=1,\quad \lambda_2=h\bar{h},\quad \lambda_3=-1-h\bar{h}.\tag{4.44}$$

To present the source of the limits which reveals Hirota Bilinear form of classical sine-Gordon equation [13], we need to rewrite (4.42) in a proper decomposition. For this purpose, we adopt the *periodicity* definition on q-numbers.

Definition 4.9. A function $f(x)$ is said to be q^n-periodic if

$$f(q^n x)=f(x),\quad q>1,\ n\in\mathbb{Z},\ x\in\mathbb{K}_q.\tag{4.45}$$

Proposition 4.10. *Hirota bilinear form of q-difference sine-Gordon equation* (4.42) *reduces to Hirota bilinear form of the continuous sine-Gordon equation* [13]

$$D_xD_t\{\bar{g}.\bar{f}\}=\bar{g}.\bar{f},\tag{4.46}$$

$$D_xD_t\{\bar{f}.\bar{f}-\bar{g}.\bar{g}\}=0,\tag{4.47}$$

in the small limit of h and $\bar{h}$.

Proof. Assume that f in (4.42) is $\bar{q}^2$-periodic function equipped with

$$f(\bar{q}z)=f(\frac{z}{\bar{q}}),$$

then the discussion includes the function f and its q-shifted version, say $\tilde{f}$, i.e. $e^{kz\partial_z}f(z)=f(\bar{q}z):=\tilde{f}(z)$, provided that $e^k=\bar{q}$. Such periodicity condition on (4.42) implies that

$$\begin{aligned}&[(h\bar{h})^{-1}\sinh(\bar{h}^2 yD_y)\sinh(h^2\tau D_\tau)-\frac{1}{2}\cosh(h^2\tau D_\tau-\bar{h}^2 yD_y)]\{f(\tau,y,z)\cdot f(\tau,y,z)\}\\&=-\frac{1}{2}\cosh(h^2\tau D_\tau+\bar{h}^2 yD_y)\{\tilde{f}(\tau,y,z).\tilde{f}(\tau,y,z)\},\end{aligned}$$

provided that $h\bar{h}\neq 0$. Equivalently one can encounter the bilinear form

$$\begin{aligned}&[(h\bar{h})^{-1}\sinh(\bar{h}^2 yD_y)\sinh(h^2\tau D_\tau)]\{f\cdot f\}\\&=\frac{1}{2}\cosh(\bar{h}^2 yD_y)\cosh(h^2\tau D_\tau)]\{f\cdot f-\tilde{f}.\tilde{f}\},\end{aligned}\tag{4.48}$$

whose continuous limit is intimately related to classical sine-Gordon equation. Setting

$$f=\bar{f}+i\bar{g}\quad \tilde{f}=\bar{f}-i\bar{g},\tag{4.49}$$

on (4.48), we have

$$\begin{aligned}&[(h\bar{h})^{-1}\sinh(\bar{h}^2 yD_y)\sinh(h^2\tau D_\tau)]\{\bar{g}\cdot\bar{f}\}\\&=\cosh(\bar{h}^2 yD_y)\cosh(h^2\tau D_\tau)]\{\bar{g}\cdot\bar{f}\}\end{aligned}\tag{4.50}$$

and

$$\sinh(\bar{h}^2 yD_y)\sinh(h^2\tau D_\tau)\{\bar{f}.\bar{f}-\bar{g}.\bar{g}\}=0. \tag{4.51}$$

Let $y=e^{\bar{h}x}$ and $\tau=e^{ht}$, then Hirota bilinear form (4.50), (4.51) turns out to be Hirota bilinear form of sine-Gordon equation (4.46), (4.47) respectively, in the limit $h,\bar{h}\to 0$. □

4.5. **Standard form of q-difference sine-Gordon equation.** To establish the ordinary form of q-difference sine-Gordon equation, we make use of its decomposed bilinear form (4.50), (4.51) on which for the sake of convenience we interchange h^2 and $\bar{h}^2$ by h and $\bar{h}$, respectively as

$$[(h\bar{h})^{-1/2}\sinh(\bar{h}yD_y)\sinh(h\tau D_\tau)]\{\bar{g}.\bar{f}\}=\cosh(\bar{h}yD_y)\cosh(h\tau D_\tau)]\{\bar{g}.\bar{f}\}, \tag{4.52}$$

$$\sinh(\bar{h}yD_y)\sinh(h\tau D_\tau)\{\bar{f}.\bar{f}-\bar{g}.\bar{g}\}=0. \tag{4.53}$$

The restrictions (4.49) imply to assume $\bar{f}:=\exp(\rho).\cos(\phi)$, $\bar{g}:=\exp(\rho).\sin(\phi)$. Then we rewrite (4.52) as

$$\begin{aligned}&(1-(h\bar{h})^{1/2})\exp(\rho(q\tau,py)+\rho(\frac{\tau}{q},\frac{y}{p})).\sin(\phi(q\tau,py)+\phi(\frac{\tau}{q},\frac{y}{p}))\\&=(1+(h\bar{h})^{1/2})\exp(\rho(q\tau,\frac{y}{p})+\rho(\frac{\tau}{q},py)).\sin(\phi(q\tau,\frac{y}{p})+\phi(\frac{\tau}{q},py)),\end{aligned}$$

and (4.53) as

$$\begin{aligned}&\exp(\rho(q\tau,py)+\rho(\frac{\tau}{q},\frac{y}{p})).\cos(\phi(q\tau,py)+\phi(\frac{\tau}{q},\frac{y}{p}))\\&=\exp(\rho(q\tau,\frac{y}{p})+\rho(\frac{\tau}{q},py)).\cos(\phi(q\tau,\frac{y}{p})+\phi(\frac{\tau}{q},py)),\end{aligned}$$

respectively by the frame of $e^h=q$ and $e^{\bar{h}}=p$. Solving the resulting equation for ϕ, we derive

$$\begin{aligned}&\sin(\phi(q\tau,py)+\phi(\frac{\tau}{q},\frac{y}{p})-\phi(q\tau,\frac{y}{p})-\phi(\frac{\tau}{q},py))\\&=(h\bar{h})^{1/2}\sin(\phi(q\tau,py)+\phi(\frac{\tau}{q},\frac{y}{p})+\phi(q\tau,\frac{y}{p})+\phi(\frac{\tau}{q},py)).\end{aligned} \tag{4.54}$$

Definition 4.11. The q-sum operator Γ_τ, operating on any function $u(\tau)$ is defined as

$$\Gamma_\tau u(\tau):=u(q\tau)+u(\frac{\tau}{q}),\quad \tau\in\mathbb{R},\ q\neq 1. \tag{4.55}$$

We introduce the standard form of the q-difference sine-Gordon equation as

$$\sin[\delta_y\delta_\tau\phi(\tau,y)]=(h\bar{h})^{1/2}\sin[\Gamma_y\Gamma_\tau\phi(\tau,y)], \tag{4.56}$$

by rewriting (4.54) in the frame of q-sum operator (4.55) and q-difference operator (4.30).

Proposition 4.12. *The q-difference sine-Gordon equation* (4.56) *admits one-q-soliton solution*

$$\phi=4\tan^{-1}(\eta\tau^\alpha y^\beta z^\gamma), \tag{4.57}$$

provided that the dispersion relation

$$(q^\alpha-q^{-\alpha})(p^\beta-p^{-\beta})+h\bar{h}[q^\alpha(p^\beta(\bar{q})^\gamma-p^{-\beta})+q^{-\alpha}(p^{-\beta}(\bar{q})^{-\gamma}-p^\beta)]=0, \tag{4.58}$$

is satisfied.

Proof. The identifications

$$t=\tau,\quad a_1=a_2=a_3=h,\quad b_1=b_2=\bar{h}, \tag{4.59}$$

$$c_2=k\quad b_3=-\bar{h},\quad c_1=c_3=0, \tag{4.60}$$

resulting from the reductions (4.43), (4.44) leads the starting solution of $f^{(1)}=\eta\tau^{\alpha}y^{\beta}z^{\gamma}$. If we expand $\bar{f}$ and $\bar{g}$ in series of ε,

$$\bar{f}=1+\varepsilon^2 f^{(2)}+\varepsilon^4 f^{(4)}+\dots$$

$$\bar{g}=\varepsilon f^{(1)}+\varepsilon^3 f^{(3)}+\dots$$

it is possible to write $\bar{g}/\bar{f}=f^{(1)}$, equipped with $\varepsilon=1$. To be more intense, we derive one-q-soliton solution

$$\phi=4\arctan(\frac{\bar{g}}{\bar{f}})=4\arctan(\eta\tau^{\alpha}y^{\beta}z^{\gamma}).$$

Further, the dispersion relation (4.58) results from the reductions (4.59), (4.60) on the dispersion relation (3.11). □

In the same way, it is possible to rewrite two-q-soliton (3.17) and three-q-soliton solutions (3.23) using the above reductions and identifications.

Conclusion. In the present article, we have introduced a general unifying framework for integrable q-discrete equations and their multi-soliton solutions. We presented a generalized q-difference equation, which reproduces q-discretized soliton equations such as Toda, KdV and sine-Gordon equations by proper transformations. We showed that Hirota direct method produces three-q-soliton solutions of this generalized q-difference equation. However, the classical method fails to construct multi-soliton solutions for Δ-differential equations on arbitrary time scales with non-constant graininess. We plan to devote our next investigation to concerning Wronskian technique to construct soliton and other special (*e.g.* complexiton) solutions of bilinear equations on arbitrary time scales. Such a work is in progress and will be communicated in a separate paper.

Acknowledgements. The authors are grateful to Prof. Maciej Błaszak for useful discussions. This work is partially supported by the Scientific and Technological Research Council of Turkey (TÜBİTAK).

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