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BOUNDARY VALUE PROBLEMS WITH AN INTEGRAL CONSTRAINT

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Abstract. We show the existence of solutions for a second-order ordinary differential equation coupled with a boundary-value condition and an integral condition.

1. Introduction and preliminaries

Ordinary differential equations are usually associated with further conditions, such as prescribed initial or boundary values, periodicity etc. (see e.g. [1, 5]). Apart from the previous ones, other classes of problems where ordinary differential equations are coupled with more elaborate conditions have been studied (see e.g. [2, 6, 7, 8, 9, 10] and the references quoted in the survey [4]). The aim in these cases is usually to get existence and uniqueness results from the assigned conditions.

In the present note we investigate a problem in which a general ordinary differential equation of the second order is coupled with a boundary quasi-linearity condition and an integral condition. More precisely we consider the problem

$$y''(x) = f(x, y(x), y'(x)) \quad a \le x \le b \tag{1.1}$$

$$\alpha y(a) + \beta y(b) = \gamma, \tag{1.2}$$

$$\int_a^b y'(t)^2 dt = \delta, \tag{1.3}$$

where $-\infty < a < b < +\infty$, $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ with $\delta > 0$ and $f : [a, b] \times \mathbb{R}^2 \to \mathbb{R}$ is a continuous function such that

- (H1) There exist $\sigma_1, \sigma_2 \in \mathbb{R}$ such that $0 < \sigma_1 \le f(x, u, v) \le \sigma_2$ for all $(x, u, v) \in [a, b] \times \mathbb{R}^2$;
- (H2) there exists $L_f > 0$ such that for all $x \in [a, b]$ and all $(u_1, v_1), (u_2, v_2) \in \mathbb{R}^2$, we have
$$|f(x, u_1, v_1) - f(x, u_2, v_2)| \le L_f(|u_1 - u_2| + |v_1 - v_2|).$$

We will prove that, under some additional condition on f, problem (1.1)–(1.3) has at least two solutions if $\alpha+\beta \ne 0$, while it has at least one solution if $\alpha+\beta = 0$. The main tool will be the classical Shaefer's Fixed Point Theorem (see e.g. [3, Chapter 9]):

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Theorem 1.1 (Schaefer's theorem)**.** *Let T be a continuous and compact mapping of a Banach space X into itself, such that the set*

$$\{x \in X : x = \lambda T x \text{ for some } 0 \le \lambda \le 1\}$$

is bounded. Then T has a fixed point.

We start with some preliminary observations.

Lemma 1.2. *If $y \in C^2[a,b]$ satisfies (1.1) and (1.3), then*

$$\begin{aligned}\Delta(y) &:= \Big[\int_a^b \int_a^t f(t, y(s), y'(s)) ds dt\Big]^2 \\ &\quad - (b-a)\Big[\int_a^b \Big[\int_a^t f(t, y(s), y'(s)) ds\Big]^2 dt - \delta\Big] \ge 0\end{aligned} \tag{1.4}$$

and either for every $x \in [a,b]$,

$$\begin{aligned}y(x) &= y(a) - \frac{\int_a^b \int_a^t f(s, y(t), y'(s))\, ds\, dt + \sqrt{\Delta(y)}}{b-a}(x-a) \\ &\quad + \int_a^x \int_a^t f(s, y(s), y'(s))\, ds\, dt\,,\end{aligned}$$

or for every $x \in [a,b]$,

$$\begin{aligned}y(x) &= y(a) - \frac{\int_a^b \int_a^t f(s, y(s), y'(s))\, ds\, dt - \sqrt{\Delta(y)}}{b-a}(x-a) \\ &\quad + \int_a^x \int_a^t f(s, y(s), y'(s))\, ds\, dt\,.\end{aligned}$$

Proof. By integrating (1.1), we obtain

$$y'(x)^2 = y'(a)^2 + 2y'(a)\int_a^x f(t, y(t), y'(t)) dt + \Big[\int_a^x f(t, y(t), y'(t)) dt\Big]^2.$$

Hence, by (1.3), $y'(a)$ is a solution of the equation

$$\begin{aligned}&z^2(b-a) + 2z\int_a^b \int_a^t f(s, y(s), y'(s))\, ds\, dt \\ &+ \int_a^b \Big[\int_a^t f(s, y(s), y'(s)) ds\Big]^2 dt - \delta = 0.\end{aligned} \tag{1.5}$$

Therefore,

$$\begin{aligned}\Delta(y) &:= \Big[\int_a^b \int_a^t f(s, y(s), y'(s)) ds dt\Big]^2 \\ &\quad - (b-a)\Big[\int_a^b \Big[\int_a^t f(s, y(s), y'(s)) ds\Big]^2 dt - \delta\Big] \ge 0\end{aligned}$$

and

$$y'(a) = \frac{-\int_a^b \int_a^t f(s, y(s), y'(s))\, ds\, dt \pm \sqrt{\Delta(y)}}{b-a}. \tag{1.6}$$

We obtain the assertion by observing that, if $y \in C^2[a,b]$ is a solution of (1.1), then

$$y(x) = y(a) + y'(a)(x-a) + \int_a^x \int_a^t f(s, y(s), y'(s))\, ds\, dt. \tag{1.7}$$

$\square$

2. The case $\alpha + \beta = 0$

If $\alpha + \beta = 0$, $\alpha \neq 0$ and $y \in C^2[a, b]$ is a solution of (1.1)–(1.3), then by (1.7),

$$y(b) = y(a) + y'(a)(b-a) + \int_a^b \int_a^t f(s, y(s), y'(s))ds$$

and therefore, by (1.2),

$$y'(a) = -\frac{\gamma}{\alpha(b-a)} - \frac{1}{b-a}\int_a^b \int_a^t f(s, y(s), y'(s))ds. \tag{2.1}$$

But $y'(a)$ solves (1.5), therefore comparing (1.6) and (2.1), we get that $\Delta(y) = \gamma = 0$ and

$$y'(a) = -\frac{1}{b-a}\int_a^b \int_a^t f(s, y(s), y'(s))ds.$$

Thus problem (1.1)–(1.3) turns into

$$y''(x) = f(x, y(x), y'(x)), \quad a \le x \le b, \tag{2.2}$$

$$y(a) = y(b), \tag{2.3}$$

$$\int_a^b y'(t)^2 dt = \delta. \tag{2.4}$$

Integrating by parts (2.4), we find that

$$\begin{aligned}
\delta &= \int_a^b y'(x)^2 dx \\
&= y(b)y'(b) - y(a)y'(a) - \int_a^b y(s)f(s, y(s), y'(s))ds \\
&= y(a)(y'(b) - y'(a)) - \int_a^b y(s)f(s, y(s), y'(s)ds \\
&= y(a) \cdot \int_a^b f(s, y(s), y'(s))ds - \int_a^b y(s)f(s, y(s), y'(s))ds.
\end{aligned}$$

Hence

$$y(a) \cdot \int_a^b f(s, y(s), y'(s))ds = \delta + \int_a^b y(s)f(s, y(s), y'(s))ds$$

Assuming (H1), we obtain

$$\int_a^b f(s, y(s), y'(s))ds \ge \sigma_1(b-a) > 0.$$

Therefore

$$y(a) = \frac{\delta + \int_a^b y(s)f(s, y(s), y'(s))ds}{\int_a^b f(s, y(s), y'(s))ds}$$

and

$$\begin{aligned}
y(x) = \frac{\delta + \int_a^b y(s)f(s, y(s), y'(s))ds}{\int_a^b f(s, y(s), y'(s))ds} - \frac{x-a}{b-a}\int_a^b \int_a^t f(s, y(s), y'(s))ds \\
+ \int_a^x \int_a^t f(s, y(s), y'(s))\, ds\, dt.
\end{aligned}$$

As a consequence, we easily obtain the following characterization of the solutions of (2.2)–(2.4).

Lemma 2.1. *Assume that* (H1) *holds. Then* $y \in C^2[a,b]$ *is a solution of* (2.2)–(2.4) *if and only if* $y \in C^1[a,b]$ *and for every* $x \in [a,b]$

$$\begin{aligned} y(x) = \frac{\delta + \int_a^b y(s)f(s,y(s),y'(s))ds}{\int_a^b f(s,y(s),y'(s))ds} - \frac{x-a}{b-a}\int_a^b \int_a^t f(s,y(s),y'(s))ds \\ + \int_a^x \int_a^t f(s,y(s),y'(s))\,ds\,dt. \end{aligned}$$

We consider now the following assumptions:

(H1') There exists $\sigma_3 > 0$ such that $|uf(x,u,v)| \le \sigma_3$ for all $(x,u,v) \in [a,b]\times\mathbb{R}^2$.

(H2)' There exists $L > 0$ such that fore all $x \in [a,b]$ and all $(u_1,v_1),(u_2,v_2) \in \mathbb{R}^2$,

$$|u_1 f(x,u_1,v_1) - u_2 f(x,u_2,v_2)| \le L(|u_1-u_2| + |v_1-v_2|).$$

Theorem 2.2. *If* (H1), (H2), (H1'), (H2') *hold, then there exists at least one solution of* (2.2)–(2.4).

Proof. Consider the map $T : C^1[a,b] \to C^1[a,b]$ defined by

$$\begin{aligned} Ty(x) = \frac{\delta + \int_a^b y(s)f(s,y(s),y'(s))ds}{\int_a^b f(s,y(s),y'(s))ds} + \frac{x-a}{b-a}\int_a^b \int_a^t f(s,y(s),y'(s))ds \\ + \int_a^x \int_a^t f(s,y(s),y'(s))\,ds\,dt. \end{aligned}$$

By Lemma 2.1, a function $y \in C^2[a,b]$ is a solution of (2.2)–(2.4) if and only if $y \in C^1[a,b]$ is a fixed point of T.

Observe that for every $y \in C^1[a,b]$ and every $x \in [a,b]$:

$$(Ty)'(x) = \frac{1}{b-a}\int_a^b \int_a^t f(s,y(s),y'(s))ds + \int_a^x f(s,y(s),y'(s))ds$$
$$(Ty)''(x) = f(x,y(x),y'(x)),$$

hence for every $x \in [a,b]$,

$$|Ty(x)| \le \frac{\delta + \sigma_3(b-a)}{\sigma_1(b-a)} + \sigma_2(b-a)^2 \tag{2.5}$$

$$|(Ty)'(x)| \le \frac{3}{2}(b-a)\sigma_2 \tag{2.6}$$

$$|(Ty)''(x)| \le \sigma_2. \tag{2.7}$$

Moreover, for every $y,z \in C^1[a,b]$, $x \in [a,b]$:

$$\begin{aligned} &|Ty(x) - Tz(x)| \\ &\le \Big|\Big(\Big(\delta + \int_a^b y(s)f(s,y(s),y'(s))ds\Big)\int_a^b f(s,z(s),z'(s))ds \\ &\quad - \Big(\delta + \int_a^b z(s)f(s,z(s),z'(s))ds\Big)\int_a^b f(s,y(s),y'(s))ds\Big) \\ &\quad \div \Big(\int_a^b f(s,y(s),y'(s))ds \int_a^b f(s,z(s),z'(s))ds\Big)\Big| + L_f(b-a)^2\|y-z\|_{C^1} \end{aligned}$$

$$\le \frac{\delta L_f}{(b-a)\sigma_1^2}\|y-z\|_{C^1} + \frac{L_f\sigma_3 + L\sigma_2}{\sigma_1^2}\|y-z\|_{C^1} + L_f(b-a)^2\|y-z\|_{C^1}$$

and

$$|(Ty)'(x) - (Tz)'(x)| \le \frac{3}{2}L_f(b-a)\cdot\|y-z\|_{C^1},$$

hence

$$\|Ty - Tz\|_{C^1} \le \Big[\frac{L_f\sigma_3 + L\sigma_2}{\sigma_1^2} + L_f\Big(\frac{3}{2}(b-a) + (b-a)^2 + \frac{\delta}{(b-a)\sigma_1^2}\Big)\Big]\|y-z\|_{C_1}.$$

Thus T is continuous on $C^1[a,b]$.

We prove that T is also compact. Let (y_n) be a sequence in $C^1[a,b]$. Then $((Ty_n)')_n$ is a bounded sequence of continuous functions such that $((Ty_n)'')_n$ is also bounded. By Ascoli-Arzelà's theorem there exists a subsequence $(y_{k_n})_n$ such that $((Ty_{k_n})')_n$ is uniformly convergent on $[a,b]$. On the other hand, $(Ty_n)_n$ is a bounded sequence in $C^1[a,b]$, hence, passing to a subsequence if necessary, we can assume that $(Ty_{k_n}(a))_n$ is convergent. It follows that $(Ty_{k_n})_n$ converges in $C^1[a,b]$.

Thus, observing that the set

$$\{y \in C^1[a,b] : y = \lambda Ty \text{ for some } \lambda \in [0,1]\}$$

is clearly bounded by (2.5) and (2.6), by Schaefer's theorem, T has a fixed point. □

An immediate application of the Schauder's Fixed Point Theorem gives the following result.

Corollary 2.3. *If*

$$\frac{L_f\sigma_3 + L\sigma_2}{\sigma_1^2} + L_f\Big(\frac{3}{2}(b-a) + (b-a)^2 + \frac{\delta}{(b-a)\sigma_1^2}\Big) <, 1$$

then problem (2.2)–(2.4) *has a unique solution in* $C^2[a,b]$.

3. The case $\alpha + \beta \neq 0$

The main result of this section is the following Theorem.

Theorem 3.1. *If* $\alpha+\beta \neq 0$, (H1) *and* (H2) *hold and*

(H3) $(3\sigma_1^2 - 4\sigma_2^2)(b-a)^3 + 12\delta > 0$,

then there exist at least two solutions to (1.1)–(1.3).

We need first some lemmas about $\Delta(y)$, defined for every $y \in C^1[a,b]$ as in (1.4).

Lemma 3.2. *If* (H1), (H2) *hold, then for every* $y \in C^1[a,b]$,

$$\begin{aligned} M_1 &:= \frac{(3\sigma_1^2 - 4\sigma_2^2)(b-a)^4 + 12\delta(b-a)}{12} \\ &\le \Delta(y) \le M_2 := \frac{(3\sigma_2^2 - 4\sigma_1^2)(b-a)^4 + 12\delta(b-a)}{12} \end{aligned} \tag{3.1}$$

and for every $y, z \in C^1[a,b]$,

$$|\Delta(y) - \Delta(z)| \le \frac{7}{6}\sigma_2 L(b-a)^4\|y-z\|_{C^1}. \tag{3.2}$$

Proof. By (H1), it holds that

$$\begin{aligned}\Delta(y) &\geq \sigma_1^2\Big[\int_a^b (t-a)dt\Big]^2 + \delta(b-a) - (b-a)\sigma_2^2\int_a^b (t-a)^2dt\\ &= \sigma_1^2\frac{(b-a)^4}{4} + \delta(b-a) - \sigma_2^2\frac{(b-a)^4}{3}\\ &= \frac{(3\sigma_1^2-4\sigma_2^2)(b-a)^4+12\delta(b-a)}{12}.\end{aligned}$$

On the other hand,

$$\Delta(y) \leq \sigma_2^2\frac{(b-a)^4}{2} + \delta(b-a) - \sigma_1^2\frac{(b-a)^4}{3} = \frac{(3\sigma_2^2-4\sigma_1^2)(b-a)^4+12\delta(b-a)}{12}.$$

Moreover,

$$\begin{aligned}&|\Delta(y)-\Delta(z)|\\ &\leq \Big|\Big[\int_a^b\int_a^t f(s,y(s),y'(s))dsdt\Big]^2 - \Big[\int_a^b\int_a^t f(s,z(s),z'(s))dsdt\Big]^2\Big|\\ &\quad + (b-a)\Big|\int_a^b \Big[\int_a^t f(s,z(s),z'(s))ds\Big]^2dt - \int_a^b \Big[\int_a^t f(s,y(s),y'(s))ds\Big]^2dt\Big|\\ &= \Big|\int_a^b\int_a^t [f(s,y(s),y'(s)) - f(s,z(s),z'(s))]\,ds\,dt\Big|\\ &\quad\times\Big|\int_a^b\int_a^x [f(s,y(s),y'(s)) + f(s,z(s),z'(s))]\,ds\,dt\Big|\\ &\quad + (b-a)\Big|\int_a^b \Big[\int_a^t [f(s,y(s),y'(s)) - f(s,z(s),z'(s))]ds\Big]\\ &\quad\times\Big[\int_a^t [f(s,y(s),y'(s)) + f(s,z(s),z'(s))]ds\Big]dt\Big|\\ &\leq \Big(\sigma_2L_f\frac{(b-a)^4}{2} + \frac{2}{3}\sigma_2L_f(b-a)^4g\Big)\|y-z\|_{C^1}\\ &= \frac{7}{6}\sigma_2L_f(b-a)^4\|y-z\|_{C^1}.\end{aligned}$$

□

As immediate consequence we have the lemma.

Lemma 3.3. *Assume that* (H1), (H2), (H3) *hold, and for every* $y \in C^1[a,b]$ *define*

$$A_1(y) = \frac{-\int_a^b\int_a^x f(t,y(t),y'(t))\,dt\,dx + \sqrt{\Delta(y)}}{b-a} \tag{3.3}$$

$$A_2(y) = \frac{-\int_a^b\int_a^x f(t,y(t),y'(t))\,dt\,dx - \sqrt{\Delta(y)}}{b-a} \tag{3.4}$$

Then, for $i=1,2$*, for every* $y,z \in C^1[a,b]$,

$$|A_i(y)| \leq \frac{1}{b-a}\Big[\sigma_2\frac{(b-a)^2}{2} + \sqrt{M_2}\Big] \tag{3.5}$$

$$|A_i(y)-A_i(z)| \leq L_f(b-a)\Big[\frac{1}{2} + \frac{7\sigma_2}{12\sqrt{M_1}}(b-a)^2\Big]\|y-z\|_{C^1}, \tag{3.6}$$

where M_1 *and* M_2 *are defined in Lemma 3.2.*

Proof. First observe that A_1 and A_2 are well-defined on $C^1[a,b]$, since $\Delta(y) \geq 0$ for every $y \in C^1[a,b]$ by Lemma 3.2 and (H3). For every $y \in C^1[a,b]$ and $i = 1,2$, by (3.1), we have

$$\begin{aligned}|A_i(y)| &\leq \frac{1}{b-a}\Big(\Big|\int_a^b \int_a^t f(s,y(s),y'(s))\,ds\,dt\Big| + |\sqrt{\Delta_i(y)}|\Big)\\ &\leq \frac{1}{b-a}\Big(\sigma_2\frac{(b-a)^2}{2} + \sqrt{M_2}\Big).\end{aligned}$$

Moreover, by (3.2), for every $y,z \in C^1[a,b]$,

$$\begin{aligned}&|A_i(y) - A_i(z)|\\ &= \Big|\frac{-\int_a^b\int_a^t [f(s,y(s),y'(s)) - f(s,z(s),z'(s))]\,ds\,dt \pm \Big(\sqrt{\Delta(y)} - \sqrt{\Delta(z)}\Big)}{b-a}\Big|\\ &\leq \frac{1}{b-a}\Big[L_f\|y-z\|_{C^1}\frac{(b-a)^2}{2} + \frac{|\Delta(y)-\Delta(z)|}{|\sqrt{\Delta(y)}+\sqrt{\Delta(z)}|}\Big]\\ &\leq \frac{1}{b-a}\Big[L_f\|y-z\|_{C^1}\frac{(b-a)^2}{2} + \frac{|\Delta(y)-\Delta(z)|}{2\sqrt{M_1}}\Big]\\ &\leq \frac{1}{b-a}\Big[L_f\|y-z\|_{C^1}\frac{(b-a)^2}{2} + \frac{7}{12\sqrt{M_1}}\sigma_2 L_f (b-a)^4\|y-z\|_{C^1}\Big].\end{aligned}$$

□

Proof of Theorem 3.1. Observe first that if $\alpha + \beta \neq 0$ and $y \in C^2[a,b]$ is a solution of (1.1), (1.2), then

$$\begin{aligned}y(x) &= y'(a)(x-a) + \int_a^x\int_a^t f(s,y(s),y'(s)\,ds\,dt\\ &\quad + \frac{1}{\alpha+\beta}\Big[\gamma - \beta y'(a)(b-a) - \beta\int_a^b\int_a^t f(s,y(s),y'(s))\,ds\,dt\Big]\end{aligned} \tag{3.7}$$

By comparing (3.7) with (1.6) and by Lemma 3.3, we obtain that for $i = 1$ or $i = 2$,

$$\begin{aligned}y(x) &= A_i(y)(x-a) + \int_a^x\int_a^t f(s,y(s),y'(s)\,ds\,dt\\ &\quad + \frac{1}{\alpha+\beta}\Big[\gamma - \beta A_i(y)(b-a) - \beta\int_a^b\int_a^t f(s,y(s),y'(s))\,ds\,dt\Big]\end{aligned} \tag{3.8}$$

It is immediate to prove that, conversely, if $y \in C^1[a,b]$ satisfies (3.8), then $y \in C^2[a,b]$ and y is a solution of (1.1)–(1.3).

Thus $y \in C^2[a,b]$ is a solution of (1.1)–(1.3) if and only if y is a fixed point of one of the operators $T_i : C^1[a,b] \to C^1[a,b]$, $i = 1,2$ defined by

$$\begin{aligned}T_i y &= A_i(y)(x-a) + \int_a^x\int_a^t f(s,y(s),y'(s)\,ds\,dt\\ &\quad + \frac{1}{\alpha+\beta}\Big[\gamma - \beta A_i(y)(b-a) - \beta\int_a^b\int_a^t f(s,y(s),y'(s)\,ds\,dt\Big]\end{aligned}$$

Observe that for every $y \in C^1[a,b]$ and every $x \in [a,b]$,

$$
\begin{gathered}
(T_i y)'(x) = A_i(y) + \int_a^x f(s, y(s), y'(s))ds \\
(T_i y)''(x) = f(x, y(x), y'(x)).
\end{gathered}
$$

By Lemma 3.3 we get that $T_i : C^1[a,b] \to C^1[a,b]$ is bounded. Moreover, if $y, z \in C^1[a,b]$ and $x \in [a,b]$,

$$
\begin{aligned}
|T_i y(x) - T_i z(x)| &\le \Big(1 + |\frac{\beta}{\alpha+\beta}|\Big)\Big(|A_i(y) - A_i(z)|(b-a) + \frac{L_f}{2}(b-a)^2\|y-z\|_{C^1}\Big) \\
&\le \Big(1 + |\frac{\beta}{\alpha+\beta}|\Big) L_f (b-a)^2 \Big[1 + \frac{7\sigma_2}{12\sqrt{M_1}}(b-a)^2\Big]\|y-z\|_{C^1},
\end{aligned}
$$

while

$$
\begin{aligned}
|(T_i y)'(x) - (T_i z)'(x)| &\le |A_i(y) - A_i(z)| + L_f(b-a)\|y-z\|_{C^1} \\
&\le L_f(b-a)\Big[\frac{3}{2} + \frac{7\sigma_2}{12\sqrt{M_1}}(b-a)^2\Big]\|y-z\|_{C^1}.
\end{aligned}
$$

Thus T_i is Lipschitz continuous with Lipschitz constant:

$$
\begin{aligned}
L_{a,b} = &\Big(1 + |\frac{\beta}{\alpha+\beta}|\Big) L_f (b-a)^2 \big[1 + \frac{7\sigma_2}{12\sqrt{M_1}}(b-a)^2\big] \\
&+ L_f(b-a)\big[\frac{3}{2} + \frac{7\sigma_2}{12\sqrt{M_1}}(b-a)^2\big].
\end{aligned} \tag{3.9}
$$

With the same argument as in the proof of Theorem 2.2, we can prove that T_i is compact and that, by Schaefer's fixed point theorem, T_i has a fixed point. □

Corollary 3.4. *For every $a \in \mathbb{R}$, $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that $\alpha + \beta \neq 0$, for every $\sigma_1, \sigma_2 > 0, L_f > 0$ such that (H2), (H1) hold, there exists $b > a$ such that the problem* (1.1)–(1.3) *has two solutions in $C^2[a,b]$.*

Proof. First observe that

$$
\lim_{b \to a^+} (3\sigma_1^2 - 4\sigma_2^2)(b-a)^3 + 12\delta = \delta > 0,
$$

thus b can be chosen in such a way that (H3) is satisfied. Moreover, considering the Lipschitz constant $L_{a,b}$ in (3.9) and observing that

$$
M_1 = \frac{(3\sigma_1^2 - 4\sigma_2^2)(b-a)^4 + 12\delta(b-a)}{12} \sim \delta(b-a) \quad \text{as } b \to a^+,
$$

we obtain $\lim_{b\to a^+} L_{a,b} = 0$. Thus we can choose b in such a way that T_1 and T_2 are contractions from $C^1[a,b]$ into itself and therefore have a unique fixed point. □

We conclude this article pointing out some problems that can be approached with similar considerations.

Remark 3.5. Consider the problem

$$
\begin{gathered}
y''(x) = f(x, y(x), y'(x)) \quad a \le x \le b \\
\alpha y(a) + \beta y(b) = \gamma \\
\int_a^b \exp(y'(t))dt = \delta,
\end{gathered}
$$

where $-\infty < a < b+\infty$, $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ with $\delta > 0$, $\alpha+\beta \neq 0$, and $f : [a,b]\times\mathbb{R}^2 \to \mathbb{R}$ continuous. Then

$$y'(x) = y'(a) + \int_a^x f(s, y(s), y'(s))ds$$

hence

$$\delta = \exp(y'(a)) \int_a^b \exp\Big(\int_a^x f(s, y(s), y'(s))ds\Big)dx.$$

Thus

$$y'(a) = \log\Big(\frac{\delta}{\int_a^b \exp\Big(\int_a^t f(s, y(s), y'(s))ds\Big)\,dt}\Big).$$

It is immediate to prove that $y \in C^2[a,b]$ is a solution of the problem if and only if $y \in C^1[a,b]$ and y is a fixed point of the operator $T : C^1[a,b] \to C^1[a,b]$ defined by

$$\begin{aligned}
Ty(x) = &-\frac{\beta}{\alpha+\beta}(b-a)\log\Big(\frac{\delta}{\int_a^b \exp\big(\int_a^x f(s, y(s), y'(s))ds\big)\,dx}\Big)\\
&-\frac{\beta}{\alpha+\beta}\int_a^b\int_a^x f(s, y(s), y'(s))ds\\
&+(x-a)\log\Big(\frac{\delta}{\int_a^b \exp\big(\int_a^x f(s, y(s), y'(s))ds\big)\,dx}\Big)\\
&+\int_a^x\int_a^t f(s, y(s), y'(s))ds + \frac{\gamma}{\alpha+\beta}.
\end{aligned}$$

Remark 3.6. The approach we have used for the case $\alpha+\beta = 0$ can be used also to study the case $\alpha+\beta \neq 0$. Indeed if we integrate by parts condition (1.3), we find that

$$\begin{aligned}
\delta &= \int_a^b y'(x)^2dx\\
&= y(b)y'(b) - y(a)y'(a) - \int_a^b y(s)f(s, y(s), y'(s)ds\\
&= \frac{\gamma-\alpha y(a)}{\beta}y'(b) - y(a)y'(a) - \int_a^b y(s)f(s, y(s), y'(s)ds\\
&= \frac{\gamma-\alpha y(a)}{\beta}\Big(y'(a) + \int_a^b f(s, y(s), y'(s))ds\Big) - y(a)y'(a)\\
&\quad-\int_a^b y(s)f(s, y(s), y'(s)ds.
\end{aligned}$$

Hence

$$\begin{aligned}
&y(a)\Big[(\alpha+\beta)y'(a) + \alpha\int_a^b f(s, y(s), y'(s))ds\Big]\\
&= -\beta\delta + \gamma y'(a) - \beta\int_a^b y(s)f(s, y(s), y'(s))ds
\end{aligned}$$

If

$$(\alpha+\beta)y'(a) + \alpha\int_a^b f(s, y(s), y'(s))ds \neq 0$$

then

$$y(a) = \frac{-\beta\delta + \gamma y'(a) - \beta \int_a^b y(s) f(s, y(s), y'(s)) ds}{(\alpha + \beta) y'(a) + \alpha \int_a^b f(s, y(s), y'(s)) ds}$$

and therefore

$$\begin{aligned} y(x) = &\frac{-\beta\delta + \gamma y'(a) - \beta \int_a^b y(s) f(s, y(s), y'(s)) ds}{(\alpha + \beta) y'(a) + \alpha \int_a^b f(s, y(s), y'(s)) ds} + y'(a)(x-a) \\ &+ \int_a^x \int_a^t f(s, y(s), y'(s)) ds \end{aligned}$$

As a consequence, we easily obtain that for $y \in C^1[a,b]$ such that

$$(\alpha + \beta) y'(a) + \alpha \int_a^b f(s, y(s), y'(s)) ds \neq 0,$$

the following conditions are equivalent:

(i) $y \in C^2[a,b]$ is a solution of (1.1)–(1.3)
(ii) $y \in C^1[a,b]$ and either for $i = 1$ or $i = 2$,

$$\begin{aligned} y(x) = &\frac{-\beta\delta + \gamma A_i(y) - \beta \int_a^b y(s) f(s, y(s), y'(s)) ds}{(\alpha + \beta) A_i(y) + \alpha \int_a^b f(s, y(s), y'(s)) ds} \\ &+ A_i(y)(x-a) + \int_a^x \int_a^t f(s, y(s), y'(s)) ds. \end{aligned}$$

Anyway, with this approach one has to require additional conditions on f such as (H1') and (H2').

Remark 3.7. Similar results can be obtained if (1.3) is replaced with

$$\int_a^b [\alpha(x)[y'(x)]^2 + \beta(x) y'(x) + \gamma(x)] dx = \delta \in \mathbb{R}$$

with suitable conditions on the functions α, β, γ and on δ. Moreover it could be of interest the study of systems such as

$$\begin{gathered} y^{(n)}(x) = f(x, y(x), y'(x), \dots, y^{(n-1)}(x)), \quad a \leq x \leq b, \\ \alpha y(a) + \beta y(b) = \gamma \\ \int_a^b y'(x)^2 dx = \delta_1 \\ \int_a^b y''(x)^2 dx = \delta_2 \\ \dots \\ \int_a^b y^{(n-1)}(x) dx = \delta_n \end{gathered}$$

or analogous problems.

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