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POSITIVE GROUND STATE SOLUTION FOR KIRCHHOFF EQUATIONS WITH SUBCRITICAL GROWTH AND ZERO MASS

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Abstract. In this article, we study the Kirchhoff equation

$$-\Big(a+b\int_{\mathbb{R}^N}|\nabla u|^2dx\Big)\Delta u=K(x)f(u),\quad x\in\mathbb{R}^N,$$
$$u\in D^{1,2}(\mathbb{R}^N),$$

where $a>0$, $b>0$ and $N\geq 3$. Under suitable conditions on K and f, we obtain four existence results and two nonexistence results, using variational methods.

1. Introduction and statement of main results

We consider the Kirchhoff equation with zero mass

$$\begin{gathered}-\Big(a+b\int_{\mathbb{R}^N}|\nabla u|^2dx\Big)\Delta u=K(x)f(u),\quad x\in\mathbb{R}^N,\\ u\in D^{1,2}(\mathbb{R}^N),\end{gathered}\tag{1.1}$$

where $a>0$, $b>0$ and $N\geq 3$. The potential function $K:\mathbb{R}^N\to\mathbb{R}$ satisfies:

- (K1) K be a nonnegative function and $K\in L^r(\mathbb{R}^N)\backslash\{0\}$, where $r=\frac{2^*}{2^*-p}$, $2^*=\frac{2N}{N-2}$ and $2<p<2^*$;
- (K2) $|(\nabla K(x),x)|\leq K(x)$ for a.e. $x\in\mathbb{R}^N$, where $(\cdot,\cdot)$ denotes the scalar product in $\mathbb{R}^N$.

The nonlinear term $f\in C(\mathbb{R}_+,\mathbb{R}_+)$ satisfies:

- (F1) $\limsup_{s\to 0^+}\frac{f(s)}{s^{p-1}}<+\infty$;
- (F2) $\limsup_{s\to +\infty}\frac{f(s)}{s^{p-1}}<+\infty$;
- (F3) $\lim_{s\to +\infty}\frac{f(s)}{s}=+\infty$.

When $\Omega\subset\mathbb{R}^N$ is a smooth bounded domain, the equation

$$\begin{gathered}-\Big(a+b\int_{\Omega}|\nabla u|^2dx\Big)\Delta u=f(x,u),\quad x\in\Omega,\\ u=0,\quad x\in\partial\Omega.\end{gathered}\tag{1.2}$$

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is related to the stationary analogue of the Kirchhoff equation

$$u_{tt} - \Big(a + b\int_\Omega |\nabla u|^2 dx\Big)\Delta u = f(x,u) \tag{1.3}$$

which was proposed by Kirchhoff [11] in 1883 as an extension of the classical D'Alembert wave equation for free vibrations of elastic strings. Kirchhoff's model takes into account the changes in length of the string produced by transverse vibrations. Some early classical studies of Kirchhoff equations were those of Bernstein [4] and Pohozaev [18]. However, equation (1.3) received great attention only after Lions [16] proposed an abstract framework for the problem. Some interesting results can be found in [1], [5], [6] and the references therein.

There are many recent articles studying the Kirchhoff equations with subcritical growth in $\mathbb{R}^N$, see for example [7, 8, 9, 12, 13, 15, 17, 20, 21] and so on. But for the Kirchhoff equations with subcritical growth and zero mass, to our best knowledge, there are very few results up to now except [2] and [14]. Azzollini [2] obtained the existence of positive radial solution under $K = 1$ and f satisfies the Berestycki-Lions conditions [3]. In [14], Li et al. obtain a positive solution for $b > 0$ small enough when the potential K satisfies

(K3) $K \in L^\infty(\mathbb{R}^N)$,

and other conditions similar to (K1), (K3), and the nonlinear term f satisfies (F3), and

(F5) $\lim_{s\to+\infty} \frac{f(s)}{s^{2^*-1}} = 0$,

(F6) $\lim_{s\to 0^+} \frac{f(s)}{s^{2^*-1}} = 0$.

Obviously, condition (F6) is stronger than (F1) and there exist functions which satisfy (F1), but do not satisfy (F6), such as $f(s) = s^{p-1}$. Thus, in the present paper, we will remove the assumption (F6) to study equation (1.1). Inspired by [14], we will use the monotonicity trick to investigate it. Set $F(s) = \int_0^s f(\tau)d\tau$. Our results read as follows.

Theorem 1.1. *Assume that $a > 0$ and $b > 0$. Suppose that* (F1)–(F3) *hold. Then there exists $b_0 > 0$ such that for any $b \in (0, b_0)$, equation* (1.1) *has a positive ground state solution, under one of the following conditions:*

1. *$N \geq 4$ and* (K1) *holds,*
2. *$N = 3$, $2 < p < 4$ and* (K1) *holds,*
3. *$N = 3$, $4 \leq p < 6$,* (K1) *and* (K2) *hold.*

Theorem 1.2. *Assume that $a > 0$, $b > 0$ and $N = 3$. Suppose that* (K1) *with $4 \leq p < 6$,* (F1) *and* (F2) *hold. In addition, f satisfies*

(F4) $\lim_{s\to+\infty} \frac{F(s)}{s^4} = +\infty$ *and $sf(s) \geq 4F(s)$ for all $s \geq 0$.*

Then equation (1.1) *has a positive ground state solution.*

Theorem 1.3. *Assume that $a > 0$ and $b > 0$. Suppose that* (F1), (F3), (F5) *hold. Then there exists $b_0 > 0$ such that for any $b \in (0, b_0)$, equation* (1.1) *has a positive ground state solution, under one of the following conditions:*

1. *$N \geq 4$,* (K1) *and* (K3) *hold,*
2. *$N = 3$,* (K1)–(K3) *hold.*

Theorem 1.4. *Assume that $a > 0$, $b > 0$ and $N = 3$. Suppose that (K1) with $4 \leq p < 6$, (K3),* (F1), (F4), (F5) *hold. Then* (1.1) *has a positive ground state solution.*

Remark 1.5. Just like the example which was given by Li et al. in [14], when $p \in (\max\{2, \frac{22^*}{3}\}, 2^*)$, one can easily verify that the function $K(x) = \frac{1}{1+|x|^\alpha}$, $\alpha \in (\frac{32^*-3p}{2^*}, 1]$ satisfies $(K_1) - (K_3)$.

For equation (1.1) with $b > 0$ large enough, we have the following nonexistence results.

Theorem 1.6. *Assume that $a > 0$, $b > 0$, $N \geq 3$ and f satisfies* (F1) *and* (F2). *Suppose that when $N = 3$,* (K1) *with $2 < p < 4$ holds and when $N \geq 4$,* (K1) *holds. Then there exists $B > 0$ such that for any $b > B$, equation* (1.1) *has only zero solution.*

Theorem 1.7. *Assume that $a > 0$, $b > 0$ and $N \geq 4$. Suppose that* (K1), (K3), (F1), (F5) *hold. Then there exists $B > 0$ such that for any $b > B$, equation* (1.1) *has only zero solution.*

Theorems 1.1–1.4 and 1.6–1.7 can be seen as an extension of the results in [14]. This article is organized as follows. In Section 2 we give some preliminary knowledge. Section 3 is devoted to the proofs of Theorem 1.1 and 1.2. Finally, in Section 4 we complete the proofs of Theorems 1.3, 1.4, 1.6, and 1.7.

2. Preliminaries

In what follows, we use the following notation.

- $E := D^{1,2}(\mathbb{R}^N)$ is the closure of the compactly supported smooth functions with respect to the norm
$$\|u\| = \Big(\int_{\mathbb{R}^N} |\nabla u|^2 dx\Big)^{1/2}.$$
- $L^s(\mathbb{R}^N)$ is the usual Lebesgue space endowed with the norm
$$|u|_s = \Big(\int_{\mathbb{R}^N} |u|^s dx\Big)^{1/s}, \quad \forall s \in [1, +\infty), \quad |u|_\infty = \operatorname{ess\,sup}_{x\in\mathbb{R}^N} |u(x)|.$$
- S denotes the best constant of Sobolev embedding $D^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$, that is,
$$S|u|_{2^*}^2 \leq \|u\|^2, \quad \text{for all } u \in D^{1,2}(\mathbb{R}^N). \tag{2.1}$$
- $\langle \cdot, \cdot \rangle$ denotes the dual pairing.
- E^* is the dual space of E.
- C, C_i denote various positive constants.

Since we are looking for positive solution, we assume that $f(s) = 0$ for all $s \leq 0$. By (F1) and (F2), there exists $C_1 > 0$ such that
$$|f(s)| \leq C_1|s|^{p-1}, \quad \text{for all } s \in \mathbb{R}, \tag{2.2}$$
$$|F(s)| \leq \frac{C_1}{p}|s|^p, \quad \text{for all } s \in \mathbb{R}. \tag{2.3}$$
By (F1) and (F5), for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that
$$|f(s)| \leq \varepsilon|s|^{2^*-1} + C_\varepsilon|s|^{p-1}, \quad \text{for all } s \in \mathbb{R}, \tag{2.4}$$
$$|F(s)| \leq \frac{\varepsilon}{2^*}|s|^{2^*} + C_\varepsilon|t|^p, \quad \text{for all } s \in \mathbb{R}. \tag{2.5}$$

By $f \in C(\mathbb{R}_+, \mathbb{R}_+)$ and (F3), for any $L > 0$, there exists $C_L > 0$ such that

$$F(s) \geq L|s|^2 - C_L, \text{ for all } s \in \mathbb{R}_+. \tag{2.6}$$

By (F4), one has

$$F(s) \geq L|s|^4 - C_L, \text{ for all } s \in \mathbb{R}_+. \tag{2.7}$$

The energy functional $I : E \to \mathbb{R}$ defined by

$$I(u) = \frac{a}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx + \frac{b}{4}\Big(\int_{\mathbb{R}^N} |\nabla u|^2 dx\Big)^2 - \int_{\mathbb{R}^N} K(x)F(u)dx.$$

Obvious, I is of class C^1 and has the derivative given by

$$\langle I'(u), v\rangle = \Big(a + b\int_{\mathbb{R}^N} |\nabla u|^2 dx\Big)\int_{\mathbb{R}^N} (\nabla u, \nabla v)dx - \int_{\mathbb{R}^N} K(x)f(u)vdx,$$

for all $u, v \in E$. As well known, the critical point of the functional I is solution of equation (1.1). For proving our theorems, we need the following proposition.

Proposition 2.1. *Let X be a Banach space equipped with a norm $\|\cdot\|_X$ and let $J \subset \mathbb{R}^+$ be an interval. We consider a family $\{\Phi_\lambda\}_{\lambda\in J}$ of C^1-functionals on X of the form*

$$\Phi_\lambda(u) = A(u) - \lambda B(u), \quad \forall \lambda \in J,$$

where $B(u) \geq 0$ for all $u \in X$ and such that either $A(u) \to +\infty$ or $B(u) \to +\infty$, as $\|u\|_X \to +\infty$. We assume that there are two points v_1, v_2 in X such that

$$c_\lambda = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} \Phi_\lambda(\gamma(t)) > \max\{\Phi_\lambda(v_1), \Phi_\lambda(v_2)\}$$

where

$$\Gamma = \{\gamma \in C([0, 1], X) : \gamma(0) = v_1, \gamma(1) = v_2\}.$$

Then for almost every $\lambda \in J$, there is a bounded $(PS)_{c_\lambda}$ sequence for Φ_λ, that is, there exists a sequence $\{u_n\} \subset X$ such that

(i) *$\{u_n\}$ is bounded in X,*
(ii) *$\Phi_\lambda(u_n) \to c_\lambda$,*
(iii) *$\Phi'_\lambda(u_n) \to 0$ in X^*, where X^* is the dual of X.*

Remark 2.2. The above result corresponds to [10, Theorem 1.1] which is reminiscent of Struwe's monotonicity trick (see [19]) and can be viewed as its generalization. In [10, Lemma 2.3] it is also proved that under the assumptions of Proposition 2.1, the map $\lambda \to c_\lambda$ is continuous from the left.

Let $X := E$, $J := [1/2, 1]$ and $\Phi_\lambda(u) := I_\lambda(u) = A(u) - \lambda B(u)$, where

$$A(u) = \frac{a}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx + \frac{b}{4}\Big(\int_{\mathbb{R}^N} |\nabla u|^2 dx\Big)^2,$$
$$B(u) = \int_{\mathbb{R}^N} K(x)F(u)dx.$$

Then $I_1(u) = I(u)$. It is obvious that $B(u) \geq 0$ for all $u \in E$ and $A(u) \to +\infty$ as $\|u\| \to +\infty$.

3. Proofs of Theorem 1.1 and Theorem 1.2

First we give some lemmas.

Lemma 3.1. *Assume that $a > 0$, $b > 0$ and $N \geq 3$. Suppose that* (K1), (F1), (F2) *hold. Then there exist $\rho > 0$ and $\alpha > 0$ such that $I_\lambda(u)|_{\|u\|=\rho} \geq \alpha$ for all $\lambda \in [1/2, 1]$.*

Proof. By (2.3), the Hölder and Sobolev inequalities, for all $u \in E$ and all $\lambda \in [1/2, 1]$, we have

$$\begin{aligned}
I_\lambda(u) &\geq \frac{a}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx - \int_{\mathbb{R}^N} K(x)F(u)dx \\
&\geq \frac{a}{2}\|u\|^2 - \frac{C_1}{p}\int_{\mathbb{R}^N} K(x)|u|^p dx \\
&\geq \frac{a}{2}\|u\|^2 - \frac{C_1}{p}|K|_r\Big(\int_{\mathbb{R}^N} |u|^{2^*} dx\Big)^{p/2^*} \\
&\geq C_2\|u\|^2 - C_3\|u\|^p \\
&= \|u\|^2(C_2 - C_3\|u\|^{p-2}).
\end{aligned}$$

Since $p > 2$, we can choose $\rho = \big(\frac{C_2}{2C_3}\big)^{\frac{1}{p-2}}$. Then $I_\lambda(u) \geq \frac{C_2}{2}\big(\frac{C_2}{2C_3}\big)^{\frac{2}{p-2}} := \alpha$ for all $\|u\| = \rho$. The proof is complete. $\square$

For c_λ in Proposition 2.1, we have the following lemma.

Lemma 3.2. *Assume that $a > 0$, $b > 0$ and $N \geq 3$. Suppose that* (K1) *and* (F1)–(F3) *hold. Then there exists $b_0 > 0$ such that for any $b \in (0, b_0)$, there are two points v_1, v_2 in E such that $c_\lambda > \max\{I_\lambda(v_1), I_\lambda(v_2)\}$ for all $\lambda \in [1/2, 1]$.*

Proof. Choose a nonnegative function $\varphi \in C_0^\infty(\mathbb{R}^N)$ such that $\|\varphi\| = 1$, anf

$$\int_{\operatorname{supp}\varphi} K(x)\varphi^2 dx \neq 0, \quad \int_{\operatorname{supp}\varphi} K(x)dx \neq 0.$$

Let

$$L = \frac{2a}{\int_{\operatorname{supp}\varphi} K(x)\varphi^2 dx}$$

in (2.6) and take

$$t_0 = \sqrt{2}\max\Big\{\rho, \Big(\frac{C_L}{a}\int_{\operatorname{supp}\varphi} K(x)dx\Big)^{1/2}\Big\},$$

where ρ is given by Lemma 3.1. Then $\|t_0\varphi\| > \rho$ and for all $\lambda \in [1/2, 1]$, one has

$$\begin{aligned}
I_\lambda(t_0\varphi) &= \frac{at_0^2}{2} + \frac{bt_0^4}{4} - \lambda\int_{\operatorname{supp}\varphi} K(x)F(t_0\varphi)dx \\
&\leq \frac{at_0^2}{2} + \frac{bt_0^4}{4} - \frac{1}{2}\int_{\operatorname{supp}\varphi} K(x)(Lt_0^2\varphi^2 - C_L)dx \\
&= \frac{at_0^2}{2} + \frac{bt_0^4}{4} + \frac{C_L}{2}\int_{\operatorname{supp}\varphi} K(x)dx - \frac{Lt_0^2}{2}\int_{\operatorname{supp}\varphi} K(x)\varphi^2 dx \\
&= \frac{bt_0^4}{4} + \frac{C_L}{2}\int_{\operatorname{supp}\varphi} K(x)dx - \frac{a}{2}t_0^2
\end{aligned}$$

$$\leq \frac{bt_0^4}{4} - \frac{C_L}{2}\int_{\operatorname{supp}\varphi} K(x)dx,$$

which implies that there exists $b_0 > 0$ such that for any $b \in (0, b_0)$, one has $I_\lambda(t_0\varphi) < 0$. Thus letting $v_1 = 0$ and $v_2 = t_0\varphi$, by Lemma 3.1 and definition of c_λ, we have

$$0 < \alpha \leq c_1 \leq c_\lambda \leq c_{\frac{1}{2}} < +\infty.$$

and then $c_\lambda > I_\lambda(v_1) > I_\lambda(v_2)$. □

Lemma 3.3. *Assume that $a > 0$, $b > 0$ and $N = 3$. Suppose that* (K1) *with* $4 \leq p < 6$, (F1), (F2), (F4) *hold. Then there are two points v_1, v_2 in E such that $c_\lambda > \max\{I_\lambda(v_1), I_\lambda(v_2)\}$ for all $\lambda \in [1/2, 1]$.*

Proof. Choose a nonnegative function $\varphi \in C_0^\infty(\mathbb{R}^3)$ such that $\|\varphi\| = 1$, and

$$\int_{\operatorname{supp}\varphi} K(x)\varphi^4 dx \neq 0, \quad \int_{\operatorname{supp}\varphi} K(x)dx \neq 0.$$

Let $L = \frac{b}{\int_{\operatorname{supp}\varphi} K(x)\varphi^4 dx}$ in (2.7). Then for any $t > 0$ and for all $\lambda \in [1/2, 1]$, one has

$$\begin{aligned}
I_\lambda(t\varphi) &= \frac{at^2}{2} + \frac{bt^4}{4} - \lambda\int_{\operatorname{supp}\varphi} K(x)F(t\varphi)dx \\
&\leq \frac{at^2}{2} + \frac{bt^4}{4} - \frac{1}{2}\int_{\operatorname{supp}\varphi} K(x)(Lt^4\varphi^4 - C_L)dx \\
&= \frac{at^2}{2} + \frac{bt^4}{4} + \frac{C_L}{2}\int_{\operatorname{supp}\varphi} K(x)dx - \frac{Lt^4}{2}\int_{\operatorname{supp}\varphi} K(x)\varphi^4 dx \\
&= \frac{at^2}{2} + \frac{C_L}{2}\int_{\operatorname{supp}\varphi} K(x)dx - \frac{bt^4}{4},
\end{aligned}$$

which indicates that there exists $t_0 > 0$ such that $\|t_0\varphi\| > \rho$ and $I_\lambda(t_0\varphi) < 0$. Thus letting $v_1 = 0$ and $v_2 = t_0\varphi$, by Lemma 3.1 and definition of c_λ, we have

$$0 < \alpha \leq c_1 \leq c_\lambda \leq c_{\frac{1}{2}} < +\infty.$$

and then $c_\lambda > I_\lambda(v_1) > I_\lambda(v_2)$. □

Lemma 3.4. *Assume that $\{u_n\}$ is bounded in $L^s(\mathbb{R}^N)$, where $1 < s < +\infty$. Suppose that $u_n(x) \to u(x)$ a.e. in $\mathbb{R}^N$. Then up to a subsequence, $u_n \rightharpoonup u$ in $L^s(\mathbb{R}^N)$.*

Proof. Suppose that $|u_n|_s \leq M$ and $v \in L^{\frac{s}{s-1}}(\mathbb{R}^N)$ is fixed. Then for any $\varepsilon > 0$, there exists $r > 0$ such that

$$\Big|\int_{|x|\geq r} u_n v dx\Big| \leq |u_n|_s\Big(\int_{|x|\geq r} |v|^{\frac{s}{s-1}} dx\Big)^{\frac{s-1}{s}} \leq M\varepsilon.$$

Similarly, combining the Fatou's lemma, we have

$$\begin{aligned}
\Big|\int_{|x|\geq r} uv dx\Big|^s &\leq |u|_s^s\Big(\int_{|x|\geq r} |v|^{\frac{s}{s-1}} dx\Big)^{s-1} \\
&= \int_{\mathbb{R}^N} \liminf_{n\to\infty} |u_n|^s dx\Big(\int_{|x|\geq r} |v|^{\frac{s}{s-1}} dx\Big)^{s-1} \\
&\leq \liminf_{n\to\infty}\int_{\mathbb{R}^N} |u_n|^s dx\Big(\int_{|x|\geq r} |v|^{\frac{s}{s-1}} dx\Big)^{s-1}
\end{aligned}$$

$$\leq M^s \varepsilon^s.$$

Since $v \in L^{\frac{s}{s-1}}(B_r(0))$ with $B_r(0) := \{x \in \mathbb{R}^N : |x| < r\}$, there exists $\delta > 0$ such that for any $A \subset B_r(0)$, when meas$A < \delta$, one has $\left(\int_A |v|^{\frac{s}{s-1}} dx\right)^{\frac{s-1}{s}} < \varepsilon$. Thus for all n, we have

$$\Big|\int_A u_n v dx\Big| \leq |u_n|_s \Big(\int_A |v|^{\frac{s}{s-1}} dx\Big)^{\frac{s-1}{s}} < M\varepsilon.$$

By Vitali's theorem, one gets

$$\int_{B_r(0)} u_n v dx = \int_{B_r(0)} uv dx + o(1).$$

Hence we have

$$\Big|\int_{\mathbb{R}^N} (u_n - u) v dx\Big| \leq \Big|\int_{|x|\geq r} (u_n - u) v dx\Big| + \Big|\int_{B_r(0)} (u_n - u) v dx\Big| \leq 2M\varepsilon + o(1).$$

By the arbitrariness of ε, we complete the proof. □

Lemma 3.5. *For any $\lambda \in [1/2, 1]$, if $\{u_n\} \subset E$ is a bounded and nonnegative Palais-Smale sequence of the function I_λ, there exists a nonnegative function $u \in E$ such that, up to a subsequence, $u_n \to u$ in E.*

Proof. Since $\{u_n\}$ is bounded and nonnegative in E, up to a subsequence, there exists a nonnegative function $u \in E$ such that $u_n \rightharpoonup u$ in E, $u_n(x) \to u(x)$ a.e. in $\mathbb{R}^N$ and there exists $d \geq 0$ such that $d = \int_{\mathbb{R}^N} |\nabla u_n|^2 dx + o(1)$. By (2.2) and the Hölder inequality, one has

$$\begin{aligned}
\int_{\mathbb{R}^N} |f(u_n)(u_n - u)|^{\frac{2^*}{p}} dx &\leq C_1 \int_{\mathbb{R}^N} |u_n|^{\frac{2^*(p-1)}{p}} |u_n - u|^{\frac{2^*}{p}} dx \\
&\leq C_1 \Big(\int_{\mathbb{R}^N} |u_n|^{2^*} dx\Big)^{\frac{p-1}{p}} \Big(\int_{\mathbb{R}^N} |u_n - u|^{2^*} dx\Big)^{1/p} \\
&\leq C.
\end{aligned}$$

Combining $f(u_n(x))(u_n(x) - u(x)) \to 0$ a.e. in $\mathbb{R}^N$ with Lemma 3.4, up to a subsequence, we get $f(u_n)(u_n - u) \rightharpoonup 0$ in $L^{\frac{2^*}{p}}(\mathbb{R}^N)$. Since $K \in L^r(\mathbb{R}^N)$, we have

$$\int_{\mathbb{R}^N} K(x) f(u_n)(u_n - u) dx = o(1).$$

Thus by $I'_\lambda(u_n) \to 0$ in E^*, one has

$$\begin{aligned}
0 &= \langle I'_\lambda(u_n), u_n - u\rangle + o(1) \\
&= a\int_{\mathbb{R}^N} (\nabla u_n, \nabla(u_n - u)) dx + b\int_{\mathbb{R}^N} |\nabla u_n|^2 dx \int_{\mathbb{R}^N} (\nabla u_n, \nabla(u_n - u)) dx \\
&\quad - \lambda \int_{\mathbb{R}^N} K(x) f(u_n)(u_n - u) dx + o(1) \\
&= a(\|u_n\|^2 - \|u\|^2) + bd(\|u_n\|^2 - \|u\|^2) + o(1),
\end{aligned}$$

which implies $\|u_n\| \to \|u\|$. Combining $u_n \rightharpoonup u$ in E, we get $u_n \to u$ in E. The proof is complete. □

Remark 3.6. *According to Proposition 2.1 and Lemma 3.2, for almost every $\lambda \in [1/2,1]$, there exists a bounded sequence $\{u_n\} \subset E$ such that $I_\lambda(u_n) \to c_\lambda$ and $I'_\lambda(u_n) \to 0$ in E^*. Define $u^\pm = \max\{\pm u, 0\}$, then*

$$\begin{aligned}
o(1) &= \langle I'_\lambda(u_n), u_n^- \rangle \\
&= \Big(a + b\int_{\mathbb{R}^N} |\nabla u_n|^2 dx\Big) \int_{\mathbb{R}^N} (\nabla u_n, \nabla u_n^-) dx - \lambda \int_{\mathbb{R}^N} K(x) f(u_n) u_n^- dx \\
&= -\Big(a + b\int_{\mathbb{R}^N} |\nabla u_n|^2 dx\Big) \int_{\mathbb{R}^N} |\nabla u_n^-|^2 dx,
\end{aligned}$$

which implies $u_n^- \to 0$ in E. Thus one has

$$\begin{aligned}
c_\lambda &= I_\lambda(u_n) + o(1) \\
&= \frac{a}{2} \int_{\mathbb{R}^N} |\nabla u_n|^2 dx + \frac{b}{4}\left(\int_{\mathbb{R}^N} |\nabla u_n|^2 dx\right)^2 - \lambda \int_{\mathbb{R}^N} K(x) F(u_n) dx + o(1) \\
&= \frac{a}{2}\Big(\int_{\mathbb{R}^N} |\nabla u_n^+|^2 dx + \int_{\mathbb{R}^N} |\nabla u_n^-|^2 dx\Big) \\
&\quad + \frac{b}{4}\Big(\int_{\mathbb{R}^N} |\nabla u_n^+|^2 dx + \int_{\mathbb{R}^N} |\nabla u_n^-|^2 dx\Big)^2 - \lambda \int_{\mathbb{R}^N} K(x) F(u_n^+) dx + o(1) \\
&= \frac{a}{2} \int_{\mathbb{R}^N} |\nabla u_n^+|^2 dx + \frac{b}{4}\left(\int_{\mathbb{R}^N} |\nabla u_n^+|^2 dx\right)^2 - \lambda \int_{\mathbb{R}^N} K(x) F(u_n^+) dx + o(1) \\
&= I_\lambda(u_n^+) + o(1)
\end{aligned}$$

and

$$\begin{aligned}
0 &= \langle I'_\lambda(u_n), \varphi \rangle + o(1) \\
&= \Big(a + b\int_{\mathbb{R}^N} |\nabla u_n|^2 dx\Big) \int_{\mathbb{R}^N} (\nabla u_n, \nabla \varphi) dx - \lambda \int_{\mathbb{R}^N} K(x) f(u_n) \varphi dx + o(1) \\
&= \Big(a + b\int_{\mathbb{R}^N} |\nabla u_n^+|^2 dx + b\int_{\mathbb{R}^N} |\nabla u_n^-|^2 dx\Big)\Big(\int_{\mathbb{R}^N} (\nabla u_n^+, \nabla \varphi) dx \\
&\quad - \int_{\mathbb{R}^N} (\nabla u_n^-, \nabla \varphi) dx\Big) - \lambda \int_{\mathbb{R}^N} K(x) f(u_n^+) \varphi dx + o(1) \\
&= \Big(a + b\int_{\mathbb{R}^N} |\nabla u_n^+|^2 dx\Big) \int_{\mathbb{R}^N} (\nabla u_n^+, \nabla \varphi) dx - \lambda \int_{\mathbb{R}^N} K(x) f(u_n^+) \varphi dx + o(1) \\
&= \langle I'_\lambda(u_n^+), \varphi \rangle + o(1),
\end{aligned}$$

uniformly for all $\varphi \in E$ and $\|\varphi\| = 1$. That is, $\{u_n^+\}$ is a bounded Palais-Smale sequence of I_λ. By Lemma 3.5, there exists a nonnegative function $u \in E$ such that, up to a subsequence, $u_n^+ \to u$ in E. Thus $I_\lambda(u) = c_\lambda$ and $I'_\lambda(u) = 0$ in E^*.

Proof of Theorem 1.1. Set $\lambda_j \in [1/2,1]$ and $\lambda_j \to 1^-$. Then there exists a sequence nonnegative functions $u_j \in E$ such that $I_{\lambda_j}(u_j) = c_{\lambda_j}$ and $I'_{\lambda_j}(u_j) = 0$. If $N \geq 4$,

$2 < p < 2^* \leq 4$. By (K1), (2.2) and $\langle I'_{\lambda_j}(u_j), u_j\rangle = 0$, we have

$$\begin{aligned} a\|u_j\|^2 + b\|u_j\|^4 &= \lambda_j \int_{\mathbb{R}^N} K(x)f(u_j)u_j dx \\ &\leq C_1 \int_{\mathbb{R}^N} K(x)|u_j|^p dx \\ &\leq C_1|K|_r|u_j|_{2^*}^p \\ &\leq C\|u_j\|^p, \end{aligned} \tag{3.1}$$

which implies $\|u_j\| \leq C$. If $N = 3$ and $2 < p < 4$, by (3.1) with $N = 3$, we have $\|u_j\| \leq C$. If $N = 3$ and $4 \leq p < 6$, by (K1), (K2), $I'_{\lambda_j}(u_j) = 0$, one has the following Pohozaev equality (see [14, Lemma 2.2])

$$\frac{a}{2}\|u_j\|^2 + \frac{b}{2}\|u_j\|^4 = 3\lambda_j \int_{\mathbb{R}^3} K(x)F(u_j)dx + \lambda_j \int_{\mathbb{R}^3} (\nabla K(x), x)F(u_j)dx.$$

Combining (K2) with $I_{\lambda_j}(u_j) = c_{\lambda_j}$, we have

$$\begin{aligned} \frac{a}{2}\|u_j\|^2 + \frac{b}{2}\|u_j\|^4 &= 3\lambda_j \int_{\mathbb{R}^3} K(x)F(u_j)dx + \lambda_j \int_{\mathbb{R}^3} (\nabla K(x), x)F(u_j)dx \\ &\geq 2\lambda_j \int_{\mathbb{R}^3} K(x)F(u_j)dx \\ &= a\|u_j\|^2 + \frac{b}{2}\|u_j\|^4 - 2c_{\lambda_j}, \end{aligned}$$

which implies that

$$2c_{\lambda_j} \geq \frac{a}{2}\|u_j\|^2.$$

Hence $\|u_j\| \leq C$. Since $I(u_j) = I_{\lambda_j}(u_j) + o(1) = c_{\lambda_j} + o(1) = c + o(1)$ and $I'(u_j) = I'_{\lambda_j}(u_j) + o(1) = o(1)$ in E^*, according to Lemma 3.5, there exists a nonnegative function $u \in E$ such that $u_j \to u$ in E. Thereby $I(u_j) \to I(u) = c$ and $I'(u_j) \to I'(u) = 0$ in E^*. That is, u is a nonnegative solution of equation (1.1). To obtain the ground state solution, we set $\pi = \inf_{u\in\Pi} I(u)$, where $\Pi = \{u \in E\backslash\{0\} | I'(u) = 0, u \geq 0\}$ and then $\pi \leq c$. Obviously, $\pi > -\infty$. Since $\Pi \neq \emptyset$, there exist a nonnegative sequence $u_n \in E$ such that $I'(u_n) = 0$ and $I(u_n) \to \pi$. With the same method, we can obtain $\{u_n\}$ is bounded in E and then there exists a nonnegative function $u \in E$ such that $u_n \to u$ in E. Hence we have $I(u_n) \to I(u) = \pi$, $I'(u_n) \to I'(u) = 0$. Because of the strongly maximum principle, we know $u > 0$. So we complete the proof of Theorem 1.1. □

Proof of Theorem 1.2. By Lemma 3.1, Lemma 3.3 and the mountain pass theorem, there exists a sequence $\{u_n\} \subset E$ such that $I(u_n) \to c$ and $I'(u_n) \to 0$ in E^*. By (F4), for n large enough, we have

$$\begin{aligned} c + 1 + \|u_n\| &= I(u_n) - \frac{1}{4}\langle I'(u_n), u_n\rangle \\ &= \frac{a}{4}\|u_n\|^2 + \int_{\mathbb{R}^3} K(x)[\frac{1}{4}f(u_n)u_n - F(u_n)]dx \\ &\geq \frac{a}{4}\|u_n\|^2. \end{aligned}$$

Thus $\{u_n\}$ is bounded in E. By Remark 3.6, $\{u_n^+\}$ is a bounded sequence satisfying $I(u_n^+) \to c$ and $I'(u_n^+) \to 0$ in E^*. By Lemma 3.5, there exists a nonnegative function $u \in E$ such that, up to a subsequence, $u_n^+ \to u$ in E. Thus $I(u) = c$ and $I'(u) = 0$ in E^*. The rest of proof is the same as Theorem 1.1. The proof is complete. □

4. Proofs of Theorem 1.3, 1.4, 1.6 and 1.7

We establish parallel steps as Lemmas 3.1, 3.2, 3.3 and 3.5.

Lemma 4.1. *Assume that $a > 0$, $b > 0$ and $N \geq 3$. Suppose that* (K1), (K3), (F1), (F5) *hold. Then there exist $\rho > 0$ and $\alpha > 0$ such that $I_\lambda(u)|_{\|u\|=\rho} \geq \alpha$ for all $\lambda \in [1/2, 1]$.*

Proof. By (2.5) with $\varepsilon = 1$, the Hölder and Sobolev inequalities, for all $u \in E$ and all $\lambda \in [1/2, 1]$, we have

$$
\begin{aligned}
I_\lambda(u) &\geq \frac{a}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx - \int_{\mathbb{R}^N} K(x)F(u)dx \\
&\geq \frac{a}{2}\|u\|^2 - \frac{1}{2^*}\int_{\mathbb{R}^N} K(x)|u|^{2^*} dx - C\int_{\mathbb{R}^N} K(x)|u|^p dx \\
&\geq \frac{a}{2}\|u\|^2 - \frac{1}{2^*}|K|_\infty \int_{\mathbb{R}^N} |u|^{2^*} dx - C|K|_r|u|_{2^*}^p \\
&\geq C_2\|u\|^2 - C_3\|u\|^{2^*} - C_4\|u\|^p \\
&= \|u\|^2(C_2 - C_3\|u\|^{2^*-2} - C_4\|u\|^{p-2}).
\end{aligned}
$$

Since $p > 2$, we can choose $\rho > 0$ small enough such that $C_2 - C_3\rho^{2^*-2} - C_4\rho^{p-2} > 0$. Then there exists $\alpha > 0$ such that $I_\lambda(u) \geq \alpha$ for all $\|u\| = \rho$. The proof is complete. □

The proofs of the following two lemmas are the same as Lemma 3.2 and 3.3.

Lemma 4.2. *Assume that $a > 0$, $b > 0$ and $N \geq 3$. Suppose that* (K1), (K3), (F1), (F3), (F5) *hold. Then there exists $b_0 > 0$ such that for any $b \in (0, b_0)$, there are two points v_1, v_2 in E such that $c_\lambda > \max\{I_\lambda(v_1), I_\lambda(v_2)\}$ for all $\lambda \in [1/2, 1]$.*

Lemma 4.3. *Assume that $a > 0$, $b > 0$ and $N = 3$. Suppose that (K1) with $4 \leq p < 6$,* (K3), (F1), (F4), (F5) *hold. Then there are two points v_1, v_2 in E such that $c_\lambda > \max\{I_\lambda(v_1), I_\lambda(v_2)\}$ for all $\lambda \in [1/2, 1]$.*

Lemma 4.4. *For any $\lambda \in [1/2, 1]$, if $\{u_n\} \subset E$ is a bounded and nonnegative Palais-Smale sequence of the function I_λ, there exists a nonnegative function $u \in E$ such that, up to a subsequence, $u_n \to u$ in E.*

Proof. Since $\{u_n\}$ is bounded and nonnegative in E, up to a subsequence, there exists a nonnegative function $u \in E$ such that $u_n \rightharpoonup u$ in E, $u_n(x) \to u(x)$ a.e. in $\mathbb{R}^N$ and there exists $d \geq 0$ such that $d = \int_{\mathbb{R}^N} |\nabla u_n|^2 dx + o(1)$. Since $\{|u_n|^{p-1}|u_n - u|\}$ is bounded in $L^{\frac{2^*}{p}}(\mathbb{R}^N)$ and $|u_n(x)|^{p-1}|u_n(x) - u(x)| \to 0$ a.e. in $\mathbb{R}^N$, by Lemma 3.4, up to a subsequence, we have $|u_n|^{p-1}|u_n - u| \rightharpoonup 0$ in $L^{\frac{2^*}{p}}(\mathbb{R}^N)$. In view of $K \in L^r(\mathbb{R}^N)$, one has

$$
\int_{\mathbb{R}^N} K(x)|u_n|^{p-1}|u_n - u|dx = o(1).
$$

Combining this with (2.4), we have

$$
\begin{aligned}
&\Big|\int_{\mathbb{R}^N} K(x)f(u_n)(u_n-u)dx\Big|\\
&\le \varepsilon\int_{\mathbb{R}^N} K(x)|u_n|^{2^*-1}|u_n-u|dx + C_\varepsilon\int_{\mathbb{R}^N} K(x)|u_n|^{p-1}|u_n-u|dx\\
&\le \varepsilon|K|_\infty|u_n|_{2^*}^{2^*-1}|u_n-u|_{2^*}+o(1)=C\varepsilon+o(1).
\end{aligned}
$$

Hence by $I'_\lambda(u_n)\to 0$ in E^*, we deduce that

$$
\begin{aligned}
0&=\langle I'_\lambda(u_n),u_n-u\rangle+o(1)\\
&=a\int_{\mathbb{R}^N}(\nabla u_n,\nabla(u_n-u))dx+b\int_{\mathbb{R}^N}|\nabla u_n|^2dx\int_{\mathbb{R}^N}(\nabla u_n,\nabla(u_n-u))dx\\
&\quad-\lambda\int_{\mathbb{R}^N}K(x)f(u_n)(u_n-u)dx+o(1)\\
&=a(\|u_n\|^2-\|u\|^2)+bd(\|u_n\|^2-\|u\|^2)+o(1),
\end{aligned}
$$

which implies $\|u_n\|\to\|u\|$. Combining $u_n\rightharpoonup u$ in E, we get $u_n\to u$ in E. The proof is complete. □

According to Proposition 2.1 and Lemma 4.2, for almost every $\lambda\in[1/2,1]$, there exists a bounded sequence $\{u_n\}\subset E$ such that $I_\lambda(u_n)\to c_\lambda$ and $I'_\lambda(u_n)\to 0$ in E^*. By Remark 3.6, we can assume that u_n is nonnegative. Thus from Lemma 4.4, we know that there exists a nonnegative function $u\in E$ such that $u_n\to u$ in E and then for almost every $\lambda\in[1/2,1]$, $I_\lambda(u)=c_\lambda$ and $I'_\lambda(u)=0$.

Proof of Theorem 1.3. Set $\lambda_j\in[1/2,1]$ and $\lambda_j\to 1^-$. T hen there exists a nonnegative sequence $\{u_j\}\subset E$ such that $I_{\lambda_j}(u_j)=c_{\lambda_j}$ and $I'_{\lambda_j}(u_j)=0$. If $N\ge 4$, $2<p<2^*\le 4$. By (K1), (K3), (2.4) and $\langle I'_{\lambda_j}(u_j),u_j\rangle=0$, we have

$$
\begin{aligned}
a\|u_j\|^2+b\|u_j\|^4&=\lambda_j\int_{\mathbb{R}^N}K(x)f(u_j)u_jdx\\
&\le\varepsilon\int_{\mathbb{R}^N}K(x)|u_j|^{2^*}dx+C_\varepsilon\int_{\mathbb{R}^N}K(x)|u_j|^pdx\\
&\le\varepsilon|K|_\infty\int_{\mathbb{R}^N}|u_j|^{2^*}dx+C_\varepsilon|K|_r|u_j|_{2^*}^p\\
&\le C\varepsilon\|u_j\|^{2^*}+C_\varepsilon\|u_j\|^p,
\end{aligned}
$$

which implies $\|u_j\|\le C$, for $\varepsilon=\frac{b}{2C}$. If $N=3$, by (K1), (K2), $I'_{\lambda_j}(u_j)=0$, one has the Pohozaev equality

$$
\frac{a}{2}\|u_j\|^2+\frac{b}{2}\|u_j\|^4=3\lambda_j\int_{\mathbb{R}^3}K(x)F(u_j)dx+\lambda_j\int_{\mathbb{R}^3}(\nabla K(x),x)F(u_j)dx.
$$

Combining $I_{\lambda_j}(u_j)=c_{\lambda_j}$, we have

$$
\begin{aligned}
\frac{a}{2}\|u_j\|^2+\frac{b}{2}\|u_j\|^4&=3\lambda_j\int_{\mathbb{R}^3}K(x)F(u_j)dx+\lambda_j\int_{\mathbb{R}^3}(\nabla K(x),x)F(u_j)dx\\
&\ge 2\lambda_j\int_{\mathbb{R}^3}K(x)F(u_j)dx\\
&=a\|u_j\|^2+\frac{b}{2}\|u_j\|^4-2c_{\lambda_j}
\end{aligned}
$$

which implies

$$2c_{\lambda_j} \geq \frac{a}{2}\|u_j\|^2.$$

Hence $\|u_j\| \leq C$. The rest of the proof is similar with Theorem 1.1. $\square$

The proof of Theorem 1.4 is same as that of Theorem 1.2 and it is omitted.

Proof of Theorem 1.6. Suppose that $u \in E$ is a nonzero solution of (1.1). Then combining (2.1), (2.2), the Hölder and Young inequalities, we have

$$\begin{aligned}
a\|u\|^2 + b\|u\|^4 &= \int_{\mathbb{R}^N} K(x)f(u)u dx \\
&\leq C_1 \int_{\mathbb{R}^N} K(x)|u|^p dx \\
&\leq C_1 |K|_r |u|_{2^*}^p \\
&\leq C_1 S^{-\frac{p}{2}} |K|_r \|u\|^p \\
&= \big(\frac{2a}{4-p}\big)^{\frac{4-p}{2}} \|u\|^{4-p} C_1 S^{-\frac{p}{2}} |K|_r \big(\frac{4-p}{2a}\big)^{\frac{4-p}{2}} \|u\|^{2p-4} \\
&\leq a\|u\|^2 + \frac{p-2}{2}\big[C_1 S^{-\frac{p}{2}} |K|_r \big(\frac{4-p}{2a}\big)^{\frac{4-p}{2}}\big]^{\frac{2}{p-2}} \|u\|^4 \\
&< a\|u\|^2 + b\|u\|^4,
\end{aligned}$$

for any $b > B := \frac{p-2}{2}[C_1 S^{-\frac{p}{2}} |K|_r (\frac{4-p}{2a})^{\frac{4-p}{2}}]^{\frac{2}{p-2}}$, which is a contradiction. The proof is complete. $\square$

Proof of Theorem 1.7. Suppose that $u \in E$ is a nonzero solution of (1.1). Then combining (2.1), (2.4) and Hölder's inequality, we have

$$\begin{aligned}
a\|u\|^2 + b\|u\|^4 &= \int_{\mathbb{R}^N} K(x)f(u)u dx \\
&\leq \varepsilon \int_{\mathbb{R}^N} K(x)|u|^{2^*} dx + C_\varepsilon \int_{\mathbb{R}^N} K(x)|u|^p dx \\
&\leq \varepsilon |K|_\infty S^{-\frac{2^*}{2}} \|u\|^{2^*} + C_\varepsilon |K|_r S^{-\frac{p}{2}} \|u\|^p.
\end{aligned} \tag{4.1}$$

When $N = 4$, $2^* = 4$. Choose $\varepsilon = \frac{b}{2|K|_\infty S^{-\frac{2^*}{2}}}$. Using (4.1) and the Young inequality, one has

$$\begin{aligned}
a\|u\|^2 + \frac{b}{2}\|u\|^4 &\leq C|K|_r S^{-\frac{p}{2}} \|u\|^p \\
&= \big(\frac{2a}{4-p}\big)^{\frac{4-p}{2}} \|u\|^{4-p} C S^{-\frac{p}{2}} |K|_r \big(\frac{4-p}{2a}\big)^{\frac{4-p}{2}} \|u\|^{2p-4} \\
&\leq a\|u\|^2 + \frac{p-2}{2}\big[C S^{-\frac{p}{2}} |K|_r \big(\frac{4-p}{2a}\big)^{\frac{4-p}{2}}\big]^{\frac{2}{p-2}} \|u\|^4 \\
&< a\|u\|^2 + \frac{b}{2}\|u\|^4,
\end{aligned}$$

for any $b > B := (p-2)\big[C S^{-\frac{p}{2}} |K|_r \big(\frac{4-p}{2a}\big)^{\frac{4-p}{2}}\big]^{\frac{2}{p-2}}$, which is a contradiction. When $N > 4$, $2 < 2^* < 4$. Choose $\varepsilon = 1$. Using (4.1) and the Young inequality, one has

$$a\|u\|^2 + b\|u\|^4 \leq |K|_\infty S^{-\frac{2^*}{2}} \|u\|^{2^*} + C|K|_r S^{-\frac{p}{2}} \|u\|^p$$

$$
\begin{aligned}
&= \Big(\frac{a}{4-2^*}\Big)^{\frac{4-2^*}{2}}\|u\|^{4-2^*}|K|_\infty S^{-\frac{2^*}{2}}\Big(\frac{4-2^*}{a}\Big)^{\frac{4-2^*}{2}}\|u\|^{22^*-4} \\
&\quad + \Big(\frac{a}{4-p}\Big)^{\frac{4-p}{2}}\|u\|^{4-p}CS^{-\frac{p}{2}}|K|_r\Big(\frac{4-p}{a}\Big)^{\frac{4-p}{2}}\|u\|^{2p-4} \\
&\le a\|u\|^2 + \frac{2^*-2}{2}\Big[|K|_\infty S^{-\frac{2^*}{2}}\Big(\frac{4-2^*}{a}\Big)^{\frac{4-2^*}{2}}\Big]^{\frac{2}{2^*-2}}\|u\|^4 \\
&\quad + \frac{p-2}{2}\Big[CS^{-\frac{p}{2}}|K|_r\Big(\frac{4-p}{a}\Big)^{\frac{4-p}{2}}\Big]^{\frac{2}{p-2}}\|u\|^4 \\
&< a\|u\|^2 + b\|u\|^4,
\end{aligned}
$$

for any $b > B := \frac{2^*-2}{2}\big[|K|_\infty S^{-\frac{2^*}{2}}\big(\frac{4-2^*}{a}\big)^{\frac{4-2^*}{2}}\big]^{\frac{2}{2^*-2}} + \frac{p-2}{2}\big[CS^{-\frac{p}{2}}|K|_r\big(\frac{4-p}{a}\big)^{\frac{4-p}{2}}\big]^{\frac{2}{p-2}}$, which is a contradiction. The proof is complete. $\square$

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