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SOLUTION BRANCHES FOR NONLINEAR PROBLEMS WITH AN ASYMPTOTIC OSCILLATION PROPERTY

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Abstract. In this article we employ an oscillatory condition on the nonlinear term, to prove the existence of a connected component of solutions of a nonlinear problem, which bifurcates from infinity and asymptotically oscillates over an interval of parameter values. An interesting and immediate consequence of such oscillation property of the connected component is the existence of infinitely many solutions to the nonlinear problem for all parameter values in that interval.

1. Introduction

Rabinowtz [8] obtained a well known result concerning the existence of unbounded connected components bifurcating from infinity for asymptotically linear operators, 1973. Since then, by using this result many authors have studied the existence of connected components of solutions of various boundary value problems. By using different methods-such as the blow up method, the maximum principle, the moving plane method, turning point theorem, eigenvalue theories and so on, they tried to obtain much possible information on the connected components. An interesting question concerning the connected component bifurcating from infinity is: in which manner the connected component of solutions approaches infinity? always from one side of a parameter in the parameter-norm plane, or oscillating infinitely about a parameter (even an interval of parameters)?

Let us first recall some results in the literature concerning the above problem. Schaaf and Schmitt [11] studied the existence of solutions of nonlinear Sturm Liouville problems whose linear part is at resonance. By using bifurcation methods, they studied the one parameter problem

$$\begin{gathered} u'' + \lambda u + g(u) = h(x), \quad 0 \le x \le \pi; \\ u(0) = 0 = u(\pi). \end{gathered} \tag{1.1}$$

They showed that (1.1) has a connected component of solutions which bifurcates from infinity at $\lambda = 1$, and showed that this connected component must cross the $\lambda = 1$ parameter plane infinitely often.

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Davidson and Rynne [2] studied the semilinear Sturm Liouville boundary value problem

$$\begin{gathered} -u'' = \lambda u + f(u) \quad \text{in } (0,\pi), \\ u(0) = u(\pi) = 0, \end{gathered} \tag{1.2}$$

where $f : \mathbb{R}^+ = [0,\infty) \to \mathbb{R}^1$ is Lipschitz continuous and λ is a real parameter. It is assumed in [2] that $f(s)$ oscillates, as $s \to \infty$, in such a manner that the problem (1.2) is not linearizable at $u = \infty$ but does have a connected component C of positive solutions bifurcating from infinity. Then, they investigated the relationship between the oscillations of f and those of C in the $\lambda - \|u\|$ plane at large $\|u\|$. They obtained some results about the oscillation properties of C over a single point λ, or over an interval I of λ values. An immediate consequence of such oscillations of I is the existence of infinitely many solutions, of arbitrarily large norm $\|u\|$, of problem (1.2) for all values of $\lambda \in I$. Here, as defined as in [2], a continuum $C \subset \mathbb{R}^+ \times E$ is said to oscillate over an interval $I = [\lambda_-, \lambda_+]$ if, for each $\nu \in \{+,-\}$, there exists a sequence of positive number $\{\zeta_n^\nu\}$,such that $\zeta_n^\nu \to \infty$ as $n \to \infty$, and any solution $(\lambda, u) \in C$ with $\|u\| = \zeta_n^\nu$ must have $\nu(\lambda - \lambda_\nu) \geq 0$, and such solutions do exist for all sufficiently large n. For other references concerning the connected component of solutions with asymptotic oscillation property one can refer to [3, 5, 9, 12, 13].

Consider the three-point boundary-value problem

$$\begin{gathered} -u'' = \lambda u + f(u) \quad \text{in } (0,1), \\ u(0) = 0, \quad u(1) = \alpha u(\eta), \end{gathered} \tag{1.3}$$

where $\eta \in (0,1)$, $\alpha \in [0,1)$, $f : \mathbb{R}^+ \to \mathbb{R}^1$ is Lipschitz continuous, $f(0) = 0$ and λ is a real parameter.

During the past twenty years the multi-point boundary value problems have been studied extensively. Especially, some authors studied multi-boundary value problems by using global bifurcation theories; see [4, 10, 15]. The main purpose of this paper is to extend some main results in [2] to equation (1.3). By employing an oscillatory condition on the nonlinear term f we will prove a result for the existence of a connected component of solutions of (1.3), which bifurcates from infinity and oscillates infinitely often over an interval of λ-values. There is a main difficulty to extend the main results of [2] to the three point boundary value problem (1.3). Obviously, the symmetric point of every positive solution in [2] is known and this plays an important role in the proof of [2]. For example, every positive solution of (1.2) is symmetric about $t_0 = 1/2$ and has a single maximum occurring at this point. However, the symmetric point of every positive solution of (1.3) is unknown and the positive solution of (1.3) may not be symmetric about $t_0 = 1/2$ when $\alpha \neq 0$. To overcome this difficulty in section 2 we will give a detailed analysis of positive solutions of (1.3). Note the nonlinearity f may not be of asymptotically linear type. Consequently, the corresponding nonlinear operator may be non-differentiable when one converts (1.3) into an operator equation in $C[0,1]$, the methods in Rabinowitz' well known global bifurcation theorems from [8] establishing existence results for unbounded connected components bifurcating from infinity do not seem to work in our situation. However, due to the contributions of Schmitt, Berestycki et al., during the past forty years significant progress on the nonlinear eigenvalue problems for non-differential mappings has been achieved; see [1, 6, 7, 14] and the references therein. By using the methods in [1, 6, 7, 14] we can show the existence of connected component of solutions of (1.3) bifurcating from infinity.

2. Properties of positive solutions

First let us recall some results concerning the linear eigenvalue problem

$$\begin{gathered}-u'' = \lambda u, \quad t \in (0,1),\\ u(0) = 0, \quad u(1) = \alpha u(\eta).\end{gathered} \tag{2.1}$$

According to [17], there exists a sequence of eigenvalues

$$0 < \sqrt{\lambda_1(\alpha)} < \sqrt{\lambda_2(\alpha)} < \dots < \sqrt{\lambda_n(\alpha)} < \dots,$$

where $\sqrt{\lambda_i(\alpha)}$ is the i-th positive solution of the elementary equation $\sin x = \alpha \sin x\eta$. The corresponding eigenfunction to $\sqrt{\lambda_n(\alpha)}$ is $\phi_{n,\alpha}(t) = \sin \sqrt{\lambda_n(\alpha)}t$. In what follows, for brevity, denote $\phi_{1,\alpha}$ by ϕ_α, for each $\alpha \in [0,1)$. It is easy to show the following result concerning the principal eigenvalue $\sqrt{\lambda_1(\alpha)}$.

Lemma 2.1. *Assume that $\alpha \in [0,1)$. Then*

(1) $\frac{\pi}{2} < \sqrt{\lambda_1(\alpha)} \le \pi$;
(2) $\sqrt{\lambda_1(\alpha)}$ *is non-increasing with* $\alpha \in [0,1)$;
(3) *there exists an unique* $t'_\alpha \in [\frac{1}{2}, 1)$ *such that* $\|\phi_\alpha\| = \phi_\alpha(t'_\alpha) = 1$, *and* $\phi'_\alpha(t) > 0$ *for* $t \in [0, t'_\alpha)$, $\phi'_\alpha(t) < 0$ *for* $t \in (t'_\alpha, 1]$;
(4) $\phi_\alpha \to \phi_0 = \sin \pi t$ *in the* C^1 *norm on* $[0,1]$ *as* $\alpha \to 0$.

For each $\alpha \in [0, \frac{1}{2}]$, let

$$\begin{aligned}S_\alpha = \big\{&(\lambda, u) : \lambda \in \mathbb{R}^+, u \in C[0,1], u(t) > 0 \text{ for } t \in (0,1), \text{ such that}\\ &u(t) \text{ is a solution of } (1.3)\big\}.\end{aligned}$$

Then, for each $(\lambda, u) \in S_\alpha$, using integration by parts and the boundary condition, we have

$$(\lambda_1(\alpha) - \lambda) \int_0^1 u\phi_\alpha dt = \int_0^1 f(u)\phi_\alpha dt - W(\alpha), \tag{2.2}$$

where

$$W(\alpha) = \begin{vmatrix} u(1), & u'(1)\\ \phi_\alpha(1), & \phi'_\alpha(1)\end{vmatrix} = \alpha \begin{vmatrix} u(\eta), & u'(1)\\ \phi_\alpha(\eta), & \phi'_\alpha(1)\end{vmatrix}.$$

Let

$$G(t,s) = \min\{t,s\}(1 - \max\{t,s\}), \quad \forall t, s \in [0,1],$$

and the operator $K_\alpha : C[0,1] \to C[0,1]$ be defined by

$$K_\alpha x(t) = \int_0^1 G(t,s)x(s)ds + \frac{\alpha t}{1 - \alpha\eta} \int_0^1 G(\eta, s)x(s)ds, \quad t \in [0,1].$$

Let $P = \{x \in C[0,1] : x(t) \ge 0 \text{ for } t \in [0,1]\}$. Then, for each $h \in C[0,1]$, $y = K_\alpha h$ if and only if

$$\begin{gathered}-y'' = h(t), \quad t \in (0,1),\\ y(0) = 0, \quad y(1) = \alpha y(\eta).\end{gathered}$$

Let

$$\begin{gathered}e_\alpha(t) = \frac{1}{2}\eta(1-\eta)t[1 - \alpha\eta - (1-\alpha)t], \quad t \in [0,1],\\ e(t) = \frac{1}{4}\eta(1-\eta)t(1-t), \quad \forall t \in [0,1].\end{gathered}$$

Then, we have $e_\alpha(t) \ge e(t)$ for $t \in [0,1]$ and $\alpha \in [0, 1/2]$.

Lemma 2.2. *For each $\alpha \in [0,1/2]$, let $Q_\alpha = \{x \in P : x(t) \geq \|x\| e_\alpha(t)$ for $t \in [0,1]\}$. Then $K_\alpha : P \to Q_\alpha$ is completely continuous.*

Proof. Obviously, $K_\alpha : P \to C[0,1]$ is completely continuous. Take $h \in P$, and let $y = K_\alpha h$. Using the fact that y is a concave function on $[0,1]$, by [18, Lemma 2] we easily check that $K_\alpha : P \to Q_\alpha$. The proof is complete. $\square$

In this paper we use the following symbols.

$$0 < \kappa^- = \liminf_{s\to+\infty} \frac{f(s)}{s} \leq \limsup_{s\to+\infty} \frac{f(s)}{s} = \kappa^+ < +\infty,$$

$$-\infty < \zeta^- = \liminf_{s\to+\infty} \frac{F(s)}{s^2} \leq \limsup_{s\to+\infty} \frac{F(s)}{s^2} = \zeta^+ < +\infty,$$

where $F(s) = \int_0^s f(t)dt$ for all $s \in [0,+\infty)$.

In this article we always assume that

$$\frac{2-\eta}{3-\eta} > \kappa^+ - \kappa^-. \tag{2.3}$$

Take $\delta > 0$ small enough, such that $\frac{2-\eta}{3-\eta} - (\kappa^+ - \kappa^-) > 2\delta$ and $\kappa^- - \delta > 0$. Let $\widehat{\kappa}^- = \kappa^- - \delta$, $\widehat{\kappa}^+ = \kappa^+ + \delta$, and

$$\widetilde{s}_0 = \min\big\{1, \frac{2-\eta}{2(3-\eta)} - \frac{1}{2}(\widehat{\kappa}^+ - \widehat{\kappa}^-)\big\}.$$

Take $s_0 > 0$ large enough, such that

$$\widehat{\kappa}^- s \leq f(s) \leq \widehat{\kappa}^+ s, \quad s \geq s_0. \tag{2.4}$$

Let $M_0 = \sup\{|f(s)| : s \in [0,s_0]\}$. By (2.4) we have

$$\widehat{\kappa}^- s - M_0 \leq f(s) \leq \widehat{\kappa}^+ s + M_0, \quad s \geq 0. \tag{2.5}$$

Let

$$\gamma_{0,\alpha} = \frac{\|K_\alpha\|^{-1} - \widehat{\kappa}^+ - \widehat{\kappa}^-}{2}, \quad \gamma_{1,\alpha} = 2\|K_\alpha e_\alpha\|^{-1} + \gamma_{0,\alpha}.$$

Let

$$k_0 = \max_{t\in[0,1]} \int_0^1 G(t,s)s(1-s)ds.$$

For each $t \in [0,1]$, we have

$$\begin{aligned} \frac{3-\eta}{2-\eta} &\geq K_\alpha e_\alpha(t) \geq \int_0^1 G(t,s)e_\alpha(s)ds \geq \int_0^1 G(t,s)e(s)ds \\ &\geq \frac{1}{4}\eta(1-\eta)\int_0^1 G(t,s)s(1-s)ds. \end{aligned}$$

and so,

$$\frac{3-\eta}{2-\eta} \geq \|K_\alpha e_\alpha\| \geq \frac{1}{4}k_0\eta(1-\eta).$$

On the other hand, for $t \in [0,1]$ we have

$$k_0 \leq \|K_\alpha\| \leq 1 + \frac{\alpha}{1-\alpha\eta} \leq \frac{3-\eta}{2-\eta}.$$

Thus, we have

$$a^- := \frac{1}{2}\Big(\frac{2-\eta}{3-\eta} - \widehat{\kappa}^+ - \widehat{\kappa}^-\Big) \leq \gamma_{0,\alpha} \leq \frac{k_0^{-1} - \widehat{\kappa}^+ - \widehat{\kappa}^-}{2},$$

$$a^+ := \frac{k_0^{-1} - \widehat{\kappa}^+ - \widehat{\kappa}^-}{2} + \frac{8}{k_0\eta(1-\eta)} \geq \gamma_{1,\alpha}.$$

For each $\alpha \in [0, \frac{1}{2}]$, let $\Lambda_\alpha = \bar{S}_\alpha \cap \big([a^-, a^+] \times B^c(\theta, R_0)\big)$, where $B^c(\theta, R_0) = \{x \in C[0,1], \|x\| \geq R_0\}$ and $R_0 = \frac{8M_0}{\eta(1-\eta)}$.

Lemma 2.3. *For each $(\lambda, u) \in \Lambda_\alpha$, we have*

$$u(t) \geq \frac{1}{2}\|u\| e_\alpha(t), \quad t \in [0,1]. \tag{2.6}$$

Proof. Define the operators $\widetilde{F}(\cdot,\cdot) : [a^-, +\infty) \times C[0,1] \to C[0,1]$ by

$$\widetilde{F}(\lambda, u)(t) = \lambda u(t) + f(u(t)) + M_0, \quad t \in [0,1]$$

and $B(\cdot,\cdot) : [a^-, +\infty) \times C[0,1] \to C[0,1]$ by $B(\lambda, u)(t) = K_\alpha \widetilde{F}(\lambda, u)(t)$ for $t \in [0,1]$. Let $v(t) = B(\lambda, u)(t)$ for all $t \in [0,1]$. By (2.3) and (2.5) we have $\widetilde{F}(\lambda, u(t)) \geq (\widehat{\kappa}^- + a^-)u(t) \geq 0$ for all $t \in [0,1]$. It follows from Lemma 2.2 that $v \in Q_\alpha$. Obviously, $u(t) = B(\lambda, u)(t) - M_0\omega_\alpha(t) = v(t) - M_0\omega_\alpha(t)$ for all $t \in [0,1]$, where

$$\omega_\alpha(t) = \int_0^1 G(t,s)ds + \frac{\alpha t}{1-\alpha\eta}\int_0^1 G(\eta, s)ds, \forall t \in [0,1].$$

It is easy to see that

$$\omega_\alpha(t) \leq \frac{1}{2}t\big[1 - t + \frac{\alpha\eta(1-\eta)}{\eta(1-\alpha)}\big] = \frac{e_\alpha(t)}{\eta(1-\eta)(1-\alpha)} \leq \frac{2e_\alpha(t)}{\eta(1-\eta)}, t \in [0,1].$$

Then, we have

$$\begin{aligned} u(t) &\geq \|v\| e_\alpha(t) - \frac{2M_0}{\eta(1-\eta)} e_\alpha(t) \\ &\geq \Big(\|u\| - M_0\|\omega_\alpha\| - \frac{2M_0}{\eta(1-\eta)}\Big) e_\alpha(t) \\ &\geq \frac{1}{2}\|u\| e_\alpha(t), \quad t \in [0,1]. \end{aligned}$$

This implies that (2.6) holds. The proof is complete. □

Lemma 2.4. *For each $(\lambda, u) \in \Lambda_\alpha$, there exists an unique $t_\alpha \in (0,1)$ such that*

1. *$u(t_\alpha) = \|u\|$;*
2. *$u'(t) > 0$, $t \in (0, t_\alpha)$; $u'(t) < 0$, $t \in (t_\alpha, 1)$;*
3. *$u(t_\alpha - s) = u(t_\alpha + s)$, $s \in [0, 1 - t_\alpha]$;*
4. *$1 > t_\alpha \geq 1/2$;*
5. *$t_\alpha = 1/2$ as $\alpha = 0$.*

Proof. It follows from the theorem on the unique solutions of initial value problems for differential equations (IVPU) that $u'(0) > 0$. From the boundary value condition $u(1) = \alpha u(\eta), 0 \leq \alpha < 1/2$, there must exist $t_\alpha \in (0,1)$ such that $u'(t_\alpha) = 0$. Assuming that $t_\alpha < 1/2$, it follows from the fact that f is independent of t and IVPU that

$$u(t_\alpha + s) = u(t_\alpha - s), \quad s \in [0, t_\alpha].$$

Hence, $u(2t_\alpha) = u(0) = 0$, $u'(2t_\alpha) = -u'(0) < 0$, and so, there exists $t' > 2t_\alpha$ such that $u(t') < 0$, which is a contradiction. Hence, $1 > t_\alpha \geq 1/2$, and $u(t_\alpha - s) = u(t_\alpha + s)$ for $s \in [0, 1 - t_\alpha]$. Obviously, (1) and (2) hold. When $\alpha = 0$, $u(1) = 0$ and so $t_\alpha = 1/2$. The proof is complete. □

Lemma 2.5. *Assume that $\Lambda_\alpha \cap \big([a^-, a^+] \times B^c(\theta, R_1)\big) \neq \emptyset$, where $R_1 = R_0 + 24M_0 c_1^{-1}$ and $c_1 = \frac{1}{8}\widetilde{s}_0 \eta(1-\eta)$. Let $\sigma_0(\alpha) = \frac{200\alpha}{\widetilde{s}_0\eta(1-\eta)}$ for $\alpha \in [0, \frac{\widetilde{s}_0\eta(1-\eta)}{4000}]$. For each $(\lambda, u) \in \Lambda_\alpha \cap \big([a^-, a^+] \times B^c(\theta, R_1)\big)$, let t_α be defined as in Lemma 2.4. Then $\sigma_0(\alpha) \to 0$ as $\alpha \to 0$, and for $\alpha \in [0, \widetilde{s}_0\eta(1-\eta)/4000]$,*

$$0 \leq t_\alpha - \frac{1}{2} \leq \sigma_0(\alpha). \tag{2.7}$$

Proof. For each $(\lambda, u) \in \Lambda_\alpha \cap \big([a^-, a^+] \times B^c(\theta, R_1)\big)$, let $v(t) = u(t)/\|u\|$ for $t \in [0,1]$. Then

$$\begin{gathered} -v''(t) = \lambda v(t) + \frac{1}{\|u\|} f(\|u\| v(t)), \quad t \in (0,1), \\ v(0) = 0, \quad v(1) = \alpha v(\eta). \end{gathered} \tag{2.8}$$

It follows from Lemma 2.4 that $v(t_\alpha) = 1$, $t_\alpha \geq 1/2$ and

$$v(t_\alpha - s) = v(t_\alpha + s), \quad \forall s \in [0, 1 - t_\alpha].$$

Hence, $v(2t_\alpha - 1) = v(1) = \alpha v(\eta) \leq \alpha$. It follows from Lemma 2.4 that $v'(t) > 0$ for $t \in (0, t_\alpha)$. Thus, if there exits a $t' \in [0, t_\alpha)$ such that $v(t') \geq \alpha \geq v(2t_\alpha - 1)$, then we have $t' \geq 2t_\alpha - 1$ or $t_\alpha \leq \frac{1}{2}(1 + t')$.

By Lemma 2.3, for $t \in [0, t_\alpha)$, we have

$$\begin{aligned} u'(t) &= -\int_t^{t_\alpha} u''(s) ds \\ &\geq \frac{1}{2}\widetilde{s}_0 \|u\| \int_t^{t_\alpha} e_\alpha(s) ds - M_0(t_\alpha - t) \\ &\geq \frac{1}{2}\widetilde{s}_0 \|u\| \int_t^{t_\alpha} e(s) ds - M_0(t_\alpha - t), \end{aligned} \tag{2.9}$$

and so

$$v'(t) \geq \frac{1}{2}\widetilde{s}_0 \int_t^{t_\alpha} e(s) ds - \frac{M_0}{\|u\|}(t_\alpha - t).$$

Then, we have

$$\begin{aligned} v(t) &= v(0) + \int_0^t v'(s) ds = \int_0^t v'(s) ds \\ &\geq \frac{1}{2}\widetilde{s}_0 \int_0^t ds \int_s^{t_\alpha} e(\tau) d\tau - \frac{M_0}{\|u\|} \int_0^t (t_\alpha - s) ds \\ &\geq c_1 \Big(t \int_t^{t_\alpha} s(1-s) ds + \int_0^t s^2(1-s) ds \Big) - \frac{M_0}{\|u\|} t \\ &= c_1 \Big(t \int_0^{t_\alpha} s(1-s) ds - t \int_0^t s(1-s) ds + \int_0^t s^2(1-s) ds \Big) - \frac{M_0}{\|u\|} t \\ &\geq c_1 \Big(t \int_0^{1/2} s(1-s) ds - t\big(\frac{1}{2}t^2 - \frac{1}{3}t^3\big) + \big(\frac{1}{3}t^3 - \frac{1}{4}t^4\big) \Big) - \frac{M_0}{\|u\|} t \\ &\geq \Big(\frac{1}{12} c_1 - \frac{M_0}{\|u\|} \Big) t - \frac{1}{6} c_1 t^2 \\ &\geq \frac{1}{24} c_1 t - \frac{1}{6} c_1 t^2. \end{aligned}$$

Solving the inequality
$$\frac{1}{24}c_1t-\frac{1}{6}c_1t^2\ge\alpha,$$
we obtain
$$\frac{1-\sqrt{1-384\alpha c_1^{-1}}}{8}\le t\le\frac{1+\sqrt{1-384\alpha c_1^{-1}}}{8}.$$
Let
$$t'=\frac{1-\sqrt{1-384\alpha c_1^{-1}}}{8}.$$
Then we have
$$0\le t_\alpha-\frac{1}{2}\le\frac{1-\sqrt{1-384\alpha c_1^{-1}}}{16}\le\frac{24\alpha}{c_1}\le\sigma_0(\alpha).$$
Obviously, $\sigma_0(\alpha)\to 0$ as $\alpha\to 0$. The proof is complete. ∎

Lemma 2.6. *Let $\lambda_1(\alpha)$, t'_α be defined as in Lemma 2.1 and $\sigma_1(\alpha)=\frac{100\alpha}{\eta(1-\eta)}$ for each $\alpha\in[0,\frac{\widetilde{s}_0\eta(1-\eta)}{4000}]$. Then for each $\alpha\in[0,\frac{\widetilde{s}_0\eta(1-\eta)}{4000}]$,*
$$0\le t'_\alpha-\frac{1}{2}\le\sigma_1(\alpha).$$

Proof. It follows from (2) in Lemma 2.1 that $\pi^2=\lambda_1(0)\ge\lambda_1(\alpha)\ge\lambda_1(\frac{1}{2})\ge\frac{\pi^2}{4}$ for each $\alpha\in[0,1/2]$. According to Lemma 2.2 we have $\phi_\alpha\in Q_\alpha$ for $\alpha\in[0,1/2]$, and so
$$\phi_\alpha(t)\ge\|\phi_\alpha\|e_\alpha(t)=e_\alpha(t)\ge e(t),\quad\forall t\in[0,1].$$
Note that $\frac{\widetilde{s}_0\eta(1-\eta)}{4000}\le 1/2$. As in the proof Lemma 2.5, for $t\in[0,t'_\alpha]$, we have
$$\begin{aligned}
\phi_\alpha(t)&=\phi_\alpha(0)+\int_0^t\phi'_\alpha(s)ds=\int_0^t\phi'_\alpha(s)ds\\
&\ge\lambda_1(\frac{1}{2})\int_0^t ds\int_s^{t'_\alpha}\phi_\alpha(\tau)d\tau\\
&\ge\frac{\eta(1-\eta)}{4}\lambda_1(\frac{1}{2})\int_0^t ds\int_s^{t'_\alpha}\tau(1-\tau)d\tau\\
&\ge\frac{\pi^2\eta(1-\eta)}{16}\Big(t\int_0^{1/2}s(1-s)ds-t\int_0^t s(1-s)ds+\int_0^t s^2(1-s)ds\Big)\\
&\ge\frac{\pi^2\eta(1-\eta)}{16}(\frac{1}{12}t-\frac{1}{6}t^2).
\end{aligned}$$
Solving the inequality
$$\frac{\pi^2\eta(1-\eta)}{16}\Big(\frac{1}{12}t-\frac{1}{6}t^2\Big)\ge\alpha,$$
we have
$$\frac{1-\sqrt{1-\frac{1536\alpha}{\pi^2\eta(1-\eta)}}}{4}\le t\le\frac{1+\sqrt{1-\frac{1536\alpha}{\pi^2\eta(1-\eta)}}}{4}.$$
Then we have
$$0\le t'_\alpha-\frac{1}{2}\le\frac{1-\sqrt{1-\frac{1536\alpha}{\pi^2\eta(1-\eta)}}}{8}\le\sigma_1(\alpha).$$
The proof is complete. ∎

Lemma 2.7. *Assume that* $\Lambda_\alpha \cap \big([a^-, a^+] \times B^c(\theta, R_2)\big) \neq \emptyset$, *where* $R_2 = R_1 + \frac{200M_0}{\widetilde{s}_0\eta(1-\eta)}$. *Let* c_1 *and* $\sigma_0(\alpha)$ *be defined as in Lemma 2.5,* $c_2 = \frac{c_1}{24}$. *Assume that* $\alpha \in [0, \frac{\widetilde{s}_0\eta(1-\eta)}{4000}]$. *For each* $(\lambda, u) \in \Lambda_\alpha \cap \big([a^-, a^+] \times B^c(\theta, R_2)\big)$, *let* t_α *be defined as in Lemma 2.4. Then we have*

$$|u'(t)| \geq c_2\|u\||t_\alpha - t|, t \in [0,1]. \tag{2.10}$$

Proof. Now, (2.9) holds for each $(\lambda, u) \in \Lambda_\alpha \cap \big([a^-, a^+] \times B^c(\theta, R_2)\big)$. From $\alpha \in [0, \frac{\widetilde{s}_0\eta(1-\eta)}{4000}]$, we have $\sigma_0(\alpha) < \frac{1}{10}$, and so, by (2.7) we have $t_\alpha \in [1/2, 3/5]$. On the other hand, we have for $t \in [0, t_\alpha)$,

$$\int_t^{t_\alpha} e(s)ds = \frac{\eta(1-\eta)}{4}\big[\frac{1}{2}(t_\alpha + t) - \frac{1}{3}(t_\alpha^2 + t_\alpha t + t^2)\big](t_\alpha - t). \tag{2.11}$$

Let

$$g(t) = \frac{1}{2}(t_\alpha + t) - \frac{1}{3}(t_\alpha^2 + t_\alpha t + t^2), \quad \forall t \in [0,1].$$

It is easy to see that $g(t) \geq \min\{g(0), g(t_\alpha)\} \geq \frac{1}{8}$ for $t \in [0, t_\alpha]$. It follows from (2.9) and (2.11) that for $t \in (0, t_\alpha)$,

$$u'(t) \geq \big[\frac{1}{64}\widetilde{s}_0\eta(1-\eta)\|u\| - M_0\big](t_\alpha - t) \geq c_2\|u\|(t_\alpha - t). \tag{2.12}$$

Similarly, for $t \in [t_\alpha, 1]$, we have

$$\begin{aligned} -u'(t) &= -\int_{t_\alpha}^t u''(s)ds \\ &\geq \frac{1}{2}\widetilde{s}_0\|u\| \int_{t_\alpha}^t e_\alpha(s)ds - M_0(t - t_\alpha) \\ &\geq \frac{1}{2}\widetilde{s}_0\|u\| \int_{t_\alpha}^t e(s)ds - M_0(t - t_\alpha) \\ &\geq \big(c_1\|u\|g(t) - M_0\big)(t - t_\alpha) \\ &\geq \big(\frac{1}{12}c_1\|u\| - M_0\big)(t - t_\alpha) \\ &\geq c_2\|u\|(t - t_\alpha). \end{aligned} \tag{2.13}$$

Now (2.10) follows from (2.12) and (2.13). The proof is complete. $\square$

Lemma 2.8. *Assume that* $\Lambda_\alpha \cap \big([a^-, a^+] \times B^c(\theta, R_3)\big) \neq \emptyset$, *where* $R_3 = R_2 + M_0$. *For each* $(\lambda, u) \in \Lambda_\alpha \cap \big([a^-, a^+] \times B^c(\theta, R_2)\big)$, *let* t_α *be defined as in Lemma 2.4. Then the following inequalities also hold:*

$$(a^- + \widehat{\kappa}^-)u(t) \leq -u''(t) \leq (|a^+| + \widehat{\kappa}^+)u(t), \quad \forall t \in E^+(u; s_0), \tag{2.14}$$

$$|u''(t)| \leq (|a^+| + \widehat{\kappa}^+)s_0 + 3M_0, \quad \forall t \in E^-(u; s_0), \tag{2.15}$$

$$|u'(t)| \leq c_3\|u\||t - t_\alpha|, \forall t \in [0,1]. \tag{2.16}$$

where $c_3 = |a^+| + \widehat{\kappa}^+ + 1$, $E^+(u; s_0) = \{s \in [0,1] : u(s) \geq s_0\}$ *and* $E^-(u; s_0) = \{s \in [0,1] : u(s) < s_0\}$.

Proof. It follows from (2.4) and (2.5) that (2.14) and (2.15) hold. Also, by (2.5) we have

$$
\begin{aligned}
u'(t) &= -\int_t^{t_\alpha} u''(s)ds = \int_t^{t_\alpha} (\lambda u(s) + f(u(s)))ds \\
&\le \int_t^{t_\alpha} (\lambda u(s) + \widehat{\kappa}^+ u(s) + M_0]ds \\
&\le (|a^+| + \widehat{\kappa}^+ + 1)\|u\|(t_\alpha - t), \quad t \in (0, t_\alpha).
\end{aligned}
\tag{2.17}
$$

Similarly, we have

$$
-u'(t) \le (|a^+| + \widehat{\kappa}^+ + 1)\|u\|(t - t_\alpha), \forall t \in (t_\alpha, 1). \tag{2.18}
$$

Now (2.16) follows from (2.17) and (2.18). The proof is complete. □

Recall that if $\{\Sigma_n\}$ is a sequence of sets then

$$
\begin{aligned}
\liminf_{n\to\infty} \Sigma_n = \big\{x :\ &\text{there exists a positive integer } N_0 \text{ such that every} \\
&\text{neighborhood of } x \text{ intersects } \Sigma_n \text{ for } n \ge N_0\big\},
\end{aligned}
$$

$$
\begin{aligned}
\limsup_{n\to\infty} \Sigma_n = \big\{x :\ &\text{every neighborhood of } x \text{ intersects } \Sigma_n \\
&\text{for infinitely many integers } n\big\}.
\end{aligned}
$$

The proof of the next lemma can be found in [16].

Lemma 2.9. *Let $\{\Sigma_n\}$ be a sequence of connected sets in a complete metric space M. Assume that*

(i) $\cup_{n=1}^{\infty} \Sigma_n$ *is precompact in M;*
(ii) $\liminf_{n\to\infty} \Sigma_n \neq \emptyset$.

Then $\limsup_{n\to\infty} \Sigma_n$ is non-empty, closed and connected.

3. Oscillatory bifurcation from infinity

Let $\zeta_0 = \zeta^+ + \zeta^-$, $\zeta = \frac{\zeta^+ - \zeta^-}{2}$, $c_4 = c_3(2\widehat{\kappa}^+ + \zeta_0) + 4\zeta$,

$$
c_5 = \frac{\eta(1-\eta)\zeta^2}{64 c_4}\Big(1 - \frac{\zeta}{4c_4}\Big),
$$

and

$$
\vartheta = \min\Big\{\frac{\widetilde{s}_0\eta(1-\eta)}{4000}, \frac{c_2 c_5 \widetilde{s}_0 \eta(1-\eta)}{6400 c_4}, \frac{c_5}{16(\pi + c_3)}\Big\}.
$$

Recall the definition of S_α in section 2. Now we have the following main result.

Theorem 3.1. *Suppose that (2.3) holds, and $\zeta^- < \zeta^+$. Then, for $\alpha \in [0, \vartheta/2]$, $\bar{S}_\alpha$ possesses at least one connected component $C_{\alpha,\infty}$ bifurcating from infinity and oscillating over an interval $I_\alpha := [d_-(\alpha), d_+(\alpha)]$, where*

$$
d_-(\alpha) = \lambda_1(\alpha) - \frac{3}{4}c_5 - \zeta_0, d_+(\alpha) = \lambda_1(\alpha) + \frac{3}{4}c_5 - \zeta_0.
$$

Proof. Take $\tau_n \in (0,1)$ such that $\tau_n \to 1$ as $n \to \infty$. Let

$$
c_5^{(\tau_n)} = \frac{\tau_n(2-\tau_n)\eta(1-\eta)\zeta^2}{64 c_3 c_4}\Big(1 - \frac{(2-\tau_n)\zeta}{4c_4}\Big),
$$

$$
\vartheta^{(\tau_n)} = \min\Big\{\frac{\widetilde{s}_0\eta(1-\eta)}{4000}, \frac{c_2 c_5^{(\tau_n)} \widetilde{s}_0 \eta(1-\eta)}{6400 c_4}, \frac{c_5^{(\tau_n)}}{16(\pi + c_3)}\Big\}.
$$

Except for the last paragraph, in what follows of the proof we always assume that $\alpha \in [0, \vartheta^{(\tau_n)}]$.

By using the methods showing the main results in [1, 6, 7, 14] we can prove that $\bar{S}_\alpha$ possesses at least one connected component bifurcating from infinity. For brevity, we will omit the process. Let $\widetilde{f}(s) = f(s) - \zeta_0 s$, and $\widetilde{F}(s) = F(s) - \frac{1}{2}\zeta_0 s^2$ for $s \geq 0$. Then we have

$$\limsup_{s\to+\infty} \frac{\widetilde{F}(s)}{s^2} = \zeta, \quad \liminf_{s\to+\infty} \frac{\widetilde{F}(s)}{s^2} = -\zeta.$$

Thus, there exists two sequences of positive numbers $\{\rho_n^+\}$ and $\{\rho_n^-\}$, with $\rho_n^+ \to \infty$ and $\rho_n^- \to \infty$, such that

$$\widetilde{F}(\rho_n^-) \geq \frac{\zeta}{2}(\rho_n^-)^2, \quad \widetilde{F}(\rho_n^+) \leq -\frac{\zeta}{2}(\rho_n^+)^2. \tag{3.1}$$

For $n = 1, 2, \dots,$ let

$$\begin{gathered} \varsigma_n^- = \inf\big\{s \geq 0 : \widetilde{F}(s) \geq \frac{\zeta}{2}(\rho_n^-)^2\big\}, \\ \varsigma_n^+ = \inf\big\{s \geq 0 : \widetilde{F}(s) \leq -\frac{\zeta}{2}(\rho_n^+)^2\big\}. \end{gathered}$$

Assume without loss of generality that $\{\varsigma_n^+\}$ and $\{\varsigma_n^-\}$ are strictly increasing, and $R_4^{(\tau_n)} \leq \varsigma_n^+ < \varsigma_n^- < \varsigma_{n+1}^+$ for $n \in \mathbb{N}$, where

$$\begin{gathered} R_4^{(\tau_n)} = R_3 + 3M_0(\widehat{\kappa}^+)^{-1} + 32 s_0(\eta(1-\eta))^{-1} + 1 + 1024 s_0 c_6 (\eta(1-\eta) c_5^{(\tau_n)})^{-1}, \\ c_6 = \frac{c_4[(|a^+| + |\widehat{\kappa}^+|)s_0 + 3M_0]}{2c_2^2}, \end{gathered}$$

and R_3 is defined as in Lemma 2.8. Let $(\lambda, u) \in \bar{S}_\alpha$, with $\|u\| = \varsigma_n^-$ and $\lambda \in [a^-, a^+]$. Then, (λ, u) satisfies (2.2). Writing $Z(u;t) = \widetilde{F}(\|u\|) - \widetilde{F}(u(t))$ for $t \in [0,1]$. Obviously, $Z(u;t) \geq 0$ for $t \in [0,1]$. By a direct computation, we have

$$\int_0^1 \widetilde{f}(u)\phi_\alpha dt = \int_0^1 Z(u;t)\frac{u'\phi_\alpha' - \phi_\alpha u''}{(u')^2} dt - Z(u;1)\frac{\phi_\alpha(1)}{u'(1)}. \tag{3.2}$$

By (2.5) and using the fact that $\|u\| \geq 3M_0(\widehat{\kappa}^+)^{-1}$, we have

$$|f(u(t))| \leq \widehat{\kappa}^+ u(t) + 3M_0 \leq 2\widehat{\kappa}^+\|u\|, \quad t \in [0,1]. \tag{3.3}$$

Let $t_\alpha \in [1/2, 1)$ be such that $u(t_\alpha) = \|u\|$, and $u'(t_\alpha) = 0$. It follows from Lemma 2.5 that $t_\alpha \in [\frac{1}{2}, \frac{1}{2} + \sigma_0(\alpha)]$. Since $Z_t'(u;t) = -\widetilde{f}(u(t))u'(t)$, by (2.16) and (3.3) we have

$$|Z_t'(u;t)| \leq c_4\|u\|^2|t - t_\alpha|, \quad \forall t \in [0,1], \tag{3.4}$$

and so

$$\begin{aligned} |Z(u;t)| &= \big|Z(u;t_\alpha) + \int_{t_\alpha}^t Z_s'(u;s)ds\big| \\ &\leq \int_{t_\alpha}^t |Z_s'(u;s)|ds \\ &\leq \frac{1}{2}c_4\|u\|^2(t - t_\alpha)^2 \end{aligned}$$

for each $t \in [t_\alpha, 1]$. Similarly, we also have

$$|Z(u;t)| \le \frac{1}{2} c_4 \|u\|^2 (t - t_\alpha)^2, \quad \forall t \in [0, t_\alpha].$$

Thus, we have

$$|Z(u;t)| \le \frac{1}{2} c_4 \|u\|^2 (t - t_\alpha)^2, \quad \forall t \in [0, 1]. \tag{3.5}$$

Now we give estimates for each part of the right side of the equality (3.2), and for $W(\alpha)$.

(1) Estimate for $Z(u;1)\frac{\phi_\alpha(1)}{u'(1)}$. It follows from (3.5) and Lemma 2.7 that for $\alpha \in [0, \vartheta^{(\tau_n)}]$,

$$\big|Z(u;1)\frac{\phi_\alpha(1)}{u'(1)}\big| \le \frac{1}{2} c_4 \|u\|^2 (1 - t_\alpha)^2 \frac{\alpha \phi_\alpha(\eta)}{c_2 \|u\|(1 - t_\alpha)} \le \frac{c_4 \|u\| \alpha}{2c_2} \le \frac{1}{16} c_5^{(\tau_n)} \|u\|. \tag{3.6}$$

(2) Estimate for $\int_0^1 Z(u;t)\frac{-\phi_\alpha u''}{(u')^2}dt$. Since $Z(u;t) \ge 0$ for $t \in [0,1]$, and $-u''(t) \ge 0$ for $t \in E^+(u; s_0)$, we have

$$\begin{aligned} \int_0^1 Z(u;t)\frac{-\phi_\alpha u''}{(u')^2}dt &= \Big(\int_{E^+(u;s_0)} + \int_{E^-(u;s_0)}\Big) Z(u;t)\frac{-\phi_\alpha u''}{(u')^2}dt \\ &\ge \int_{E^-(u;s_0)} Z(u;t)\frac{-\phi_\alpha u''}{(u')^2}dt. \end{aligned} \tag{3.7}$$

On the other hand, by (3.5), Lemmas 2.7 and 2.8 we have

$$\begin{aligned} &\big|\int_{E^-(u;s_0)} Z(u;t)\frac{-\phi_\alpha u''}{(u')^2}dt\big| \\ &\le \int_{E^-(u;s_0)} |Z(u;t)|\big|\frac{\phi_\alpha u''}{(u')^2}\big|dt \\ &\le \int_{E^-(u;s_0)} |Z(u;t)|\big|\frac{u''}{(u')^2}\big|dt \\ &\le \int_{E^-(u;s_0)} \frac{1}{2} c_4 \|u\|^2 (t - t_\alpha)^2 \frac{|u''|}{c_2^2 \|u\|^2 (t - t_\alpha)^2} dt \\ &= \int_{E^-(u;s_0)} \frac{c_4 |u''|}{2c_2^2} dt \\ &\le \int_{E^-(u;s_0)} \frac{c_4[(|a^+| + |\widehat{\kappa}^+|)s_0 + 3M_0]}{2c_2^2} dt \\ &\le c_6 \cdot \operatorname{meas}(E^-(u; s_0)). \end{aligned}$$

from Lemma 2.3, it follows that for $\alpha \in [0, \vartheta^{(\tau_n)}]$,

$$E^-(u;s_0) \subset \Big[0, \frac{1 - \sqrt{1 - \frac{32 s_0}{\eta(1-\eta)\|u\|}}}{2}\Big] \cup \Big[\frac{1 + \sqrt{1 - \frac{32 s_0}{\eta(1-\eta)\|u\|}}}{2}, 1\Big].$$

and so

$$\begin{aligned} \big|\int_{E^-(u;s_0)} Z(u;t)\frac{-\phi_\alpha u''}{(u')^2}dt\big| &\le c_6 \cdot \operatorname{mes}(E^-(u; s_0)) \\ &\le \frac{64 s_0 c_6}{\eta(1-\eta)\|u\|} \le \frac{1}{16} c_5^{(\tau_n)} \|u\|. \end{aligned} \tag{3.8}$$

It follows from (3.7) and (3.8) that

$$\int_0^1 Z(u;t)\frac{-\phi_\alpha u''}{(u')^2}dt \geq -\frac{1}{16}c_5^{(\tau_n)}\|u\|. \tag{3.9}$$

(3) Estimate for $\int_0^1 Z(u;t)\frac{\phi_\alpha'}{u'}dt$. According to Lemmas 2.5 and 2.6, there exist $t_\alpha, t_\alpha' \geq 1/2$ such that $\|u\| = u(t_\alpha)$, $\|\phi_\alpha\| = \phi_\alpha(t_\alpha')$, and $0 \leq t_\alpha - \frac{1}{2} \leq \sigma_0(\alpha)$, $0 \leq t_\alpha' - \frac{1}{2} \leq \sigma_1(\alpha)$. Let $\sigma(\alpha) = \max\{\sigma_0(\alpha), \sigma_1(\alpha)\}$. Obviously, we have $\sigma(\alpha) = \sigma_0(\alpha)$. Now we have

$$\int_0^1 Z(u;t)\frac{\phi_\alpha'}{u'}dt = \Big(\int_0^{1/2} + \int_{\frac{1}{2}}^{\frac{1}{2}+\sigma(\alpha)} + \int_{\frac{1}{2}+\sigma(\alpha)}^1\Big) Z(u;t)\frac{\phi_\alpha'}{u'}dt. \tag{3.10}$$

It follows from Lemmas 2.1 and 2.4 that $\phi_\alpha'(t) < 0$ and $u'(t) < 0$ for $t \in [\frac{1}{2}+\sigma(\alpha), 1]$, and so

$$\int_{\frac{1}{2}+\sigma(\alpha)}^1 Z(u;t)\frac{\phi_\alpha'}{u'}dt \geq 0. \tag{3.11}$$

From Lemma 2.1 we have $\pi/2 < |\sqrt{\lambda_1(\alpha)}| \leq \pi$, and so $\pi/2 \leq |\phi_\alpha'| \leq \pi$. It follows from (3.5), Lemmas 2.7 and 2.8 that

$$\begin{aligned}
\Big|\int_{\frac{1}{2}}^{\frac{1}{2}+\sigma(\alpha)} Z(u;t)\frac{\phi_\alpha'}{u'}dt\Big| &\leq \int_{\frac{1}{2}}^{\frac{1}{2}+\sigma(\alpha)} |Z(u;t)\frac{\phi_\alpha'}{u'}|dt \\
&\leq \int_{\frac{1}{2}}^{\frac{1}{2}+\sigma(\alpha)} \frac{1}{2}c_4\|u\|^2(t-t_\alpha)^2\frac{\pi}{c_2\|u\|(t-t_\alpha)}dt \\
&\leq \frac{c_4\pi\|u\|}{2c_2}\sigma(\alpha) \\
&\leq \frac{2c_4\|u\|}{c_2}\sigma_0(\alpha) \leq \frac{1}{16}c_5^{(\tau_n)}\|u\|.
\end{aligned} \tag{3.12}$$

Obviously, by (3.1) we have

$$Z(u;0) = \widetilde{F}(\|u\|) - \widetilde{F}(u(0)) = \widetilde{F}(\|u\|) = \frac{\zeta}{2}(\rho_n^-)^2 \geq \frac{\zeta}{2}(\varsigma_n^-)^2 = \frac{\zeta}{2}\|u\|^2. \tag{3.13}$$

By (3.4) and (3.13), for $t \in \Big[0, 1-\sqrt{1-\frac{(2-\tau_n)\zeta}{2c_4}}\Big]$, we have

$$\begin{aligned}
Z(u;t) &= Z(u;0) + \int_0^t Z_s'(u;s)ds \\
&\geq \frac{\zeta}{2}\|u\|^2 - \int_0^t c_4\|u\|^2(t_\alpha - s)ds \\
&\geq \frac{\zeta}{2}\|u\|^2 - \int_0^t c_4\|u\|^2(1-s)ds \\
&= \frac{\|u\|^2}{2}(\zeta + c_4t^2 - 2c_4t) \geq \frac{\tau_n\zeta}{4}\|u\|^2.
\end{aligned} \tag{3.14}$$

It follows from Lemmas 2.1 and 2.2 that $\phi_\alpha = \lambda_1(\alpha)K\phi_\alpha \in Q_\alpha$, and so

$$\phi_\alpha(t) \geq \|\phi_\alpha\|e_\alpha(t) = e_\alpha(t) \geq e(t), \quad \forall t \in [0,1].$$

It follows from Lemmas 2.1 and 2.4 that $\phi'_\alpha(t) > 0$ and $u'(t) > 0$ for $t \in [0, 1/2]$. By Lemma 2.7 and (3.14) we have

$$\begin{aligned}
\int_0^{1/2} Z(u;t)\frac{\phi'_\alpha}{u'}dt &= \Big(\int_0^{1-\sqrt{1-\frac{(2-\tau_n)\zeta}{2c_4}}} + \int_{1-\sqrt{1-\frac{(2-\tau_n)\zeta}{2c_4}}}^{1/2}\Big)Z(u;t)\frac{\phi'_\alpha}{u'}dt \\
&\geq \int_0^{1-\sqrt{1-\frac{(2-\tau_n)\zeta}{2c_4}}} Z(u;t)\frac{\phi'_\alpha}{u'}dt \\
&\geq \int_0^{1-\sqrt{1-\frac{(2-\tau_n)\zeta}{2c_4}}} \frac{\tau_n\zeta}{4}\|u\|^2\frac{\phi'_\alpha}{c_3\|u\|}dt \\
&\geq \frac{\tau_n\zeta}{4c_3}\phi_\alpha\big(1-\sqrt{1-\frac{(2-\tau_n)\zeta}{2c_4}}\big)\|u\| \\
&\geq \frac{\tau_n\zeta}{4c_3}\phi_\alpha\big(\frac{(2-\tau_n)\zeta}{4c_4}\big)\|u\| \\
&\geq \frac{\tau_n\eta(1-\eta)\zeta}{64c_3}\frac{(2-\tau_n)\zeta}{c_4}\big(1-\frac{(2-\tau_n)\zeta}{4c_4}\big)\|u\| \\
&= c_5^{(\tau_n)}\|u\|.
\end{aligned} \tag{3.15}$$

(4) Estimate for $W(\alpha)$. It follows from Lemma 2.8 that

$$\begin{aligned}
|W(\alpha)| &= \big|\alpha\big[u(\eta)\phi'_\alpha(1) - u'(1)\phi_\alpha(\eta)\big]\big| \\
&\leq \alpha\big(\|u\|\|\phi'_\alpha\| + c_3\|u\|(1-t_\alpha)\|\phi_\alpha\|\big) \\
&\leq \alpha\|u\|\big(\pi + c_3\big) \leq \frac{1}{16}c_5^{(\tau_n)}\|u\|.
\end{aligned} \tag{3.16}$$

From Lemmas 2.2 and 2.3 it follows that

$$0 < \frac{1}{2}\|u\|\int_0^1 e^2(t)dt \leq \frac{1}{2}\|u\|\int_0^1 e_\alpha^2(t)dt \leq \int_0^1 u\phi_\alpha dt \leq \|u\|. \tag{3.17}$$

From (3.2), (3.6), (3.9)–(3.12), (3.15)–(3.17) it follows that

$$\int_0^1 \widetilde{f}(u)\phi_\alpha dt - W(\alpha) \geq \frac{3}{4}c_5^{(\tau_n)}\|u\| \geq \frac{3}{4}c_5^{(\tau_n)}\int_0^1 u\phi_\alpha dt. \tag{3.18}$$

Thus, by (2.2),(3.17) and (3.18), we have

$$\lambda \leq \lambda_1(\alpha) - \frac{3}{4}c_5^{(\tau_n)} - \zeta_0 := d_-^{(\tau_n)}(\alpha).$$

Similarly, for each $(\lambda, u) \in \bar{S}_\alpha$ with $\|u\| = \varsigma_n^+ \geq R_4^{(\tau_n)}$ and $\lambda \in [a^-, a^+]$, we have

$$\lambda \geq \lambda_1(\alpha) + \frac{3}{4}c_5^{(\tau_n)} - \zeta_0 := d_+^{(\tau_n)}(\alpha).$$

Assume that $\vartheta^{(\tau_n)} \to \vartheta$ as $n \to \infty$. Obviously, $\vartheta > 0$. Assume without loss of generality that $\vartheta^{(\tau_n)} \geq \frac{\vartheta}{2}$ for all $n \in \mathbb{N}$. From the proof above we see that, for each $\alpha \in [0, \vartheta/2]$ and $n \in \mathbb{N}$, there exists at least one connected component $C_{\alpha,\infty}^{(\tau_n)}$ of $\bar{S}_\alpha$ oscillating over the interval $[d_-^{(\tau_n)}(\alpha), d_+^{(\tau_n)}(\alpha)]$. Note $d_-^{(\tau_n)}(\alpha) \to d_-(\alpha)$ and $d_+^{(\tau_n)}(\alpha) \to d_+(\alpha)$ as $n \to \infty$. Applying Lemma 2.9, we see that for each $\alpha \in [0, \vartheta/2]$, there exists at least one connected component $C_{\alpha,\infty}$ of $\bar{S}_\alpha$ such that

$C_{\alpha,\infty}$ contains $\limsup_{n\to\infty} C_{\alpha,\infty}^{(\tau_n)}$ and oscillates over the interval $[d_-(\alpha), d_+(\alpha)]$. The proof is complete. $\square$

Corollary 3.2. *Suppose that all condition of Theorem 3.1 hold. Then for $\alpha \in [0, \vartheta/2]$ and $\lambda \in [d_-(\alpha), d_+(\alpha)]$, (1.3) has infinitely many solutions.*

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