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PULLBACK ATTRACTOR OF A NONAUTONOMOUS FOURTH-ORDER PARABOLIC EQUATION MODELING EPITAXIAL THIN FILM GROWTH

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Abstract. We study a nonautonomous fourth-order parabolic equation modeling epitaxial thin film growth. It is shown that a pullback attractor of the model exists when the external force has exponential growth.

1. Introduction

The study of pullback attractors for nonautonomous infinite dimensional dynamical systems has attracted much attention and made fast progress in recent years; see for instance, [1, 3, 4, 10, 19]. Recently, Caraballo et al [2] introduced the notion of the pullback $\mathcal{D}$-attractor for nonautonomous dynamical systems and gave a general method to prove the existence of pullback $\mathcal{D}$-attractor. In [12], Li and Zhong proposed the concept of norm-to-weak continuous process and proved the existence of pullback attractors for the nonautonomous reaction-diffusion equation. The authors in [17] considered the existence of pullback attractor for a nonautonomous modified Swift-Hohenberg equation when its external force has exponential growth.

Suppose that Ω is an open connected bounded domain in $\mathbb{R}^3$ with a smooth boundary $\partial\Omega$, $p \in]2, \frac{10}{3}[$. We are concerned with the existence of pullback attractor for the following non-autonomous equation

$$u_t + \Delta^2 u - \nabla \cdot \big(|\nabla u|^{p-2}\nabla u - \nabla u\big) = g(x,t), \quad \text{in } \Omega \times [\tau, \infty), \tag{1.1}$$

with the boundary value conditions

$$u = \frac{\partial u}{\partial n} = 0, \quad \text{on } \partial\Omega \times [\tau, \infty), \tag{1.2}$$

and the initial condition

$$u(x,\tau) = u_\tau(x), \quad \text{in } \Omega. \tag{1.3}$$

Equation (1.1) arises in epitaxial growth of nanoscale thin films (see [8, 22]), where $u(x,t)$ denotes the height from the surface of the film in epitaxial growth, the term $\Delta^2 u$ denotes the capillarity-driven surface diffusion (see [16]), $\nabla \cdot (|\nabla u|^{p-2}\nabla u)$ corresponding to the upward hopping of atoms (see [5]) and Δu denotes the diffusion due to the evaporation-condensation (see [7]). Recently, Equation (1.1) was studied

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by several authors. By numerical simulations and heuristic arguments, Kohn and Yan [9] proved that the standard deviation of u in Equation (1.1) with $g \equiv 0$ grows as $t^{-\frac{1}{3}}$, and the energy per unit area decays as $t^{-\frac{1}{3}}$. Based on Schauder type estimates and Campanato spaces, Liu [13, 14] Studied the regularity of solutions for Equation (1.1) with nonlinear principal part and $g \equiv 0$ in 1D and 2D case. The analysis of the long-time behavior of Equation (1.1) with $g \equiv 0$ has been developed by [6, 23]. There are also some papers which has done with the numerical analysis for the discrete scheme of Equation (1.1), such as [15, 18, 21] and so on. However, the existence of pullback attractor for the fourth-order evolution equation modeling epitaxial thin film growth has not been considered yet.

In this article, we are interested in the existence of pullback attractor for the nonautonomous problem (1.1)-(1.3). This article is organized as follows. In Section 2, we recall some abstract results on pullback attractors and give the main result. In Section 3, we prove the existence of pullback attractor for problem (1.1)-(1.3).

Throughout this paper, we denote $(\cdot,\cdot)$ as the inner product of $L^2(\Omega)$ and $\|\cdot\|$ as the induced norm. $\|\cdot\|_X$ denotes the norm of a Banach space X. For simplicity, we denote $\|\cdot\|_{L^p(\Omega)}$ by $\|\cdot\|_p$, respectively. In the following, c will represent generic positive constants that may change from line to line even if in the same inequality.

2. Preliminaries

In this section, we give some basic definitions and results on the existence of pullback attractor. Suppose that X is a complete metric space and $\{U(t,\tau)\} = \{U(t,\tau) : t \geq \tau, \tau \in \mathbb{R}\}$ is a two-parameter family of mappings act on X: $U(t,\tau) : X \to X, t \geq \tau, \tau \in \mathbb{R}$.

Definition 2.1 ([11])**.** A two-parameter family of mappings $\{U(t,\tau)\}$ is said to be norm-to-weak continuous process in X if

- $U(t,s)U(s,\tau) = U(t,\tau)$ for all $t \geq s \geq \tau$,
- for all $\tau \in \mathbb{R}$, $U(t,\tau) = Id$ is the identity operator,
- $U(t,\tau)x_n \rightharpoonup U(t,\tau)x$, if $x_n \rightharpoonup x$ in X.

Let B be a bounded subset of X. The Kuratowski measure of noncompactness $\alpha(B)$ of B is defined by

$$\alpha(B) = \inf\{\delta > 0 : B \text{ has a finite open cover of sets of diameter} \leq \delta\}.$$

Suppose $\mathcal{D}$ is a nonempty class of parameterised sets $\hat{\mathcal{D}} = \{D(t) : t \in \mathbb{R}\} \subset B(X)$.

Definition 2.2 ([12])**.** A process $\{U(t,\tau)\}$ is called pullback ω-$\mathcal{D}$-limit compact if for any $\varepsilon > 0$ and $\hat{D} \in \mathcal{D}$, there exists a $\tau_0(t, \hat{\mathcal{D}} \leq t$ such that $\alpha(U_{\tau \geq \tau_0} U(t,\tau)D(\tau)) \leq \varepsilon$.

Definition 2.3 ([12])**.** *The family $\hat{A} = \{A(t) : t \in \mathbb{R}\} \subset B(X)$ is said to be a pullback $\mathcal{D}$-attractor for $U(t,\tau)$ if*

- *for all $t \in \mathbb{R}$, $A(t)$ is compact,*
- *$\hat{A}$ is invariant, i.e., $U(t,\tau)A(\tau) = A(\tau)$ for all $t \geq \tau$,*
- *$\hat{A}$ is pullback $\mathcal{D}$-attracting, i.e.,*
$$\lim_{\tau \to -\infty} \operatorname{dist}(U(t,\tau)D(\tau), A(t)) = 0, \quad \forall \hat{\mathcal{D}} \in \mathcal{D},\ t \in \mathbb{R}.$$
- *if $\{C(t)\}_{t\in\mathbb{R}}$ is another family of closed attracting sets, then $A(t) \subset C(t)$ for all $t \in \mathbb{R}$.*

In the following, we give the result on the existence of pullback $\mathcal{D}$-attractor for nonautonomous systems which can be seen in [12].

Lemma 2.4. *Let $\{U(t,\tau)\}_{\tau\le t}$ be a norm-to-weak continuous process such that $\{U(t,\tau)\}_{\tau\le t}$ is pullback ω-$\mathcal{D}$-limit compact. If there exists a family of pullback $\mathcal{D}$-absorbing sets $\{B(t):t\in\mathbb{R}\}\in\mathcal{D}$, i.e., for any $t\in\mathbb{R}$ and $\hat{D}\in\mathcal{D}$ there is a $\tau_0(t,\hat{D})\le t$ such that $U(t,\tau)D(\tau)\subset B(t)$ for all $\tau\le\tau_0$. Then, there is a pullback $\mathcal{D}$-attractor $\{A(t):t\in\mathbb{R}\}$ and*

$$A(t)=\cap_{s\le t}\overline{\cup_{\tau\le s}U(t,\tau)D(\tau)}.$$

To study the existence of pullback attractor for problem (1.1)-(1.3), we suppose that $g(x,t)$ is translation bounded in $L^2_{\rm loc}(\mathbb{R};L^2(\Omega))$; that is,

$$g(x,t)\in L^2_{\rm loc}(\mathbb{R};L^2(\Omega))\quad\text{and}\quad \sup_{t\in\mathbb{R}}\int_t^{t+1}|g(x,t)|^2ds<\infty.$$

We also suppose that, for any $t\in\mathbb{R}$, there exist $\beta\ge 0$ and $0\le\alpha<\big(\frac{20-6p}{6p-4}\big)^2\lambda$, such that

$$\|g(t)\|^2\le\beta e^{\alpha|t|}, \tag{2.1}$$

where λ is the first eigenvalue of $A=\Delta^2$. By (2.1), we have the following properties:

$$\begin{gathered}
G_1(t):=\int_{-\infty}^t e^{\lambda s}\|g(s)\|^2ds<\infty,\quad\forall t\in\mathbb{R},\\
G_2(t):=\int_{-\infty}^t\int_{-\infty}^s e^{\lambda r}\|g(r)\|^2\,dr\,ds<\infty,\quad\forall t\in\mathbb{R},\\
\int_{-\infty}^t e^{\frac{24-12p}{20-6p}\lambda s}\big[[G_1(s)]^{\frac{6p-4}{20-6p}}+[G_2(s)]^{\frac{6p-4}{20-6p}}\big]ds<\infty,\quad\forall t\in\mathbb{R}.
\end{gathered}$$

Using a slight modification of the classical results in the autonomous framework, mainly of the Faedo-Galerkin method (see [20]), we obtain the following result on the existence and uniqueness of solutions for problem (1.1)-(1.3) (see [8, 23]).

Lemma 2.5. *Suppose that $g\in L^2_{\rm loc}(\mathbb{R};L^2(\Omega))$. There is a unique solution $u(x,t)$ such that*

- *if $u_0=u_\tau\in L^2(\Omega)\Rightarrow u(x,t)\in C^0([\tau,\infty);L^2(\Omega))$;*
- *if $u_0=u(\tau)\in H^2_0(\Omega)\Rightarrow u(x,t)\in C^0([\tau,\infty);H^2_0(\Omega))$.*

Based on Lemma 2.5, we obtain that the solution $u(x,t)$ is continuous with respect to the initial value condition u_τ in the space $H^2_0(\Omega)$. In order to construct a process $\{U(t,\tau)\}$ for problem (1.1)-(1.3), we define $U(t,\tau):H^2_0(\Omega)\to H^2_0(\Omega)$ by $U(t,\tau)u_\tau$. Thus, the process $\{U(t,\tau)\}$ is a norm-to-weak continuous process in the space $H^2_0(\Omega)$.

Now, we give the main result of this article, which will be proved in the next section.

Theorem 2.6. *The process corresponding to problem* (1.1)-(1.3) *possesses a unique pullback $\mathcal{D}$-attractor in the space $H^2_0(\Omega)$.*

3. Proof of main results

In this section, we study the existence of pullback $\mathcal{D}$-attractors for non-autonomous problem (1.1)-(1.3). First of all, we derive uniform estimates of solutions for problem (1.1)-(1.3), which are necessary for proving the existence of absorbing set of $\{u(t,\tau)\}$ associated with the problem.

Lemma 3.1. *Consider the problem* (1.1)-(1.3)*, for all* $t \geq \tau$*, we have*

$$\|u(t)\|^2 \leq e^{-\lambda(t-\tau)}\|u_\tau\|^2 + \frac{2M}{\lambda} + \frac{1}{\lambda}e^{-\lambda t}G_1(t),$$

and

$$\int_\tau^t e^{\lambda s}\|\Delta u(s)\|^2 ds \leq [1+\lambda(t-\tau)]e^{\lambda\tau}\|u_\tau\|^2 + \frac{4M}{\lambda}e^{\lambda t} + \frac{1}{\lambda}G_1(t) + G_2(t), \tag{3.1}$$

where $M = \frac{p}{p-2}(\frac{p}{2})^{\frac{-2}{p-2}}$ *is a positive constant.*

Proof. Multiplying (1.1) by u, integrating it over Ω, using Young's inequality, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|u(t)\|^2 + \|\Delta u(t)\|^2 + \|\nabla u(t)\|_p^p \\
&= \|\nabla u(t)\|^2 + (g(t), u(t)) \\
&\leq \frac{1}{2}\|\nabla u(t)\|_p^p + M + \|g(t)\|\|u(t)\| \\
&\leq \frac{1}{2}\|\nabla u(t)\|_p^p + M + \frac{1}{2\lambda}\|g(t)\|^2 + \frac{1}{2}\|\Delta u(t)\|^2,
\end{aligned} \tag{3.2}$$

where $M = \frac{2p}{p-2}(\frac{p}{2})^{\frac{-2}{p-2}}$. It then follows from (3.2) that

$$\frac{d}{dt}\|u(t)\|^2 + \|\Delta u(t)\|^2 + \|\nabla u(t)\|_p^p \leq 2M + \frac{\|g(t)\|^2}{\lambda}, \tag{3.3}$$

$$\frac{d}{dt}\|u(t)\|^2 + \lambda\|u(t)\|^2 \leq 2M + \frac{\|g(t)\|^2}{\lambda}, \tag{3.4}$$

Multiplying (3.4) by $e^{\lambda(t-\tau)}$ and integrating it over (τ,t), we derive that

$$\|u(t)\|^2 \leq e^{-\lambda(t-\tau)}\|u_\tau\|^2 + \frac{2M}{\lambda} + \frac{1}{\lambda}e^{-\lambda t}\int_{-\infty}^t e^{\delta s}\|g(s)\|^2 ds. \tag{3.5}$$

Multiplying (3.5) by $e^{\lambda t}$ and integrating it over (τ,t), we deduce that

$$\int_\tau^t e^{\lambda s}\|u(s)\|^2 ds \leq (t-\tau)e^{\lambda\tau}\|u_\tau\|^2 + \frac{2M}{\lambda^2}e^{\lambda t} + \frac{1}{\lambda}\int_{-\infty}^t\int_{-\infty}^s e^{\lambda r}\|g(r)\|^2\,dr\,ds. \tag{3.6}$$

Similarly, multiplying (3.3) by $e^{\lambda t}$ and integrating it over (τ,t), we obtain

$$\begin{aligned}
&\int_\tau^t e^{\lambda s}\|\Delta u(s)\|^2 ds + \int_\tau^t e^{\lambda s}\|\nabla u(s)\|_p^p ds \\
&\leq e^{\lambda\tau}\|u_\tau\|^2 + \frac{2M}{\lambda}e^{\lambda t} + \lambda\int_\tau^t e^{\lambda s}\|u(s)\|^2 ds + \frac{1}{\lambda}\int_{-\infty}^t e^{\lambda s}\|g(s)\|^2 ds \\
&\leq [1+\lambda(t-\tau)]\,e^{\lambda\tau}\|u_\tau\|^2 + \frac{4M}{\lambda}e^{\lambda t} + \int_{-\infty}^t\int_{-\infty}^s e^{\lambda r}\|g(r)\|^2\,dr\,ds \\
&\quad + \frac{1}{\lambda}\int_{-\infty}^t e^{\lambda s}\|g(s)\|^2 ds.
\end{aligned} \tag{3.7}$$

The proof is complete. □

Lemma 3.2. *In problem* (1.1)-(1.3), *for all* $t \geq \tau$, *we have*

$$\begin{aligned} &\|\nabla u(t)\|^2 \\ &\leq c\Big\{e^{-\lambda(t-\tau)}(\|u_\tau\|^2 + \|\nabla u_\tau\|^2) + \frac{2M}{\lambda^2} + e^{-\lambda t}\big[\frac{1}{\lambda}G_2(t) + G_1(t)\big]\Big\}. \end{aligned} \tag{3.8}$$

Proof. Multiplying (1.1) by Δu, integrating it over Ω, we derive that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\nabla u(t)\|^2 + \|\nabla\Delta u(t)\|^2 + (\nabla\cdot(|\nabla u(t)|^{p-2}\nabla u(t), \Delta u(t)) \\ &= \|\Delta u(t)\|^2 - (g(t), \Delta u(t)). \end{aligned} \tag{3.9}$$

Note that

$$\begin{aligned} &(\nabla\cdot(|\nabla u(t)|^{p-2}\nabla u(t), \Delta u(t)) \\ &= \int_\Omega |\nabla u|^{p-2}|\Delta u|^2 dx + \int_\Omega \nabla u(|\nabla u|^{p-2})\Delta u dx \\ &= \int_\Omega |\nabla u|^{p-2}|\Delta u|^2 dx + \frac{p-2}{2}\int_\Omega \nabla u(|\nabla u|^2)|\nabla u|^{p-4}\Delta u\, dx \\ &= \int_\Omega |\nabla u|^{p-2}|\Delta u|^2 dx + (p-2)\int_\Omega (\sum_{i,j=1}^3 u_{ij}u_iu_j)(\sum_{k=1}^3 u_{kk})|\nabla u|^{p-4}\, dx \\ &= \int_\Omega |\nabla u|^{p-2}|\Delta u|^2 dx + (p-2)\int_\Omega (\sum_{i=1}^3 u_{ii}u_i^2)(\sum_{k=1}^3 u_{kk})|\nabla u|^{p-4} dx \\ &\quad + (p-2)\int_\Omega (\sum_{i,j=1,i\neq j}^3 u_{ij}u_iu_j)(\sum_{k=1}^3 u_{kk})|\nabla u|^{p-4} dx \\ &= \int_\Omega |\nabla u|^{p-2}|\Delta u|^2 dx + (p-2)\sum_{i=1}^3\int_\Omega |\nabla u|^{p-4}u_i^2u_{ii}^2 dx \\ &\quad + (p-2)\sum_{i,j=1,i\neq j}^3\int_\Omega |\nabla u|^{p-4}u_i^2u_{ii}u_{jj} dx \\ &\quad + (p-2)\int_\Omega |\nabla u|^{p-4}\Delta u(\sum_{i,j=1,i\neq j}^3 u_{ij}u_iu_j) dx\,. \end{aligned} \tag{3.10}$$

Using Hölder's inequality, we have

$$\begin{aligned} &(p-2)\sum_{i,j=1,i\neq j}^3\int_\Omega |\nabla u|^{p-4}u_i^2u_{ii}u_{jj} dx \\ &\geq -\frac{p-2}{2}\sum_{i,j=1,i\neq j}^3\int_\Omega |\nabla u|^{p-4}u_i^2(u_{ii}^2 + u_{jj}^2) dx \\ &= -\frac{p-2}{2}\sum_{i=1}^3\int_\Omega |\nabla u|^{p-4}u_i^2u_{ii}^2 dx - \frac{p-2}{2}\sum_{i,j=1,i\neq j}^3\int_\Omega |\nabla u|^{p-4}u_i^2u_{jj}^2 dx. \end{aligned} \tag{3.11}$$

Based on the regularity theorem of elliptic equation, we have

$$\begin{aligned}&(p-2)\int_\Omega |\nabla u|^{p-4}\Delta u\Big(\sum_{i,j=1,i\neq j}^{3} u_{ij}u_iu_j\Big)dx\\&\geq -\frac{p-2}{2}\int_\Omega |\nabla u|^{p-2}|\Delta u|^2dx.\end{aligned}\tag{3.12}$$

Combining (3.10), (3.11) and (3.12) together gives

$$\begin{aligned}&(\nabla\cdot(|\nabla u(t)|^{p-2}\nabla u(t),\Delta u(t))\\&\geq \int_\Omega |\nabla u|^{p-2}|\Delta u|^2dx+\frac{p-2}{2}\sum_{i=1}^{3}\int_\Omega |\nabla u|^{p-4}u_i^2u_{ii}^2dx\\&-\frac{p-2}{2}\int_\Omega |\nabla u|^{p-2}|\Delta u|^2dx\geq 0.\end{aligned}\tag{3.13}$$

By Nirenberg's inequality, we obtain

$$\|\Delta u(t)\|\leq c\|\nabla\Delta u(t)\|^{2/3}\|u(t)\|^{1/3}.\tag{3.14}$$

It then follows from (3.9), (3.13) and (3.14) that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\nabla u(t)\|^2+\|\nabla\Delta u(t)\|^2\\&\leq \|\Delta u(t)\|^2+\|g(t)\|\|\Delta u(t)\|\\&\leq c\|\nabla\Delta u(t)\|^{\frac{4}{3}}\|u(t)\|^{2/3}+c\|g(t)\|\|\nabla\Delta u(t)\|^{2/3}\|u(t)\|^{1/3}\\&\leq \frac{1}{2}\|\nabla\Delta u(t)\|^2+c(\|u(t)\|^2+\|g(t)\|^2);\end{aligned}\tag{3.15}$$

that is,

$$\frac{d}{dt}\|\nabla u(t)\|^2+\|\nabla\Delta u(t)\|^2\leq c(\|u(t)\|^2+\|g(t)\|^2),\tag{3.16}$$

and

$$\frac{d}{dt}\|\nabla u(t)\|^2+\lambda\|\nabla u(t)\|^2\leq c(\|u(t)\|^2+\|g(t)\|^2).\tag{3.17}$$

Multiplying (3.17) by $e^{\lambda(t-\tau)}$ and integrating over (τ,t), we derive that

$$\begin{aligned}&\|\nabla u(t)\|^2\\&\leq e^{-\lambda(t-\tau)}\|\nabla u_\tau\|^2+ce^{-\lambda t}\int_\tau^t e^{\lambda s}\|u(s)\|^2ds+ce^{-\lambda t}\int_\tau^t e^{\lambda s}\|g(s)\|^2ds\end{aligned}\tag{3.18}$$

Combining (3.6) and (3.18) gives

$$\|\nabla u(t)\|^2\leq c\Big\{e^{-\lambda(t-\tau)}(\|u_\tau\|^2+\|\nabla u_\tau\|^2)+\frac{2M}{\lambda^2}+e^{-\lambda t}\big[\frac{1}{\lambda}G_2(t)+G_1(t)\big]\Big\}.$$

Hence, the proof is complete. □

Lemma 3.3. *Consider the problem* (1.1)-(1.3), *for all* $t\geq\tau$, *we have*

$$\begin{aligned}\|\Delta u(t)\|^2\leq c\Big\{&\big(1+(t-\tau)+\frac{1}{t-\tau}\big)e^{-\lambda(t-\tau)}\big[\|u_\tau\|^2+\|u_\tau\|^{\frac{6p-4}{10-3p}}+\|\nabla u_\tau\|^{\frac{6p-4}{10-3p}}\big]\\&+\big(1+\frac{1}{t-\tau}\big)\{1+e^{-\lambda t}[G_1(t)+G_2(t)]\}\\&+e^{-\lambda t}\int_{-\infty}^{t} e^{\frac{24-12p}{20-6p}\lambda s}\Big[[G_1(s)]^{\frac{6p-4}{20-6p}}+[G_2(s)]^{\frac{6p-4}{20-6p}}\Big]ds\Big\}.\end{aligned}$$

Proof. Multiplying (1.1) by $\Delta^2 u$, integrating it over Ω, using Nirenberg's inequality and Young's inequality, we derive that

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\Delta u(t)\|^2 + \|\Delta^2 u(t)\|^2 \\
&= \big(\nabla\cdot(|\nabla u(t)|^{p-2}\nabla u(t)), \Delta^2 u(t)\big) + (g(t), \Delta^2 u(t)) - (\Delta u(t), \Delta^2 u(t)) \\
&\le \frac{1}{4}\|\Delta^2 u(t)\|^2 + c(\|\Delta u(t)\|^2 + \|g(t)\|^2 + \|\nabla\cdot(|\nabla u(t)|^{p-2}\nabla u(t))\|^2) \\
&\le \frac{1}{4}\|\Delta^2 u(t)\|^2 + c(\|\Delta u(t)\|^2 + \|g(t)\|^2 + \|\Delta u(t)\|_\infty^2 \|\nabla u(t)\|_{2p-4}^{2p-4}) \\
&\le \frac{1}{2}\|\Delta^2 u(t)\|^2 + c(\|\Delta u(t)\|^2 + \|g(t)\|^2 \\
&\quad + \|\Delta^2 u(t)\|^{\frac{5}{3}}\|\nabla u(t)\|^{1/3}\|\Delta^2 u(t)\|^{p-3}\|\nabla u(t)\|^{p-1}) \\
&\le \frac{1}{2}\|\Delta u(t)\|^2 + c(\|\Delta u(t)\|^2 + \|g(t)\|^2 + \|\nabla u(t)\|^{\frac{6p-4}{10-3p}});
\end{aligned}
\tag{3.19}
$$

that is,

$$
\frac{d}{dt}\|\Delta u(t)\|^2 + \|\Delta^2 u(t)\|^2 \le c(\|\Delta u(t)\|^2 + \|g(t)\|^2 + \|\nabla u(t)\|^{\frac{6p-4}{10-3p}}), \tag{3.20}
$$

and

$$
\frac{d}{dt}\|\Delta u(t)\|^2 + \lambda\|\Delta u(t)\|^2 \le c(\|\Delta u(t)\|^2 + \|g(t)\|^2 + \|\nabla u(t)\|^{\frac{6p-4}{10-3p}}), \tag{3.21}
$$

Multiplying this by $(t-\tau)e^{\lambda t}$ and integrating it over (τ, t), we obtain

$$
\begin{aligned}
&(t-\tau)e^{\lambda t}\|\Delta u(t)\|^2 \\
&\le c\Big[\int_\tau^t [1+(s-\tau)]e^{\lambda s}\|\Delta u(s)\|^2 ds + \int_\tau^t (s-\tau)e^{\lambda s}\|g(s)\|^2 ds \\
&\quad + \int_\tau^t (s-\tau)e^{\lambda s}\|\nabla u(t)\|^{\frac{6p-4}{10-3p}}\Big].
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\|\Delta u(t)\|^2 &\le c\Big(1+\frac{1}{t-\tau}\Big)e^{-\lambda t}\int_\tau^t e^{\lambda s}\|\Delta u(s)\|^2 ds + ce^{-\lambda t}G_1(t) \\
&\quad + ce^{-\lambda t}\int_\tau^t e^{\lambda s}\|\nabla u(t)\|^{\frac{6p-4}{10-3p}} ds \\
&=: I_1 + I_2 + I_3.
\end{aligned}
\tag{3.22}
$$

It then follows from (3.1) that

$$
\begin{aligned}
I_1 &\le c\Big(1+\frac{1}{t-\tau}\Big)e^{-\lambda t}\Big[[1+\lambda(t-\tau)]e^{\lambda\tau}\|u_\tau\|^2 + \frac{4M}{\lambda}e^{\lambda t} + \frac{1}{\lambda}G_1(t) + G_2(t)\Big] \\
&\le c\Big(1+(t-\tau)+\frac{1}{t-\tau}\Big)e^{-\lambda(t-\tau)}\|u_\tau\|^2 + c\Big(1+\frac{1}{t-\tau}\Big) \\
&\quad + c\Big(1+\frac{1}{t-\tau}\Big)e^{-\lambda t}[G_1(t)+G_2(t)].
\end{aligned}
\tag{3.23}
$$

On the other hand, using Hölder's inequality and (3.8) for sum, we obtain

$$
\begin{aligned}
I_3 &\le ce^{-\lambda t}\int_\tau^t e^{\lambda s}\Big\{e^{-\lambda(t-\tau)}(\|u_\tau\|^2+\|\nabla u_\tau\|^2)+\frac{2M}{\lambda^2} \\
&\quad + e^{-\lambda s}\big[\frac{1}{\lambda}G_2(s)+G_1(s)\big]\Big\}^{\frac{6p-4}{20-6p}}ds \\
&\le ce^{-\lambda t}\int_\tau^t e^{\lambda s}ds + ce^{-\lambda t}\int_\tau^t e^{\lambda s}e^{-\frac{6p-4}{20-6p}\lambda(s-\tau)}(\|u_\tau\|^2+\|\nabla u_\tau\|^2)^{\frac{6p-4}{20-6p}}ds \\
&\quad + ce^{-\lambda t}\int_\tau^t e^{\lambda s}e^{-\frac{6p-4}{20-6p}\lambda s}\{[G_1(t)]^{\frac{6p-4}{20-6p}}+[G_2(t)]^{\frac{6p-4}{20-6p}}\}ds \\
&\le ce^{-\lambda t}[e^{\lambda t}-e^{\lambda\tau}] + ce^{-\lambda(t-\tau)}(\|u_\tau\|^{\frac{6p-4}{10-3p}} \\
&\quad + \|\nabla u_\tau\|^{\frac{6p-4}{10-3p}})\int_\tau^t e^{-\frac{6p-4}{20-6p}\lambda(s-\tau)}ds \\
&\quad + ce^{-\lambda t}\int_\tau^t e^{\frac{24-12p}{20-6p}\lambda s}\{[G_1(t)]^{\frac{6p-4}{20-6p}}+[G_2(t)]^{\frac{6p-4}{20-6p}}\}ds \\
&\le c + c(t-\tau)e^{-\lambda(t-\tau)}(\|u_\tau\|^{\frac{6p-4}{10-3p}}+\|\nabla u_\tau\|^{\frac{6p-4}{10-3p}}) \\
&\quad + ce^{-\lambda t}\int_\tau^t e^{\frac{24-12p}{20-6p}\lambda s}\{[G_1(t)]^{\frac{6p-4}{20-6p}}+[G_2(t)]^{\frac{6p-4}{20-6p}}\}ds.
\end{aligned}
\tag{3.24}
$$

Combining (3.23) and (3.24) with (3.22), we complete the proof. □

Remark 3.4. In this article, motivated by the ideas in [12, 17], we establish the existence of pullback attractors for the nonautomous fourth-order parabolic problem (1.1)-(1.3) in 3D case. To overcome the difficulty caused by the nonlinearity term $\nabla\cdot(|\nabla u|^{p-2}\nabla u)$, we impose the exponential growth condition (2.1) on the external forcing term $g(x,t)$. On the other hand, in order to obtain the suitable a prior estimates, we have to restrict the parameter p and take advantage of the Nirenberg's inequality more times. For example, in the six line of (3.19), the exponent of $\|\Delta^2 u(t)\|$ is $\frac{5}{3}+p-3$, which should be less than 2, so we have to let the parameter $p\in]2,\frac{10}{3}[$. Furthermore, if the problem is studied in nD case, where $n\le 3$, we need only restrict the parameter $p\in]2,2+\frac{4}{n}[$. Using the same method as this article, we can obtain the result on the existence of pullback attractor for nD problem (1.1)-(1.3) when $p\in]2,2+\frac{4}{n}[$ and $n\le 3$.

Let $\mathfrak{R}$ be the set of all function $r:\mathbb{R}\to(0,\infty)$ such that

$$
\lim_{t\to-\infty} te^{\delta t}r^{\frac{6p-4}{10-3p}}(t)=0.
$$

Denote by $\mathcal{D}$ the class of all families $\hat{D}:=\{D(t): t\in\mathbb{R}\}\subset B(H_0^2(\Omega))$ such that $D(t)\subset\overline{B_0}(r(t))$ for some $r(t)\in\mathfrak{R}$, $\overline{B_0}(r(t))$ denote the closed ball in $H_0^2(\Omega)$ with radius $r(t)$. Let

$$
\begin{aligned}
r_0^2(t) &= 2c\Big[1+e^{-\lambda t}G_1(t)+e^{-\lambda t}G_2(t) \\
&\quad + e^{-\lambda t}\int_{-\infty}^t e^{\frac{24-12p}{20-6p}\lambda s}\big[[G_1(s)]^{\frac{6p-4}{20-6p}}+[G_2(s)]^{\frac{6p-4}{20-6p}}\big]ds\Big].
\end{aligned}
$$

Using the continuity of embedding $H_0^2(\Omega) \hookrightarrow L^2(\Omega)$, for any $\hat{D} \in \mathcal{D}$ and $t \in \mathbb{R}$ there exists $\tau_0(\hat{D}, t) < t$ such that

$$\|\Delta u(t)\| \le r_0(t), \quad \forall \tau < \tau_0. \tag{3.25}$$

Furthermore, since $0 \le \alpha < \left(\frac{20-6p}{6p-4}\right)^2 \lambda$, we obtain $\overline{B_0}(r_0(t)) \in \mathcal{D}$. Therefore, $\overline{B_0}(r_0(t))$ is a family of bounded pullback $\mathcal{D}$-absorbing sets in $H_0^2(\Omega)$.

Now, we present the proof of the main result.

Proof of Theorem 2.6. To prove the existence of pullback attractor for problem (1.1)-(1.3), we need only prove that the process $\{u(t, \tau)\}$ is pullback $\omega - \mathcal{D}$-limit compact (PDC). Thanks to A^{-1} is a continuous compact operator in $L^2(\Omega)$, there is a sequence $\{\lambda_j\}_{j=1}^{\infty}$ satisfying

$$\lambda_1 \le \lambda_2 \le \dots \le \lambda_j \le \dots, \quad \lambda_j \to \infty, \quad \text{as } j \to \infty,$$

and a family of elements $\{w_j\}_{j=1}^{\infty}$ of $H_0^2(\Omega)$ which are orthonormal in $L^2(\Omega)$ such that

$$Aw_j = \lambda_j w_j, \quad \text{for } j = 1, 2, \dots.$$

Write $X_n = \operatorname{span}\{w_1, w_2, \dots, w_n\} \subset H_0^2(\Omega)$ and $P_n : H_0^2(\Omega) \to X_n$ is an orthogonal projector. Hence

$$u = P_n u + (I - P_n)u := u_1 + u_2.$$

Taking the scalar product of (1.1) with $\Delta^2 u_2$, using Young's inequality, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\Delta u_2(t)\|^2 + \|\Delta^2 u_2(t)\|^2 \le \frac{1}{2}\|\Delta^2 u_2(t)\|^2 + c(\|\Delta u(t)\|^2 + \|g(t)\|^2 + \|\nabla u(t)\|^{\frac{6p-4}{10-3p}}),$$

which means

$$\frac{d}{dt}\|\Delta u_2(t)\|^2 + \lambda_n\|\Delta u_2(t)\|^2 \le c(\|\Delta u(t)\|^2 + \|g(t)\|^2 + \|\nabla u(t)\|^{\frac{6p-4}{10-3p}}). \tag{3.26}$$

Multiplying (3.26) by $(t-\tau)e^{\lambda_n t}$ and integrating it over (τ, t), we deduce that

$$\begin{aligned}
&(t-\tau)e^{\lambda_n t}\|\Delta u_2(t)\|^2 \\
&\le \int_\tau^t e^{\lambda_n s}\|\Delta u_2(s)\|^2 dx + c\int_\tau^t (s-\tau)e^{\lambda_n s}\|\Delta u(s)\|^2 ds \\
&\quad + c\int_\tau^t (s-\tau)e^{\lambda_n s}\|g(s)\|^2 ds + c\int_\tau^t (s-\tau)e^{\lambda_n s}\|\nabla u(s)\|^{\frac{6p-4}{10-3p}} ds \\
&\le \int_\tau^t e^{\lambda_n s}\|\Delta u(s)\|^2 dx + c(t-\tau)\int_\tau^t e^{\lambda_n s}\|\Delta u(s)\|^2 ds \\
&\quad + c(t-\tau)\int_\tau^t e^{\lambda_n s}\|g(s)\|^2 ds + c\int_\tau^t (s-\tau)e^{\lambda_n s}\|\nabla u(s)\|^{\frac{6p-4}{10-3p}} ds.
\end{aligned} \tag{3.27}$$

It then follows from (3.25) and (3.27) that

$$\begin{aligned}
&\|\Delta u_2(t)\|^2\\
&\le (t-\tau)^{-1}e^{-\lambda_n t}\int_\tau^t e^{\lambda_n s}\|\Delta u(s)\|^2dx + ce^{-\lambda_n t}\int_\tau^t e^{\lambda_n s}\|\Delta u(s)\|^2ds\\
&\quad + ce^{-\lambda_n t}\int_\tau^t e^{\lambda_n s}\|g(s)\|^2ds + ce^{-\lambda_n t}\int_\tau^t e^{\lambda_n s}\|\nabla u(s)\|^{\frac{6p-4}{10-3p}}ds\\
&\le c(t-\tau)^{-1}e^{-\lambda_n t}\int_\tau^t e^{\lambda_n s}r_0^2(s)dx + ce^{-\lambda_n t}\int_\tau^t e^{\lambda_n s}r_0^2(s)ds\\
&\quad + ce^{-\lambda_n t}\int_\tau^t e^{\lambda_n s}\|g(s)\|^2ds + ce^{-\lambda_n t}\int_\tau^t e^{\lambda_n s}r_0^{\frac{6p-4}{10-3p}}(s)ds, \quad \forall \tau\le\tau_0.
\end{aligned} \tag{3.28}$$

Note that

$$\begin{aligned}
e^{-\lambda_n t}\int_\tau^t e^{\lambda_n s}r_0^2(s)dx &= ce^{-\lambda_n t}\int_\tau^t e^{\lambda_n s}[1+e^{-\lambda s}G_1(s)+e^{-\lambda s}G_2(s)]ds\\
&\le c\lambda_n^{-1} + c(\lambda_n-\lambda)^{-1}e^{-\lambda t}[G_1(t)+G_2(t)].
\end{aligned} \tag{3.29}$$

and

$$\begin{aligned}
&e^{-\lambda_n t}\int_\tau^t e^{\lambda_n s}r_0^{\frac{6p-4}{10-3p}}(s)ds\\
&= ce^{-\lambda_n t}\int_\tau^t e^{\lambda_n s}\Big[1+e^{-\lambda s}G_1(s)+e^{-\lambda s}G_2(s)\\
&\quad + e^{-\lambda s}\int_{-\infty}^s e^{\frac{24-12p}{20-6p}\lambda r}\big[[G_1(r)]^{\frac{6p-4}{20-6p}} + [G_2(r)]^{\frac{6p-4}{20-6p}}\big]dr\Big]^{\frac{6p-4}{20-6p}}ds\\
&\le c\lambda_n^{-1} + c\Big(\lambda_n - \frac{6p-4}{20-6p}\lambda\Big)^{-1}e^{-\frac{6p-4}{20-6p}\lambda t}\big\{[G_1(t)]^{\frac{6p-4}{20-6p}} + [G_2(t)]^{\frac{6p-4}{20-6p}}\big\}\\
&\quad + c\Big(\lambda_n - \frac{6p-4}{20-6p}\lambda\Big)^{-1}e^{-\frac{6p-4}{20-6p}\lambda t}\Big[\Big(\int_{-\infty}^t e^{\frac{24-12p}{20-6p}\lambda s}[G_1(s)]^{\frac{6p-4}{20-6p}}\Big)^{\frac{6p-4}{20-6p}}\\
&\quad + \Big(\int_{-\infty}^t e^{\frac{24-12p}{20-6p}\lambda s}[G_2(s)]^{\frac{6p-4}{20-6p}}\Big)^{\frac{6p-4}{20-6p}}\Big].
\end{aligned} \tag{3.30}$$

On the other hand, simple calculations show that

$$\begin{aligned}
e^{-\lambda_n t}\int_\tau^t e^{\lambda_n s}\|g(s)\|^2ds &\le \begin{cases}\beta e^{-\lambda_n t}\int_\tau^t e^{\lambda_n s}e^{-\alpha s}ds, & t\le 0,\\ \beta e^{-\lambda_n t}\int_\tau^t e^{\lambda_n s}e^{\alpha|s|}ds, & t\ge 0,\end{cases}\\
&\le \begin{cases}\frac{\beta e^{-\lambda_n t}}{\lambda_n-\alpha}, & t\le 0,\\ \frac{\beta e^{-\lambda_n t}}{\lambda_n-\alpha} + \frac{\beta e^{\alpha t}}{\lambda_n+\alpha}, & t\ge 0.\end{cases}
\end{aligned} \tag{3.31}$$

Adding (3.28), (3.29), (3.30) and (3.31), we can obtain for any $\varepsilon>0$, there exist $\tau_0<t$ and $N\in\mathbb{N}$ such that

$$\|\Delta u_2(t)\|\le\varepsilon, \quad \forall \tau<\tau_0.$$

It indicates that the process $\{U(t,\tau)\}$ is pullback $\omega-\mathcal{D}$-limit compact. Then, by Lemma 2.4, the process corresponding to problem (1.1)-(1.3) possesses a unique pullback $\mathcal{D}$-attractor in $H_0^2(\Omega)$. $\square$

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