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MULTIPLE HOMOCLINIC SOLUTIONS FOR INDEFINITE SECOND-ORDER DISCRETE HAMILTON SYSTEM WITH SMALL PERTURBATION

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Abstract. In this article, we sutdy the multiplicity of homoclinic solutions to the perturbed second-order discrete Hamiltonian system

$$\Delta[p(n)\Delta u(n-1)] - L(n)u(n) + \nabla W(n, u(n)) + \theta\nabla F(n, u(n)) = 0,$$

where $L(n)$ and $W(n, x)$ are neither autonomous nor periodic in n. Under the assumption that $W(n, x)$ is only locally superquardic as $|x| \to \infty$ and even in x and $F(n, x)$ is a perturbation term, we establish some existence criteria to guarantee that the above system has multiple homoclinic solutions by minimax method in critical point theory.

1. Introduction and statement of main results

In this article, we consider the second-order perturbed discrete Hamilton system

$$\Delta[p(n)\Delta u(n-1)] - L(n)u(n) + \nabla W(n, u(n)) + \theta\nabla F(n, u(n)) = 0, \tag{1.1}$$

where $n \in \mathbb{Z}$, $u \in \mathbb{R}^N$, $\Delta u(n) = u(n+1) - u(n)$ is the forward difference operator, $p(n)$ and $L(n)$ are $N \times N$ real symmetric positive definite matrices for all $n \in \mathbb{Z}$, and W, F: $\mathbb{Z} \times \mathbb{R}^{N\times N} \to \mathbb{R}$. As usual, we say that a solution $u(n)$ of (1.1) is homoclinic (to 0) if $u(n) \to 0$ as $n \to \pm\infty$. In addition, if $u(n) \not\equiv 0$ then $u(n)$ is called a nontrivial homoclinic solution.

System (1.1) does have its applicable setting as evidenced by the excellent monographs (see [1, 3]), and some authors studied the existence of periodic solutions and subharmonic solutions of (1.1) using the critical point theory (see [2, 4, 21, 22, 23]). Moreover, the existence and multiplicity results of boundary value problems for discrete inclusions, such as fourth-order discrete inclusion and partial difference inclusions, have been established by the application of non-smooth version of critical point theory (see [8, 12, 13]). It is obvious that system (1.1) with $\theta = 0$ is a discretization of the following second-order Hamiltonian system:

$$\frac{d}{dt}(p(t)\dot{u}(t)) - L(t)u(t) + \nabla W(t, u(t)) = 0. \tag{1.2}$$

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In recent years, the study of homoclinic solution of system (1.2) is rapid by variational methods (see [5, 7, 9, 14, 17, 18]). It is well known that homoclinic orbits play an important role in analyzing the chaos of dynamical systems. If a system has the smoothly connected homoclinic orbits, then it can not stand the perturbation, and its perturbed system probably produce chaotic phenomenon.

For system (1.1) with $\theta = 0$, the existence and multiplicity of homoclinic solutions of system (1.1) or its special forms have been investigated by the use of critical point theory (see [6, 10, 11, 19, 24]). If $p(n)$, $L(n)$ and $W(n, x)$ are periodic in n, some authors dealt with the periodic case in [6, 11]. When the periodicity is lost, this case is quite different from the one mentioned above, because of lack of compactness of the Sobolev embedding. If $W(n, x)$ is superquadratic as $|x| \to \infty$ uniformly for $n \in \mathbb{Z}$, the following well known global Ambrosetti-Rabinowitz superquadratic condition is often required:

(A1) there exists $\mu > 2$ such that
$$0 < \mu W(n, x) \le (\nabla W(n, x), x), \quad (n, x) \in \mathbb{Z} \times \mathbb{R}^N \setminus \{0\},$$
where and in the sequel, $(\cdot, \cdot)$ denotes the standard inner product in $\mathbb{R}^N$, and $|\cdot|$ is the induced norm.

However, there are many indefinite functions not satisfying (A1). For example, let
$$W(n, x) = (n - 1)|x|^s, \quad 2 < s < \infty. \tag{1.3}$$
It is obvious that $W(n, x)$ is only locally superquadratic as $|x| \to \infty$. If $W(n, x)$ is even in x, the classical multiple critical point theorems can be applied to obtain multiplicity results for system (1.1) with $\theta = 0$. When $\theta \ne 0$ and $F(n, x)$ is not even in x, then the perturbation term $F(n, x)$ breaks the symmetry of the energy functional of system (1.1). This case becomes different and more complicated. A natural question is that whether multiple homoclinic solutions exist for system (1.1) with indefinite functions $W(n, x)$ under broken symmetry situation. As far as the authors are aware, there are few papers discussing this question. In this paper, we give a positive answer to this question. In detail, we obtain the following theorems.

Theorem 1.1. *Assume that L, W and F satisfy the following conditions:*

(A2) *$L(n)$ is an $N \times N$ real symmetric positive definite matrix for all $n \in \mathbb{Z}$ and there exists a function $l : \mathbb{Z} \to (0, \infty)$ such that $l(n) \to +\infty$, $|n| \to \infty$, and*
$$(L(n)x, x) \ge l(n)|x|^2, \quad (n, x) \in \mathbb{Z} \times \mathbb{R}^N;$$

(A3) *$W(n, 0) \equiv 0$, and there exist constants $\mu > 2$ such that*
$$\mu W(n, x) \le (\nabla W(n, x), x), \quad (n, x) \in \mathbb{Z} \times \mathbb{R}^N;$$

(A4) *for every $n \in \mathbb{Z}$, W is continuously differentiable in x, and there exists constants $a_1 > 0$ and $1 < \nu_1 \le \nu_2 < \infty$ such that*
$$|\nabla W(n, x)| \le a_1 l(n)(|x|^{\nu_1} + |x|^{\nu_2}), \quad (n, x) \in \mathbb{Z} \times \mathbb{R}^N;$$

(A5) *there exists an infinite subset $\Lambda \subset \mathbb{Z}$ such that*
$$\lim_{|x| \to \infty} \frac{W(n, x)}{|x|^2} = \infty, \quad n \in \Lambda,$$
and there exists $r_0 \ge 0$ such that
$$W(n, x) \ge 0, \quad (n, x) \in \Lambda \times \mathbb{R}^N \text{ and } |x| \ge r_0;$$

(A6) *$W(n,x) = W(n,-x)$, $(n,x) \in \mathbb{Z} \times \mathbb{R}^N$;*

(A7) *for every $n \in \mathbb{Z}$, F is continuously differentiable in x, and there exists a function $\gamma_1 \in l^1(\mathbb{Z}, [0,+\infty))$ such that*

$$|F(n,x)| \le \gamma_1(n), \quad (n,x) \in \mathbb{Z} \times \mathbb{R}^N;$$

(A8) *there exists a function $\gamma_2 \in l^2(\mathbb{Z}, [0,+\infty))$ such that*

$$|\nabla F(n,x)| \le \gamma_2(n), \quad (n,x) \in \mathbb{Z} \times \mathbb{R}^N.$$

Then for any $j \in \mathbb{N}$, there exists $\varepsilon_j > 0$ such that if $|\theta| \le \varepsilon_j$, system (1.1) *possesses at least j distinct homoclinic solutions.*

Theorem 1.2. *Assume that L, W satisfy* (A2)–(A6)*. Then there exists an unbounded sequence of homoclinic solutions for system* (1.1) *with $\theta = 0$.*

Remark 1.3. We would like to point out that even in the symmetric case, our results are also new. In fact, the condition (A5) implies that $W(n,x)$ is only of locally superquadratic growth as $|x| \to \infty$, and our assumption (A5) is weaker than the conditions presented in the reference.

Since $F(n,x)$ is not even in x in Theorem 1.1, the classical multiple critical point theorems fail to obtain multiplicity results for system (1.1). The main difficulty is to find an appropriate class of sets due to indefinite character of the function $W(n,x)$ which is used to construct multiple critical values for the perturbed functional of system (1.1). To overcome this difficulty, we construct an orthogonal sequence by which a sequence of sets are introduced, then multiple critical values will be obtained by minimax procedure over these sets, which correspond to multiple homoclinic solutions of system (1.1).

The article is organized as follows. In Section 2, we present some preliminary results and useful lemmas. The proof of Theorem 1.1 and Corollary 1.1 are given in Section 3. In Section 4, we present an example to illustrate our results.

Throughout the article, we denote by C_n various positive constants which may vary from line to line and are not essential to the proof.

2. Variational setting and preliminaries

Let

$$S = \big\{\{u(n)\}_{n\in\mathbb{Z}} : \ u(n) \in \mathbb{R}^N, \ n \in \mathbb{Z}\big\},$$
$$E = \Big\{u \in S : \ \sum_{n\in\mathbb{Z}} \Big[\big(p(n+1)\Delta u(n), \Delta u(n)\big) + \big(L(n)u(n), u(n)\big)\Big] < +\infty\Big\}.$$

For $u, v \in E$, let

$$\langle u, v\rangle = \sum_{n\in\mathbb{Z}} \big[\big(p(n+1)\Delta u(n), \Delta v(n)\big) + \big(L(n)u(n), v(n)\big)\big].$$

Then E is a Hilbert space with the above inner product, and the corresponding norm is

$$\|u\| := \Big(\sum_{n\in\mathbb{Z}} \big[\big(p(n+1)\Delta u(n), \Delta u(n)\big) + \big(L(n)u(n), u(n)\big)\big]\Big)^{1/2}, \quad u \in E.$$

Moreover, we use E^* to denote the topological dual space with norm $\|\cdot\|_{E^*}$. As usual, for $1 \le p < \infty$, $k = 1$ or N, set

$$l^p(\mathbb{Z},\mathbb{R}^k) = \big\{\{u(n)\}_{n\in\mathbb{Z}} : u(n) \in \mathbb{R}^k,\ n \in \mathbb{Z},\ \sum_{n\in\mathbb{Z}} |u(n)|^p < +\infty\big\},$$
$$l^\infty(\mathbb{Z},\mathbb{R}^k) = \big\{\{u(n)\}_{n\in\mathbb{Z}} : u(n) \in \mathbb{R}^k,\ n \in \mathbb{Z},\ \sup_{n\in\mathbb{Z}} |u(n)| < +\infty\big\},$$

and their norms are defined by

$$\|u\|_p = \Big(\sum_{n\in\mathbb{Z}} |u(n)|^p\Big)^{1/p}, \quad u \in l^p(\mathbb{Z},\mathbb{R}^k); \quad \|u\|_\infty = \sup_{n\in\mathbb{Z}} |u(n)|, \quad u \in l^\infty(\mathbb{Z},\mathbb{R}^k).$$

If the condition (A2) holds, E is continuously embedded in $l^p(\mathbb{Z},\mathbb{R}^N)$ for all $p \in [2,+\infty]$. Consequently, there exists $\tau_p > 0$ such that

$$\|u\|_p \le \tau_p \|u\|, \quad u \in E. \tag{2.1}$$

Lemma 2.1. *If condition* (A2) *holds. Then E is compactly embedded in $l^\infty(\mathbb{Z},\mathbb{R}^N)$.*

Proof. Let $\{u_k\}$ be a bounded sequence in E, that is, there is a constant A such that

$$\|u_k\| \le A, \quad k \in \mathbb{N}.$$

Since E is a reflexive space, passing to a subsequence, also denoted by $\{u_k\}$, it can be assumed that $u_k \rightharpoonup u_0$, $k \to \infty$. Next we only need to prove

$$u_k \to u_0 \quad \text{in } l^\infty(\mathbb{Z},\mathbb{R}^N). \tag{2.2}$$

For any given number $\varepsilon > 0$, by (A2), we can choose a positive integer Π_0 such that

$$l(n) > \frac{4A^2}{\varepsilon^2}, \quad |n| \ge \Pi_0. \tag{2.3}$$

By (A2) and (2.3), we have

$$|u_k(n)|^2 \le \frac{1}{l(n)}\big(L(n)u_k(n), u_k(n)\big) \le \frac{\varepsilon^2}{4A^2}\|u_k\|^2 \le \frac{\varepsilon^2}{4}, \quad |n| \ge \Pi_0,\ k \in \mathbb{N}. \tag{2.4}$$

Since $u_k \rightharpoonup u_0$ in E, it is easy to verify that $u_k(n)$ converges to $u_0(n)$ pointwise for all $n \in \mathbb{Z}$; that is,

$$\lim_{k\to\infty} u_k(n) = u_0(n), \quad n \in \mathbb{Z}. \tag{2.5}$$

In view of (2.4) and (2.5), we have

$$|u_0(n)| \le \varepsilon/2, \quad |n| \ge \Pi_0. \tag{2.6}$$

By (2.5), there exists $k_0 \in \mathbb{N}$ such that

$$|u_k(n) - u_0(n)| \le \varepsilon, \quad k \ge k_0,\ |n| < \Pi_0. \tag{2.7}$$

In combination with (2.4), (2.6) and (2.7)

$$|u_k(n) - u_0(n)| \le \varepsilon, \quad k \ge k_0,\ n \in \mathbb{Z},$$

which implies that (2.2) holds. The proof is complete. $\square$

Next we introduce a functional $I : \mathbb{R} \times E \to \mathbb{R}$

$$I_\theta(u) := \frac{\|u\|^2}{2} - \sum_{n\in\mathbb{Z}} W(n,u(n)) - \theta \sum_{n\in\mathbb{Z}} F(n,u(n)). \tag{2.8}$$

By (A2), (A4), (A7) and (A8), for fixed $\theta_0 \in \mathbb{R}$, $I_{\theta_0}(u)$ is well defined and of class $C^1(E,\mathbb{R})$. For $u, v \in E$,

$$\begin{aligned} \langle I'_{\theta_0}(u), v\rangle = \sum_{n\in\mathbb{Z}} &\big[\big(p(n+1)\Delta u(n), \Delta v(n)\big) + \big(L(n)u(n), v(n)\big)\big] \\ &- \sum_{n\in\mathbb{Z}} \big(\nabla W(n,u(n)), v(n)\big) - \theta_0 \sum_{n\in\mathbb{Z}} \big(\nabla F(n,u(n)), v(n)\big). \end{aligned} \tag{2.9}$$

Furthermore, if $u_0 \in E$ is a critical point of $I_{\theta_0}(u)$, then u_0 is a homoclinic solution for system (1.1) with $\theta = \theta_0$.

Lemma 2.2. *Assume that all the hypotheses of Theorem 1.1 hold. Then*

(1) *for any fixed $\theta_0 \in \mathbb{R}$, I_{θ_0} satisfies the Palais-Smale condition;*
(2) *there exists a positive constant C_0 such that*

$$|I_\theta(u) - I_0(u)| \le C_0|\theta|, \quad (\theta,u) \in \mathbb{R} \times E.$$

where $C_0 := \sum_{n\in\mathbb{Z}} |\gamma_1(n)|$.

Proof. To prove (1), we first show that there exists a constant M such that $\{u_k\} \subset E$ is a sequence for which

$$|I_{\theta_0}(u_k)| \le M \quad \text{and} \quad I'_{\theta_0}(u_k) \to 0, \tag{2.10}$$

then $\{u_k\}$ is bounded. For large k, it follows (2.8) and (2.9) that

$$\begin{aligned} 2\mu^{-1}\|u_k\| + M &\ge I_{\theta_0}(u_k) - \frac{1}{\mu}\langle I'_{\theta_0}(u_k), u_k\rangle \\ &> \frac{\mu-2}{2\mu}\|u_k\|^2 - C_1\|u_k\| - C_2, \end{aligned} \tag{2.11}$$

which implies that $\|u_k\|$ is bounded in E, that is, there exists a constant $A' > 0$ such that

$$\|u_k\| \le A', \quad k \in \mathbb{N}.$$

Since E is a reflexive space, passing to a subsequence, also denoted by $\{u_k\}$, it can be assumed that

$$u_k \rightharpoonup u_0, \quad k \to \infty. \tag{2.12}$$

Moreover, $\|u_0\| \le A'$ and it is easy to verify that

$$\lim_{k\to\infty} u_k(n) = u_0(n), \quad n \in \mathbb{Z}. \tag{2.13}$$

For any given number $\varepsilon > 0$, by (A4), there exists a positive constant $\delta < 1$ such that

$$|\nabla W(n,x)| \le \varepsilon l(n)|x|, \quad (n,x) \in \mathbb{Z} \times \mathbb{R}^N, \ |x| \le \delta. \tag{2.14}$$

Arguing as in Lemma 2.1, there exists a positive integer Π_0 such that

$$|u_k(n)| \le \delta \quad \text{and} \quad |u_0(n)| \le \delta, \quad k \in \mathbb{N}, \ |n| > \Pi_0. \tag{2.15}$$

It follows (2.13) and the continuity of $\nabla W(n,x)$ on x that there exists $k_0 \in \mathbb{N}$ such that

$$\sum_{|n|\le\Pi_0} |\nabla W(n,u_k(n)) - \nabla W(n,u_0(n))||u_k(n) - u_0(n)| < \varepsilon, \quad k \ge k_0. \tag{2.16}$$

On the other hand, by (2.14) and (2.15),

$$
\begin{aligned}
&\sum_{|n|>\Pi_0} |\nabla W(n,u_k(n)) - \nabla W(n,u_0(n))||u_k(n)-u_0(n)| \\
&\le \sum_{|n|>\Pi_0} (|\nabla W(n,u_k(n))| + |\nabla W(n,u_0(n))|)(|u_k(n| + |u_0(n)|) \\
&\le \varepsilon \sum_{|n|>\Pi_0} l(n)(|u_k(n)| + |u_0(n)|)(|u_k(n)| + |u_0(n)|) \\
&\le 2\varepsilon \sum_{|n|>\Pi_0} l(n)(|u_k(n)|^2 + |u_0(n)|^2) \\
&\le 2\varepsilon \sum_{|n|>\Pi_0} [(L(n)u_k(n), u_k(n)) + (L(n)u_0(n), u_0(n))] \\
&\le 2\varepsilon(\|u_k\|^2 + \|u_0\|^2) \\
&\le 4\varepsilon A'^2, \quad k\in\mathbb{N}.
\end{aligned}
\tag{2.17}
$$

Since ε is arbitrary, combing (2.16) and (2.17),

$$
\sum_{n\in\mathbb{Z}} |\nabla W(n,u_k(n)) - \nabla W(n,u_0(n)||u_k(n)-u_0(n)| \to 0, \quad k\to\infty. \tag{2.18}
$$

By (A8), there exists a positive integer Π_1 such that

$$
\Big(\sum_{|n|>\Pi_1} |\gamma_2(n)|^2\Big)^{1/2} < \varepsilon. \tag{2.19}
$$

In view of (A8), (2.15) and (2.19), we have

$$
\begin{aligned}
&\sum_{|n|>\Pi_2} |\nabla F(n,u_k(n)) - \nabla F(n,u_0(n))||u_k(n)-u_0(n)| \\
&\le 2\Big(\sum_{|n|>\Pi_2} |\gamma_2(n)|^2\Big)^{1/2}\Big(\sum_{|n|>\Pi_2} |u_k(n)-u_0(n)|^2\Big)^{1/2} \\
&\le 2\tau_2^2\|u_k-u_0\|^2\varepsilon \\
&\le 4\tau_2^2 A'^2\varepsilon, \quad k\in\mathbb{N}.
\end{aligned}
\tag{2.20}
$$

where $\Pi_2 := \max\{\Pi_0, \Pi_1\}$. Moreover, it follows from the continuity of $\nabla F(n,x)$ on x that there exists $k_1\in\mathbb{N}$ such that

$$
\sum_{|n|\le\Pi_2} |\nabla F(n,u_k(n)) - \nabla F(n,u_0(n))||u_k(n)-u_0(n)| < \varepsilon, \quad k\ge k_1. \tag{2.21}
$$

Since ε is arbitrary, for any fixed $\theta_0\in\mathbb{R}$, in combination with (2.15) and (2.20),

$$
\theta_0\sum_{n\in\mathbb{Z}} |\nabla F(n,u_k(n)) - \nabla F(n,u_0(n))||u_k(n)-u_0(n)| \to 0, \quad k\to\infty. \tag{2.22}
$$

It follows from (2.10) and (2.12) that

$$
\langle I'_{\theta_0}(u_k) - I'_{\theta_0}(u_0), u_k-u_0\rangle := \epsilon_k \to 0, \quad k\to\infty. \tag{2.23}
$$

It follows from (2.18), (2.22) and (2.23) that

$$
\|u_k-u_0\|^2 \le \sum_{n\in\mathbb{Z}} |\nabla W(n,u_k(n)) - \nabla W(n,u_0(n))||u_k(n)-u_0(n)|
$$

$$+\theta_0\sum_{n\in\mathbb{Z}}|\nabla F(n,u_k(n))-\nabla F(n,u_0(n))||u_k(n)-u_0(n)|+|\epsilon_k|,$$

which implies that $u_k\to u_0$ in E. Hence, I_{θ_0} satisfies Palais-Smale condition.

To prove (2), by (A7) and direct computations,

$$|I_\theta(u)-I_0(u)|\le C_0|\theta|,\quad (\theta,u)\in\mathbb{R}\times E.$$

The proof is complete. □

Lemma 2.3. *Suppose that* (A5) *holds. Then there exists a normalized orthogonal sequence* $\{\phi_i\}_{i=1}^\infty\subset E$.

Proof. Since $\Lambda\subset\mathbb{Z}$ is an infinite set, there exist a strictly increasing sequence or a strictly decreasing sequence $\{n_k\}_{k=1}^\infty\subset\Lambda$. Without loss of generality, we assume

$$n_1<n_2<\dots<n_k<\dots\to+\infty.$$

Define

$$\phi_i(n)=\begin{cases}(1,0,\dots,0)^\top\in\mathbb{R}^N, & n=n_i,\\ 0, & n\ne n_i.\end{cases}\tag{2.24}$$

It is obvious that $\{\phi_i\}_{i=1}^\infty$ forms a linearly independent sequence in E. By Gram-Schmidt orthogonalization process, also denoted by $\{\phi_i\}$, we can get a normalized orthogonal sequence. The proof is complete. □

Let $D_m=\operatorname{span}\{\phi_1,\dots,\phi_m\}$, $m\in\mathbb{N}$. It is obvious that D_m is a finite dimensional subspace in E. Next we prove that there exists a strictly increasing sequence of numbers R_m such that

$$I_0(u)\le 0,\quad u\in D_m\backslash B_{R_m},\tag{2.25}$$

where B_{R_m} denotes the open ball of radius R_m centered at 0 in E, and $\bar B_{R_m}$ denotes the closure of B_{R_m} in E.

Lemma 2.4. *Under assumptions* (A5)*, for any finite dimensional subspace* $D_m\subset E$*,*

$$I_0(u)\to-\infty,\quad \|u\|\to\infty,\ u\in D_m.\tag{2.26}$$

Proof. We prove (2.26) by contradiction. If (2.26) is false, there exists a sequence $\{u_k\}\subset D_m$ with $\|u_k\|\to\infty$, there exists $M>0$ such that $I_0(u_k)\ge -M$ for all $k\in\mathbb{N}$. Set $v_k=u_k/\|u_k\|$, then $\|v_k\|=1$. Passing to subsequence, we may assume $v_k\rightharpoonup v$ in E. Since D_m is a finite dimensional space, then $v_k\to v\in D_m$, then $\|v\|=1$. Set

$$\Pi=\{n\in\mathbb{Z}:v(n)\ne 0\}\quad\text{and}\quad \Theta=\{n_1,n_2,\dots,n_m\},$$

then

$$\Pi\ne\emptyset\quad\text{and}\quad \Pi\subset\Theta,\tag{2.27}$$

moreover,

$$\lim_{k\to\infty}|u_k(n)|=\infty,\quad n\in\Pi.\tag{2.28}$$

It follows from (A3) and (A4) that

$$|W(n,x)|\le a_1 l(n)(|x|^{\nu_1+1}+|x|^{\nu_2+1}),\quad (n,x)\in\mathbb{Z}\times\mathbb{R}^N.\tag{2.29}$$

For $0\le a<b$, let

$$\Omega_k(a,b)=\big\{n\in\Theta:a\le|u_k(n)|<b\big\},$$

it follows from (2.28) that $\Pi \subset \Omega_k(r_0,\infty)$ for large $k \in \mathbb{N}$. By (A3), (A5), (2.27), (2.28) and (2.29) that

$$\begin{aligned}
0 &\le \lim_{k\to\infty} \frac{I_0(u_k)}{\|u_k\|^2} = \lim_{k\to\infty}\Big[\frac{1}{2} - \sum_{n\in\mathbb{Z}} \frac{W(n,u_k)}{\|u_k\|^2}\Big] \\
&= \lim_{k\to\infty}\Big[\frac{1}{2} - \sum_{n\in\Theta} \frac{W(n,u_k)}{\|u_k\|^2}\Big] \\
&= \lim_{k\to\infty}\Big[\frac{1}{2} - \sum_{n\in\Omega_k(0,r_0)} \frac{W(n,u_k)}{\|u_k\|^2} - \sum_{n\in\Omega_k(r_0,\infty)} \frac{W(n,u_k)}{|u_k|^2}|v_k|^2\Big] \\
&\le \limsup_{k\to\infty}\Big[\frac{1}{2} + ma_m a_1\big(r_0^{\nu_1+1} + r_0^{\nu_2+1}\big)\|u_k\|^{-2} - \sum_{n\in\Omega_k(r_0,\infty)} \frac{W(n,u_k)}{|u_k|^2}|v_k|^2\Big] \\
&\le \frac{1}{2} - \liminf_{k\to\infty} \sum_{n\in\Omega_k(r_0,\infty)} \frac{W(n,u_k)}{|u_k|^2}|v_k|^2 = -\infty,
\end{aligned}$$

where $a_m := \max\{l(n), n \in \Theta\}$. But the above inequality can not hold. Thus (2.26) holds. The proof is complete. □

3. Proofs of main results

Next we introduce some continuous maps in E. Set

$$\Gamma_m = \{h \in C(F_m, E)|\ h \text{ is odd and } h = \mathrm{id} \text{ on } \partial B_{R_m} \cap D_m\}, \tag{3.1}$$

where $F_m := \bar{B}_{R_m} \cap D_m$. By (3.1), we define a sequences of minimax values

$$b_m = \inf_{h\in\Gamma_m} \max_{u\in F_m} I_0(h(u)). \tag{3.2}$$

Since E is a separable Hilbert space, there exists a total orthonormal basis $\{e_j\}$ of E. Define $X_j = \mathbb{R}e_j$, $j \in \mathbb{N}$ and

$$Y_k = \oplus_{j=1}^k X_j, \quad Z_k = \overline{\oplus_{j=k+1}^\infty X_j}, \quad k \in \mathbb{N}. \tag{3.3}$$

It is obvious that

$$E = Y_k \oplus Z_k, \quad Z_k = Y_k^\perp, \quad k \in \mathbb{N}.$$

Next we give an intersection property which has been essentially proved by Rabinowitz in Proposition 9.23 of [16].

Lemma 3.1. *For any $m \in \mathbb{N}$, $\rho < R_m$ and $h \in \Gamma_m$,*

$$h(F_m) \cap \partial B_\rho \cap Z_{m-1} \ne \emptyset.$$

Lemma 3.2. *Suppose that* (A2) *hold. Then*

$$\beta_k := \sup_{u\in Z_k,\quad \|u\|=1} \|u\|_\infty \to 0, \quad k \to \infty. \tag{3.4}$$

Proof. In fact, it is obvious that $\beta_k \ge \beta_{k+1} > 0$, so $\beta_k \to \beta \ge 0$ as $k \to \infty$. For $k \in \mathbb{N}$, there exists $u_k \in Z_k$ such that

$$\|u_k\| = 1 \quad \text{and} \quad \|u_k\|_\infty > \beta_k/2. \tag{3.5}$$

By a similar proof in [20, Lemma 3.8], $u_k \rightharpoonup 0$ in E. By Lemma 2.1, we have

$$u_k \to 0 \quad \text{in } l^\infty(\mathbb{Z}, \mathbb{R}^N). \tag{3.6}$$

In combination with (3.5) and (3.6), (3.4) holds. The proof is complete. □

Lemma 3.3. *Assume* (A3) *and* (A4) *hold. Then*

$$b_m \to \infty, \quad m \to \infty. \tag{3.7}$$

Proof. By Lemma 3.1, for any $h \in \Gamma_m$ and $\rho < R_m$, there exists $u_m \in h(F_m) \cap \partial B_\rho \cap Z_{m-1}$, then

$$\max_{u \in F_m} I_0(h(u)) \geq I_0(u_m) \geq \inf_{u \in \partial B_\rho \cap Z_{m-1}} I_0(u). \tag{3.8}$$

In view of (A3) and (A4),

$$|W(n,x)| \leq a_1 l(n)(|x|^{\nu_1+1} + |x|^{\nu_2+1}), \quad (n,x) \in \mathbb{Z} \times \mathbb{R}^N. \tag{3.9}$$

By (A2), (2.8), (3.4) and (3.9), for $u \in Z_{m-1}$,

$$\begin{aligned} I_0(u) &= \frac{\|u\|^2}{2} - \sum_{n\in\mathbb{Z}} W(n,u(n)) \\ &\geq \frac{\|u\|^2}{2} - a_1 \sum_{n\in\mathbb{Z}} l(n)(|u(n)|^{\nu_1+1} + |u(n)|^{\nu_2+1}) \\ &\geq \frac{\|u\|^2}{2} - a_1\beta_{m-1}^{\nu_1-1}\|u\|^{\nu_1+1} - a_1\beta_{m-1}^{\nu_2-1}\|u\|^{\nu_2+1}. \end{aligned} \tag{3.10}$$

In view of (3.4) and (3.10), when m is large enough, for $u \in Z_{m-1}$,

$$I_0(u) \geq \frac{\|u\|^2}{2} - 2a_1\beta_{m-1}^{\nu_1-1}\|u\|^{\nu_2+1} - C_3. \tag{3.11}$$

Choose $\rho := (8a_1\beta_{m-1}^{\nu_1-1})^{\frac{1}{1-\nu_2}}$, if $u \in Z_{m-1}$ and $\|u\| = \rho$,

$$I_0(u) \geq \frac{1}{4}(8a_1\beta_{m-1}^{\nu_1-1})^{\frac{2}{1-\nu_2}} - C_3. \tag{3.12}$$

In combination with (3.8) and (3.12), when m is large enough,

$$b_m \geq \frac{1}{4}(8a_1\beta_{m-1}^{\nu_1-1})^{\frac{2}{1-\nu_2}} - C_3,$$

which implies that (3.7) holds by (3.4). The proof is complete. $\square$

Next we introduce some continuous maps in E. Set

$$\begin{aligned} \Lambda_m := \big\{&H \in C(U_m, E)|\ H|_{F_m} \in \Gamma_m \text{ and } H = \text{id for} \\ &u \in Q_m := (\partial B_{R_{m+1}} \cap D_{m+1}) \cup \big((B_{R_{m+1}} \backslash \bar{B}_{R_m}) \cap D_m\big)\big\}, \end{aligned} \tag{3.13}$$

where

$$U_m := \big\{u = t\phi_{m+1} + \omega : t \in [0, R_{m+1}],\ \omega \in \bar{B}_{R_{m+1}} \cap D_m,\ \|u\| \leq R_{m+1}\big\}. \tag{3.14}$$

In view of Lemma 3.3, it is impossible that $b_{m+1} = b_m$ for all large m. Next we can construct critical values of $I_\theta(u)$ as follows.

Lemma 3.4. *Suppose $b_{m+1} > b_m > 0$. For any $\delta \in (0, b_{m+1} - b_m)$, define*

$$\Lambda_m(\delta) = \big\{H \in \Lambda_m | I_0(H(u)) \leq b_m + \delta \textit{ for } u \in F_m\big\}. \tag{3.15}$$

For any $|\theta| < 2C_0^{-1}(b_{m+1} - b_m - \delta)$, where C_0 is given in Lemma 2.2, let

$$c_m(\theta) = \inf_{H \in \Lambda_m(\delta)} \max_{u \in U_m} I_\theta(H(u)). \tag{3.16}$$

Then $c_m(\theta)$ is a critical value of $I_\theta(u)$.

Proof. By (2) in Lemma 2.2, we have

$$I_0(u) - C_0|\theta| \le I_\theta(u) \le I_0(u) + C_0|\theta|, \quad (\theta, u) \in \mathbb{R} \times E. \tag{3.17}$$

For any $H \in \Lambda_m(\delta)$, since $F_{m+1} = U_m \cup (-U_m)$, then H can be continuously extended to F_{m+1} as an odd function $\bar{H}$. Moreover, $\bar{H} \in \Gamma_{m+1}$. Since $I_0(u)$ is even, by the construction of $\bar{H}$, we have

$$\max_{x \in U_m} I_0(H(x)) = \max_{x \in F_{m+1}} I_0(\bar{H}(x)). \tag{3.18}$$

It follows from (3.2), (3.17) and (3.18) that

$$\begin{aligned} \max_{x \in U_m} I_\theta(H(x)) &\ge \max_{x \in U_m} I_0(H(x)) - C_0|\theta| \\ &= \max_{x \in F_{m+1}} I_0(\bar{H}(x)) - C_0|\theta| \\ &\ge b_{m+1} - C_0|\theta|. \end{aligned} \tag{3.19}$$

In view of (3.16) and (3.19), we obtain

$$c_m(\theta) \ge b_{m+1} - C_0|\theta| > b_m + \delta + C_0|\theta|. \tag{3.20}$$

If we choose $H_m \in \Lambda_m(\delta)$, then H_m can be continuously extended to F_{m+1} as an odd function $\bar{H}_m$. Moreover, $\bar{H}_m \in \Gamma_{m+1}$. Define

$$c_m = \max_{x \in U_m} I_0(H_m(x)). \tag{3.21}$$

It is obvious that $c_m < +\infty$ and c_m is independent of θ. It follows from (3.2) and (3.21) that

$$c_m = \max_{x \in U_m} I_0(H_m(x)) = \max_{x \in F_{m+1}} I_0(\bar{H}_m(x)) \ge b_{m+1}. \tag{3.22}$$

Moreover, by (3.16), (3.17) and (3.21),

$$c_m(\theta) \le c_m + C_0|\theta|. \tag{3.23}$$

Next we show that $c_m(\theta)$ is a critical value of $I_\theta(u)$. If $c_m(\theta)$ is a regular value of $I_\theta(u)$, by (3.20), choose

$$\bar{\varepsilon} = (c_m(\theta) - b_m - \delta - C_0|\theta|)/2, \tag{3.24}$$

By the Deformation Theorem in [16], there exists $\varepsilon \in (0, \bar{\varepsilon})$ and $\eta \in C([0,1] \times E, E)$ such that

$$\eta(1, u) = u, \quad I_\theta(u) \notin [c_m(\theta) - \bar{\varepsilon}, c_m(\theta) + \bar{\varepsilon}], \tag{3.25}$$

and if $I_\theta(u) \le c_m(\theta) + \varepsilon$, then

$$I_\theta(\eta(1, u)) \le c_m(\theta) - \varepsilon. \tag{3.26}$$

By (3.16), there exists $H_0 \in \Lambda_m(\delta)$ such that

$$\max_{u \in U_m} I_\theta(H_0(u)) < c_m(\theta) + \varepsilon. \tag{3.27}$$

Define

$$\bar{H}_0(\cdot) = \eta(1, H_0(\cdot)). \tag{3.28}$$

Next we prove $\bar{H}_0 \in \Lambda_m(\delta)$. It is obvious that $\bar{H}_0 \in C(U_m, E)$. In view of $H_0 \in \Lambda_m(\delta)$, (3.15), (3.17) and (3.24),

$$I_\theta(H_0(u)) \le I_0(H_0(u)) + C_0|\theta| \le b_m + \delta + C_0|\theta| < c_m(\theta) - \bar{\varepsilon}, \quad u \in F_m. \tag{3.29}$$

In combination with (3.25), (3.28) and (3.29),

$$\bar{H}_0(u) = \eta(1, H_0(u)) = H_0(u), \quad u \in F_m,$$

which yields

$$\bar{H}_0|_{F_m} \in \Gamma_m \quad \text{and} \quad I_0(\bar{H}_0(u)) = I_0(H_0(u)) \le b_m + \delta,\ u \in F_m. \tag{3.30}$$

In view of $H_0 \in \Lambda_m(\delta)$ and the definitions of R_m and R_{m+1}

$$H_0(u) = u \quad \text{and} \quad I_0(H_0(u)) \le 0, \quad u \in Q_m. \tag{3.31}$$

By (3.17), (3.24) and (3.31), we have

$$I_\theta(H_0(u)) \le I_0(H_0(u)) + C_0|\theta| \le C_0|\theta| < c_m(\theta) - \bar{\varepsilon}, \quad u \in Q_m. \tag{3.32}$$

It follows (3.25), (3.31) and (3.32) that

$$\bar{H}_0(u) = \eta(1, H_0(u)) = H_0(u) = u, \quad u \in Q_m. \tag{3.33}$$

In view of (3.30) and (3.33), $\bar{H}_0 \in \Lambda_m(\delta)$. Moreover, it follows (3.26) and (3.27) that

$$\max_{u \in U_m} I_\theta\big(\bar{H}_0(u)\big) = \max_{u \in U_m} I_\theta\big(\eta(1, H_0(u))\big) \le c_m(\theta) - \varepsilon,$$

which is a contradiction to (3.16). The proof is complete. □

Proof of Theorem 1.1. First we can choose a subsequence $\{n_k\} \subset \mathbb{N}$ such that $b_{n_k+1} > b_{n_k} > 0$. In view of Lemma 3.4, there exist two sequences $\{\theta_k\}$ and $\{c_{n_k}(\theta)\}$ such that $\theta_k > 0$ and $c_{n_k}(\theta)$ is a critical value for $I_\theta(u)$ with $|\theta| \le \theta_k$. Moreover, by (3.20) and (3.23),

$$b_{n_k} - C_0|\theta| \le c_{n_k}(\theta) \le c_{n_k} + C_0|\theta|. \tag{3.34}$$

For any $j \in \mathbb{N}$, choose strictly increasing integers p_i such that for $1 \le i \le j$,

$$p_i \in \{n_k\} \quad \text{and} \quad c_{p_i} < b_{p_{(i+1)}}.$$

Next we can choose $\varepsilon_j > 0$ small enough such that $c_{p_i}(\theta)$ with $1 \le i \le j$ are defined for $|\theta| \le \varepsilon_j$. Moreover, if $|\theta| \le \varepsilon_j$, for $1 \le i \le j$,

$$c_{p_i} + C_0|\theta| < b_{p_{(i+1)}} - C_0|\theta|. \tag{3.35}$$

In view of (3.34) and (3.35), for $|\theta| \le \varepsilon_j$, $I_\theta(u)$ has at least j critical values and

$$c_{p_1}(\theta) < c_{p_2}(\theta) < \dots < c_{p_j}(\theta).$$

Therefore system (1.1) has at least j distinct solutions. The proof is complete. □

Proof of Theorem 1.2. By the Deformation Theorem and Lemma 3.4, we can prove that $\{b_m\}$ is a sequence of critical values of $I_0(u)$ which converge to $+\infty$. Hence the corresponding critical points are solutions of system (1.1) with $\theta = 0$. The proof is complete. □

4. Examples

In this section, we give an example to illustrate our results. In system (1.1), let $p(n)$ be an $N \times N$ real symmetric positive definite matrix for all $n \in \mathbb{Z}$, $L(n) = (n^2 + 1)I_N$, and let

$$W(n, x) = (n^2 - 10)|x|^3 \quad \text{and} \quad F(n, x) = \frac{\sin x_1}{1 + n^2},$$

where $x = (x_1, x_2, \dots x_N)$. Thus all conditions of Theorem 1.1 are satisfied with

$$\mu = 3, \quad \nu_1 = \nu_2 = 2, \quad \gamma_1(n) = \gamma_2(n) = \frac{1}{1 + n^2}.$$

By Theorem 1.1, for any $j \in \mathbb{N}$, there exists $\varepsilon_j > 0$ such that if $|\theta| \le \varepsilon_j$, then system (1.1) possesses at least j distinct solutions. Since $F(n,x)$ in our example is not even in x, the results in [6, 10, 11, 19, 24] can't be applied to this example.

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