*Electronic Journal of Differential Equations*, Vol. 2015 (2015), No. 283, pp. 1–11.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

# EXPONENT OF CONVERGENCE OF SOLUTIONS TO LINEAR DIFFERENTIAL EQUATIONS IN THE UNIT DISC

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Abstract. In this article, we study the exponent of convergence of $f^{(i)} - \varphi$ where $f \not\equiv 0$ is a solution of linear differential equations with analytic and meromorphic coefficients in the unit disc and $\varphi$ is a small function of $f$. From this results we deduce the fixed points of $f^{(i)}$ by taking $\varphi(z) = z$. We will see the similarities and differences between the complex plane and the unit disc.

## 1. Introduction and statement of results

Throughout this paper, we assume that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna value distribution theory of meromorphic function on the complex plane $\mathbb{C}$ and in the unit disc $D = \{z \in \mathbb{C} : |z| < 1\}$ (see [10, 19]). In addition, we will use $\sigma(f)$, $\sigma_2(f)$ and $\tau(f)$ to denote respectively the order, hyper-order and type of a meromorphic function $f(z)$ in $D$, that are defined respectively by

$$\sigma(f) = \limsup_{r\to 1^-} \frac{\log^+ T(r,f)}{-\log(1-r)}, \quad \sigma_2(f) = \limsup_{r\to 1^-} \frac{\log^+\log^+ T(r,f)}{-\log(1-r)},$$
$$\tau(f) = \limsup_{r\to 1^-} (1-r)^{\sigma} T(r,f),$$

where $T(r,f)$ is the Nevanlinna characteristic function of $f$, $0 < \sigma = \sigma(f) < +\infty$. For an analytic function $f(z)$ in $D$, we have also the definitions

$$\sigma_M(f) = \limsup_{r\to 1^-} \frac{\log^+\log^+ M(r,f)}{-\log(1-r)}, \quad \sigma_{M,2}(f) = \limsup_{r\to 1^-} \frac{\log^+\log^+\log^+ M(r,f)}{-\log(1-r)},$$
$$\tau_M(f) = \limsup_{r\to 1^-} (1-r)^{\sigma_M} \log^+ M(r,f),$$

where $M(r,f) = \max\{|f(z)| : |z| = r\}$, $0 < \sigma_M = \sigma_M(f) < +\infty$.

M. Tsuji [16, p.205], shows that

$$\sigma(f) \le \sigma_M(f) \le \sigma(f) + 1. \tag{1.1}$$

2010 *Mathematics Subject Classification.* 34M10, 30D35.
*Key words and phrases.* Linear differential equations; exponent of convergence; growth of solutions, unit disc.

Submitted May 21, 2015. Published November 11, 2015.

For example, the function $f(z) = \exp\{\frac{1}{(1-z)^\mu}\}$, $(\mu \geq 1)$, satisfies $\sigma(f) = \mu - 1$ and $\sigma_M(f) = \mu$. Obviously, we have

$$\sigma(f) < \infty \quad \text{if and only if} \quad \sigma_M(f) < \infty.$$

Inequalities (1.1) are the best possible in the sense that there are analytic functions $g$ and $h$ such that $\sigma_M(g) = \sigma(g)$ and $\sigma_M(h) = \sigma(h) + 1$, see [7]. However, it follows by [14, Prop. 2.2.2] that $\sigma_{M,2}(f) = \sigma_2(f)$.

We use $\lambda(f)$, $(\overline{\lambda}(f))$ to denote the exponent of convergence of the zero-sequence (distinct zero-sequence) of meromorphic function $f(z)$ and $\lambda_2(f)$, $(\overline{\lambda_2}(f))$ to denote the hyper-exponent of convergence of zero-sequence (distinct zero-sequence) of $f(z)$, which are defined as follows:

$$\lambda(f) = \limsup_{r\to 1^-} \frac{\log N(r, \frac{1}{f})}{-\log(1-r)}, \quad \overline{\lambda}(f) = \limsup_{r\to 1^-} \frac{\log \overline{N}(r, \frac{1}{f})}{-\log(1-r)},$$

$$\lambda_2(f) = \limsup_{r\to 1^-} \frac{\log\log N(r, \frac{1}{f})}{-\log(1-r)}, \quad \overline{\lambda_2}(f) = \limsup_{r\to 1^-} \frac{\log\log \overline{N}(r, \frac{1}{f})}{-\log(1-r)}.$$

**Definition 1.1** ([11]). Let $f$ be an analytic function in the unit disc $D$ and let $q \in [0, \infty)$. Then $f$ is said to belong to the weighted Hardy space $H_q^\infty$ provided that

$$\sup_{z\in D} (1 - |z|^2)^q |f(z)| < \infty.$$

And we say that $f$ is an $\mathcal{H}$-function when $f \in H_q^\infty$ for some $q$.

**Definition 1.2** ([11]). A meromorphic function $f$ in the unit disc $D$ is called admissible if

$$\limsup_{r\to 1^-} \frac{T(r, f)}{-\log(1-r)} = \infty$$

and nonadmissible if

$$\limsup_{r\to 1^-} \frac{T(r, f)}{-\log(1-r)} < \infty.$$

The complex oscillation and fixed points of solutions and their derivatives of linear differential equations is an interesting area of research and have been investigated by many authors in the complex plane (see for example [1, 2, 3, 6]). In 2012, Xu, Tu and Zheng investigated the relationship between small function and derivatives of solutions of the higher order differential equation

$$f^{(k)} + A_{k-1}(z) f^{(k-1)} + \cdots + A_1(z) f' + A_0(z) f = 0 \tag{1.2}$$

where $A_j(z)$ are entire or meromorphic functions in the complex plane, and obtained the following results.

**Theorem 1.3** ([18]). *Let $A_j(z)$ $j = 0, 1, \dots, k-1$ be entire functions with finite order and satisfy one of the following conditions:*

(i) $\max\{\sigma(A_j) : j = 1, 2, \dots, k-1\} < \sigma(A_0) < \infty$;
(ii) $0 < \sigma(A_{k-1}) = \dots \sigma(A_1) = \sigma(A_0) < \infty$ *and* $\max\{\tau(A_j) : j = 1, 2, \dots, k-1\} = \tau_1 < \tau(A_0) = \tau$,

*then for every solution $f \not\equiv 0$ of (1.2) and for any entire function $\varphi(z) \not\equiv 0$ satisfying $\sigma_2(\varphi) < \sigma(A_0)$, we have*

$$\overline{\lambda_2}(f - \varphi) = \overline{\lambda_2}(f' - \varphi) = \overline{\lambda_2}(f'' - \varphi) = \overline{\lambda_2}(f^{(i)} - \varphi) = \sigma_2(f) = \sigma(A_0) \quad (i \in \mathbb{N}).$$

**Theorem 1.4** ([18]). *Let $A_j(z)$ $j = 1, 2, \dots, k-1$ be polynomials, $A_0(z)$ be a transcendental entire function, then for every solution $f \not\equiv 0$ of (1.2) and for any entire function $\varphi(z)$ of finite order, we have*

(i) $\overline{\lambda}(f - \varphi) = \lambda(f - \varphi) = \sigma(f) = \infty$;
(ii) $\overline{\lambda}(f^{(i)} - \varphi) = \lambda(f^{(i)} - \varphi) = \sigma(f^{(i)} - \varphi) = \infty \quad (i \geq 1, i \in \mathbb{N})$.

**Theorem 1.5** ([18]). *Let $A_j(z)$ $j = 0, 1, \dots, k-1$ be meromorphic functions satisfying $\max\{\sigma(A_j) : j = 1, 2, \dots, k-1\} < \sigma(A_0)$ and $\delta(\infty, A_0) > 0$. Then, for every meromorphic solution $f \not\equiv 0$ of (1.2) and for any meromorphic function $\varphi(z) \not\equiv 0$ satisfying $\sigma_2(\varphi) < \sigma(A_0)$, we have*

$$\overline{\lambda_2}(f^{(i)} - \varphi) = \lambda_2(f^{(i)} - \varphi) \geq \sigma(A_0) \quad (i \in \mathbb{N}),$$

*where $f^{(0)} = f$.*

Recently, Xu and Tu improved some of these results by making use the notion of $[p, q]$-order in the complex plane, see [17].

A natural question is how about the case of the unit disc? In this paper, we will answer this question and we will see the similarities and differences between the complex plane and the unit disc.

**Theorem 1.6.** *Let $A_j(z)$ $j = 0, 1, \dots, k-1$ be analytic functions in the unit disc $D$ with finite order and satisfy one of the following conditions:*

(1) $\max\{\sigma_M(A_j) : j = 1, \dots, k-1\} < \sigma_M(A_0) < \infty$;
(2) $0 < \sigma_M(A_{k-1}) = \dots \sigma_M(A_1) = \sigma_M(A_0) < \infty$ *and* $\max\{\tau_M(A_j) : j = 1, 2, \dots, k-1\} = \tau_1 < \tau_M(A_0) = \tau$,

*then for every solution $f \not\equiv 0$ of (1.2) and for any analytic function $\varphi(z) \not\equiv 0$ in the unit disc $D$ satisfying $\sigma_{M,2}(\varphi) < \sigma_M(A_0)$, we have*

$$\overline{\lambda_2}(f - \varphi) = \overline{\lambda_2}(f^{(i)} - \varphi) = \lambda_2(f^{(i)} - \varphi) = \sigma_{M,2}(f) = \sigma_M(A_0) \quad (i \in \mathbb{N}). \tag{1.3}$$

**Remark 1.7.** We get the same result of Theorem 1.6 if some coefficients $A_j(z)$ $j = 1, 2, \dots, k-1$, satisfy (1) and the others satisfy (2), i.e. there exists $J \subset \{1, \dots, k-1\}$ such that $\sigma(A_j) < \sigma(A_0)$ for $j \in J$ and $\max\{\tau(A_j) : \sigma(A_j) = \sigma(A_0)\} < \tau(A_0)$.

If we replace $\sigma_M(A_j)$ and $\tau_M(A_j)$ by $\sigma(A_j)$ and $\tau(A_j)$ in Theorem 1.6, we cannot get the same result. For example, we can see the difference between [9, Theorem 3] and the following result.

**Theorem 1.8.** *Let $A_j(z)$ $j = 0, 1, \dots, k-1$ be analytic functions in the unit disc $D$ with finite order such that $0 < \sigma(A_0) < \infty$, $\sigma(A_j) = \sigma(A_0)$ for $j \in J \subset \{1, \dots, k-1\}$ and $\sum_{j \in J} \tau(A_j) < \tau(A_0)$, and $\sigma(A_j) < \sigma(A_0)$ for $j \notin J$. Then, every solution $f \not\equiv 0$ of (1.2) satisfies $\sigma(A_0) \leq \sigma_2(f) \leq \alpha_M = \max_{0 \leq j \leq k-1}\{\sigma_M(A_j)\}$.*

If we replace, in Theorem 1.8, the condition $\sum_{j \in J} \tau(A_j) < \tau(A_0)$ by $\max\{\tau(A_j) : \sigma(A_j) = \sigma(A_0)\} < \tau(A_0)$, we cannot get the result, except for the second order linear differential equations. Theorem 1.8 is also an improvement of [15, Theorem 2.1].

**Corollary 1.9.** *Let $A_j(z)$ $j = 0, 1$ be analytic functions in the unit disc $D$ with finite order and satisfy $\sigma(A_1) < \sigma(A_0) < \infty$ or $0 < \sigma(A_1) = \sigma(A_0) < \infty$ and $\tau(A_1) < \tau(A_0)$. Then, for every solution $f \not\equiv 0$ of the differential equation*

$$f'' + A_1(z)f' + A_0(z)f = 0,$$

*satisfies* $\sigma(A_0) \le \sigma_2(f) \le \max\{\sigma_M(A_0), \sigma_M(A_1)\}$.

In Theorem 1.6, for $k = 2$, If we replace $\sigma_M(A_j)$ and $\tau_M(A_j)$ by $\sigma(A_j)$ and $\tau(A_j)$, we can get the following result:

$$\sigma(A_0) \le \overline{\lambda_2}(f-\varphi) = \overline{\lambda_2}(f^{(i)}-\varphi) = \sigma_2(f) \le \max\{\sigma_M(A_0), \sigma_M(A_1)\}. \tag{1.4}$$

But for $k \ge 3$, (1.4) remains valid only for the condition (1).

**Theorem 1.10.** *Let $A_j(z)$ $j = 1, 2, \dots, k-1$ be $\mathcal{H}$- functions while $A_0(z)$ is analytic not being an $\mathcal{H}$- function. Then for every solution $f \not\equiv 0$ of* (1.2) *and for any analytic function $\varphi(z) \not\equiv 0$ of finite order, we have*

(1) $\overline{\lambda}(f-\varphi) = \lambda(f-\varphi) = \sigma(f) = \infty$;
(2) $\overline{\lambda}(f^{(i)}-\varphi) = \sigma(f^{(i)}-\varphi) = \infty$ $(i \ge 1, i \in \mathbb{N})$.

**Theorem 1.11.** *Let $A_j(z)$ $j = 0, 1, \dots, k-1$ be meromorphic functions in the unit disc $D$ satisfying $\max\{\sigma(A_j) : j = 1, 2, \dots, k-1\} < \sigma(A_0)$ and $\delta(\infty, A_0) > 0$. Then, for every meromorphic solution $f \not\equiv 0$ of* (1.2) *and for any meromorphic function $\varphi(z) \not\equiv 0$ in the unit disc $D$ satisfying $\sigma_2(\varphi) < \sigma(A_0)$, we have*

$$\overline{\lambda_2}(f^{(i)}-\varphi) = \lambda_2(f^{(i)}-\varphi) = \sigma_2(f) \ge \sigma(A_0) \quad (i \in \mathbb{N}), \tag{1.5}$$

*where $f^{(0)} = f$.*

## 2. Preliminaries

Throughout this paper, we use the following notation that are not necessarily the same at each occurrence:
$E \subset (0,1)$ is a set of finite logarithmic measure, that is $\int_E \frac{dr}{1-r} < \infty$.
$F \subset (0,1)$ is a set of infinite logarithmic measure, that is $\int_F \frac{dr}{1-r} = \infty$.
$c > 0,\ \varepsilon > 0,\ \sigma \ge 0,\ \sigma_1 \ge 0,\ \tau \ge 0,\ \tau_1 \ge 0$, are real constants.

**Lemma 2.1** ([18])**.** *Assume that $f \not\equiv 0$ is a solution of* (1.2). *Set $g = f - \varphi$; then $g$ satisfies the equation*

$$g^{(k)} + A_{k-1}g^{(k-1)} + \dots + A_0 g = -[\varphi^{(k)} + A_{k-1}\varphi^{(k-1)} + \dots + A_0\varphi]. \tag{2.1}$$

**Lemma 2.2** ([18])**.** *Assume that $f \not\equiv 0$ is a solution of* (1.2). *Set $g_i = f^{(i)} - \varphi$, $(i \in \mathbb{N} - \{0\})$; then $g_i$ satisfies*

$$g_i^{(k)} + U_{k-1}^i g_i^{(k-1)} + \dots + U_0^i g_i = -[\varphi^{(k)} + U_{k-1}^i \varphi^{(k-1)} + \dots + U_0^i \varphi], \tag{2.2}$$

*where*

$$U_j^i = (U_{j+1}^{i-1})' + U_j^{i-1} - \frac{(U_0^{i-1})'}{U_0^{i-1}} U_{j+1}^{i-1}, \tag{2.3}$$

$j = 0, 1, \dots, k-1$, $U_j^0 = A_j$ *and* $U_k^i \equiv 1$.

The next lemma is a consequence of [7, Theorem 3.1].

**Lemma 2.3.** *Let $f$ be a meromorphic function in the unit disc $D$ such that $f^{(j)}$ does not vanish identically. Let $\varepsilon > 0$ be a constant; $k$ and $j$ be integers satisfying $k > j \ge 0$ and $d \in (0,1)$. Then, we have*

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \le \Big( \big(\frac{1}{1-|z|}\big)^{(2+\varepsilon)} \max\{\log\frac{1}{1-|z|}, T(s(|z|), f)\}\Big)^{k-j}, \quad |z| \notin E,$$

*where $s(|z|) = 1 - d(1 - |z|)$. As a particular case, if $\sigma_1(f) < \infty$, then*

$$|\frac{f^{(k)}(z)}{f^{(j)}(z)}| \leq \Big(\frac{1}{1-|z|}\Big)^{(k-j)(\sigma_1+2+\varepsilon)}, \quad |z| \notin E. \tag{2.4}$$

**Lemma 2.4** ([9]). *Let $f(z)$ be an analytic function in the unit disc $D$ with $\sigma_M(f) = \sigma$, $\tau_M(f) = \tau$, $0 < \sigma < \infty$, $0 < \tau < \infty$, then for any given $0 < \beta < \tau$, there exists a set $F \subset (0,1)$ that has infinite logarithmic measure such that for all $r \in F$ we have*

$$\log^+ M(r,f) > \frac{\beta}{(1-r)^\sigma}.$$

By the same method of the proof of Lemma 2.4, we obtain the followings two lemmas.

**Lemma 2.5.** *Let $f(z)$ be an analytic function in the unit disc $D$ with $\sigma_M(f) = \sigma$, $0 < \sigma < \infty$, then for any given $0 < \beta < \sigma$, there exists a set $F \subset (0,1)$ that has infinite logarithmic measure such that for all $r \in F$ we have*

$$\log^+ M(r,f) > \frac{1}{(1-r)^\beta}.$$

**Lemma 2.6.** *Let $f(z)$ be meromorphic function in the unit disc $D$ with $\sigma(f) = \sigma$, $\tau(f) = \tau$, $0 < \sigma < \infty$, $0 < \tau < \infty$, then for any given $0 < \beta < \tau$, there exists a set $F \subset (0,1)$ that has infinite logarithmic measure such that for all $r \in F$ we have*

$$T(r,f) > \frac{\beta}{(1-r)^\sigma}.$$

**Lemma 2.7.** *Let $A_j(z)$ $j = 0,1,\dots,k-1$ be analytic functions in the unit disc $D$ with finite order and satisfy $0 < \sigma_M(A_j) \leq \sigma_M(A_0) = \sigma$ for all $j = 1,\dots,k-1$, and $\max\{\tau_M(A_j) : j \neq 0\} = \tau_1 < \tau_M(A_0) = \tau$, and $U_j^i$ $(j = 0,1,\dots,k)$ $(i \in \mathbb{N})$ be stated as in (2.3). Then, for any given $\varepsilon$ $(0 < 2\varepsilon < \tau - \tau_1)$, there exists a set $F$ of infinite logarithmic measure such that for $r \in F$, we have*

$$|U_0^i| \geq \exp\{\frac{\tau-\varepsilon}{(1-r)^\sigma}\} \quad \textit{and} \quad |U_j^i| \leq \exp\{\frac{\tau_1+\varepsilon}{(1-r)^\sigma}\}, \tag{2.5}$$

*where $j = 1,2,\dots,k-1$.*

*Proof.* If we want to prove (2.5) for $i = m \in \mathbb{N} - \{0,1\}$, we start by

$$|A_0| \geq \exp\{\frac{\tau-\varepsilon/2^m}{(1-r)^\sigma}\} \quad \text{and} \quad |A_j| \leq \exp\{\frac{\tau_1+\varepsilon/2^m}{(1-r)^\sigma}\}.$$

We have $U_j^1 = A'_{j+1} + A_j - \frac{A'_0}{A_0}A_{j+1} = A_j + A_{j+1}(\frac{A'_{j+1}}{A_{j+1}} - \frac{A'_0}{A_0})$ $(j = 0,1,\dots,k-1)$ and $A_k \equiv 1$. So

$$|U_0^1| \geq |A_0| - |A_1|\Big(|\frac{A'_1}{A_1}| + |\frac{A'_0}{A_0}|\Big), \tag{2.6}$$

$$|U_j^1| \leq |A_j| + |A_{j+1}|(|\frac{A'_{j+1}}{A_{j+1}}| + |\frac{A'_0}{A_0}|). \tag{2.7}$$

By Lemma 2.3, Lemma 2.4 and (2.6)-(2.7), there exists a set $F$ with infinite logarithmic measure such that

$$\begin{aligned} |U_0^1| &\geq \exp\{\frac{\tau-\varepsilon/2^m}{(1-r)^\sigma}\} - 2\exp\{\frac{\tau_1+\varepsilon/2^m}{(1-r)^\sigma}\}\frac{1}{(1-r)^c} \\ &\geq \exp\{\frac{\tau-\varepsilon/2^{m-1}}{(1-r)^\sigma}\}, \end{aligned} \tag{2.8}$$

$$\begin{aligned} |U_j^1| &\leq \exp\{\frac{\tau_1+\varepsilon/2^m}{(1-r)^\sigma}\} + 2\exp\{\frac{\tau_1+\varepsilon/2^m}{(1-r)^\sigma}\}\frac{1}{(1-r)^c} \\ &\leq \exp\{\frac{\tau_1+\varepsilon/2^{m-1}}{(1-r)^\sigma}\}, \quad j \neq 0, \end{aligned} \tag{2.9}$$

where $c > 0$ is a constant. Now for $i = 2$ in (2.3), we have

$$|U_0^2| \geq |U_0^1| - |U_1^1|\Big(|\frac{(U_1^1)'}{U_1^1}| + |\frac{(U_0^1)'}{U_0^1}|\Big), \tag{2.10}$$

$$|U_j^2| \leq |U_j^2| + |U_{j+1}^2|\Big(|\frac{(U_{j+1}^2)'}{U_{j+1}^2}| + |\frac{(U_0^2)'}{U_0^2}|\Big), \quad j \neq 0. \tag{2.11}$$

From (2.8)-(2.11), we obtain

$$|U_0^2| \geq \exp\{\frac{\tau-\varepsilon/2^{m-2}}{(1-r)^\sigma}\} \quad \text{and} \quad |U_j^2| \leq \exp\{\frac{\tau_1+\varepsilon/2^{m-2}}{(1-r)^\sigma}\}. \tag{2.12}$$

By (2.12) and for $i = 3$ in (2.3), we obtain

$$|U_0^3| \geq \exp\{\frac{\tau-\varepsilon/2^{m-3}}{(1-r)^\sigma}\} \quad \text{and} \quad |U_j^3| \leq \exp\{\frac{\tau_1+\varepsilon/2^{m-3}}{(1-r)^\sigma}\}.$$

By the same method until $i = m$, we obtain

$$|U_0^i| \geq \exp\{\frac{\tau-\varepsilon}{(1-r)^\sigma}\} \quad \text{and} \quad |U_j^i| \leq \exp\{\frac{\tau_1+\varepsilon}{(1-r)^\sigma}\}.$$

Thus, the proof is complete. □

**Lemma 2.8.** *Let $H_j(z)$ $j = 0, 1, \dots, k-1$ be meromorphic functions in the unit disc $D$ of finite order satisfying* $\max\{|H_j(z)|, j = 1, \dots, k-1\} \leq \exp\{\frac{\beta_1}{(1-r)^\sigma}\}$ *and* $|H_0(z)| \geq \exp\{\frac{\beta}{(1-r)^\sigma}\}$ *where $0 < \beta_1 < \beta$, $\sigma > 0$ and $|z| = r \in F \subset (0,1)$ with $F$ is of infinite logarithmic measure. Then, every meromorphic solution $f$ of the differential equation*

$$f^{(k)} + H_{k-1}(z)f^{(k-1)} + \dots + H_1(z)f' + H_0(z)f = 0 \tag{2.13}$$

*satisfies $\sigma_2(f) \geq \sigma$.*

*Proof.* Let $f \not\equiv 0$ be a meromorphic solution of (2.13) of finite order $\sigma(f) = \sigma < \infty$. From (2.13), we obtain

$$|H_0(z)| \leq |\frac{f^{(k)}}{f}| + \sum_{j=1}^{k-1} |H_j(z)||\frac{f^{(j)}}{f}|. \tag{2.14}$$

By Lemma 2.3, for a given $\varepsilon > 0$ there exists a set $E \subset [0,1)$ of finite logarithmic measure such that for all $z \in D$ satisfying $|z| \notin E$, we have

$$|\frac{f^{(j)}(z)}{f(z)}| \leq \frac{1}{(1-|z|)^{j(\sigma+2+\varepsilon)}}, \quad (j = 1, \dots, k). \tag{2.15}$$

From (2.14)-(2.15) and the assumptions of Lemma 2.8, we obtain

$$\exp\{\frac{\beta}{(1-r)^{\sigma}}\} \le \frac{c}{(1-r)^{k(\sigma+2+\varepsilon)}} \exp\{\frac{\beta_1}{(1-r)^{\sigma}}\}, \tag{2.16}$$

where $c > 0$ is a constant. Since $\beta_1 < \beta$, a contradiction follows from (2.16) as $r \to 1^-$. So, $\sigma(f) = \infty$; and by Lemma 2.3, we obtain

$$|\frac{f^{(j)}(z)}{f(z)}| \le \frac{1}{(1-r)^{j(2+\varepsilon)}} (T(s(r), f))^j, \quad r \notin E. \tag{2.17}$$

From (2.14), (2.17) and the assumptions of Lemma 2.8, we obtain

$$\exp\{\frac{\beta}{(1-r)^{\sigma}}\} \le \frac{c}{(1-r)^{k(2+\varepsilon)}} (T(s(r), f))^k \exp\{\frac{\beta_1}{(1-r)^{\sigma}}\}. \tag{2.18}$$

Set $s(r) = R$. We have $1 - r = \frac{1}{d}(1 - R)$ and ( 2.18) becomes

$$(\frac{1-R}{d})^{k(2+\varepsilon)} \exp\{\frac{(\beta-\beta_1)d^{\sigma}}{(1-R)^{\sigma}}\} \le M(T(R, f))^k, \quad R \notin E. \tag{2.19}$$

From (2.19), we conclude that $\sigma_2(f) \ge \sigma$. $\square$

By the same reasoning of Lemma 2.7 and using Lemma 2.5, we obtain the following lemma.

**Lemma 2.9.** *Let $A_j(z)$ $j = 0, 1, \dots, k-1$ be analytic functions in the unit disc $D$ with finite order and satisfy $\max\{\sigma_M(A_j) : j = 1, 2, \dots, k-1\} < \sigma_M(A_0) = \sigma < \infty$, and $(U_j^i)$ $(j = 0, 1, \dots, k)$ $(i \in \mathbb{N})$ be sequences of functions satisfying (2.3). Then, for any given $\varepsilon$ $(0 < 2\varepsilon < \sigma - \sigma_1)$, there exists a set $F$ of infinite logarithmic measure such that for $r \in F$, we have*

$$|U_0^i| \ge \exp\{\frac{1}{(1-r)^{\sigma-\varepsilon}}\} \quad \text{and} \quad |U_j^i| \le \exp\{\frac{1}{(1-r)^{\sigma_1+\varepsilon}}\}. \tag{2.20}$$

By using the same method of the proof of Lemma 2.8, we obtain the following lemma.

**Lemma 2.10.** *Let $H_j(z)$ $j = 0, 1, \dots, k-1$ be meromorphic functions of finite order in the unit disc $D$ satisfying $\max\{|H_j(z)|,\ j = 1, \dots, k-1\} \le \exp\{\frac{1}{(1-r)^{\sigma_1}}\}$ and $|H_0(z)| \ge \exp\{\frac{1}{(1-r)^{\sigma}}\}$ where $0 < \sigma_1 < \sigma$ and $|z| = r \in F \subset (0, 1)$ with $F$ is of infinite logarithmic measure. Then, every meromorphic solution $f \not\equiv 0$ of (2.13) satisfies $\sigma_2(f) \ge \sigma$.*

**Lemma 2.11.** *Let $f(z)$ be an admissible meromorphic function in the unit disc $D$ with $\sigma(f) = \sigma \ge 0$, then there exists a set $F \subset (0, 1)$ with infinite logarithmic measure such that for all $r \in F$, we have*

$$\lim_{r \to 1^-} \frac{\log T(r, f)}{-\log(1-r)} = \sigma.$$

*Proof.* By the definition of $\sigma(f)$, there exists an increasing sequence $\{r_m\} \to 1^-$ satisfying $1 - (1 - \frac{1}{m})(1 - r_m) < r_{m+1}$ and

$$\lim_{r_m \to 1^-} \frac{\log T(r_m, f)}{-\log(1-r_m)} = \sigma.$$

Then, there exists $m_0$ such that for all $m \geq m_0$ and $r \in I_m = [r_m, 1-(1-\frac{1}{m})(1-r_m)]$, we have

$$\frac{\log T(r_m, f)}{-\log[(1-\frac{1}{m})(1-r_m)]} \leq \frac{\log T(r, f)}{-\log(1-r)} \leq \frac{\log T(1-(1-\frac{1}{m})(1-r_m), f)}{-\log(1-r_m)}. \quad (2.21)$$

The limit of both sides of (2.21), when $r_m \to 1^-$, is equal to $\sigma$; so for $r \in I_m$, we have

$$\lim_{r\to 1^-} \frac{\log T(r, f)}{-\log(1-r)} = \sigma.$$

Set $F = \cup_{m=m_0}^{\infty} I_m$. Then

$$m_l(F) = \sum_{m=m_2}^{\infty} \int_{I_m} \frac{dr}{1-r} = \sum_{m=m_2}^{\infty} \log\big(\frac{m}{m-1}\big) = \infty.$$

□

**Lemma 2.12.** *Let $H_j(z)$ $j = 0, 1, \dots, k-1$ be meromorphic functions in the unit disc $D$ with $\max\{\sigma(H_j), j = 1, \dots, k-1\} = \sigma_1 < \sigma(H_0) = \sigma$ and $\delta(\infty, H_0) > 0$. Then, every meromorphic solution $f$ of (2.13) satisfies $\sigma_2(f) \geq \sigma$.*

*Proof.* Let $f$ be a meromorphic solution of (2.13). From (2.13) and the logarithmic derivative lemma, we have

$$\begin{aligned} m(r, H_0) &\leq m\big(r, \frac{f^{(k)}}{f}\big) + m\big(r, \frac{f^{(k-1)}}{f}\big) + \dots + m\big(r, \frac{f'}{f}\big) \\ &\quad + \sum_{j=1}^{k-1} m(r, H_j) + \log(k+1) \\ &\leq c\big(\log^+ T(r, f) + \log\frac{1}{1-r}\big) + \sum_{j=1}^{k-1} m(r, H_j), \quad r \notin E, \end{aligned} \quad (2.22)$$

where $E \subset (0,1)$ of finite logarithmic measure and $c > 0$. By Lemma 2.11, there exists a set $F$ of infinite logarithmic measure such that for all $r \in F$, we have

$$\lim_{r\to 1^-} \frac{\log T(r, H_0)}{-\log(1-r)} = \sigma. \quad (2.23)$$

Since $\delta(\infty, A_0) = \liminf_{r\to+\infty} \frac{m(r,H_0)}{T(r,H_0)} > 0$, then by (2.23), for given $\varepsilon$ $(0 < 2\varepsilon < \sigma - \sigma_1)$ and for all $r \in F$, we have

$$m(r, H_0) \geq \frac{1}{(1-r)^{\sigma-\varepsilon}}.$$

From (2.22) and (2.23), for $r \in F - E$, we have

$$\frac{1}{(1-r)^{\sigma-\varepsilon}} \leq c\big(\log^+ T(r, f) + \log\frac{1}{1-r}\big) + (k-1)\frac{1}{(1-r)^{\sigma_1+\varepsilon}}. \quad (2.24)$$

From (2.24), we obtain $\sigma_2(f) \geq \sigma$. □

**Lemma 2.13.** *Let $A_j(z)$, $j = 0, 1, \dots, k-1$ be meromorphic functions in the unit disc $D$ satisfying $\max\{\sigma(A_j) : j = 1, 2, \dots, k-1\} = \sigma_1 < \sigma(A_0) = \sigma$ and*

*$\delta(\infty, A_0) > 0$ and $U_j^i$ $(j = 0, 1, \dots, k)$ $(i \in \mathbb{N})$ be stated as in (2.3). Then, for any given $\varepsilon$ satisfying $0 < 2\varepsilon < \sigma - \sigma_1$, we have*

$$m(r, U_j^i) \le \frac{1}{(1-r)^{\sigma_1+\varepsilon}}, \quad (j = 1, 2, \dots, k-1),\ r \notin E,$$

$$m(r, U_0^i) \ge \frac{1}{(1-r)^{\sigma-\varepsilon}}, \quad r \in F.$$

By using the same reasoning as above we can get the conclusion of this lemma; so we omit the proof here.

**Lemma 2.14.** *Let $G \not\equiv 0, H_j(z)$ $j = 0, 1, \dots, k-1$ be meromorphic functions in the unit disc $D$. If $f$ is a meromorphic solution of the differential equation*

$$f^{(k)} + H_{k-1}(z)f^{(k-1)} + \dots + H_1(z)f' + H_0(z)f = G(z), \tag{2.25}$$

*satisfying $\max\{\sigma_n(G), \sigma_n(H_j);\ j = 0, 1, \dots, k-1\} < \sigma_n(f) = \sigma_n$, then $\overline{\lambda_n}(f) = \lambda_n(f) = \sigma_n(f)$, $(n \in \mathbb{N} - \{0\})$.*

*Proof.* The same reasoning of the proof of Lemma 3.5 in [4] when $G \not\equiv 0, H_j(z)$ $j = 0, 1, \dots, k-1$ are analytic in the unit disc $D$. □

**Lemma 2.15** ([9, Thm. 3])**.** *Let $n \in \mathbb{N} - \{0\}$. If the coefficients $A_0(z), \dots, A_{k-1}(z)$ are analytic in $D$ such that $\sigma_{M,n}(A_j) \le \sigma_{M,n}(A_0)$ for all $j = 1, \dots, k-1$, and*

$$\max\{\tau_{M,n}(A_j) : \sigma_{M,n}(A_j) = \sigma_{M,n}(A_0)\} < \tau_{M,n}(A_0),$$

*then all solutions $f \not\equiv 0$ of (1.2) satisfy $\sigma_{M,n+1}(f) = \sigma_{M,n}(A_0)$.*

## 3. Proofs of theorems

*Proof of Theorem 1.6.* **Case (1).** $\max\{\sigma_M(A_j) : j = 1, \dots, k-1\} < \sigma_M(A_0) < \infty$. Suppose that $f \not\equiv 0$ is a solution of (1.2) and $\varphi(z) \not\equiv 0$ is an analytic function in the unit disc $D$ satisfying $\sigma_2(\varphi) < \sigma(A_0)$. We start to prove (1.3) for $i = 0$, i.e. $\overline{\lambda_2}(f - \varphi) = \lambda_2(f - \varphi) = \sigma_2(f) = \sigma_M(A_0)$. From [5], we have $\sigma_2(f) = \sigma_M(A_0)$. Set $g = f - \varphi$. Since $\sigma_2(\varphi) < \sigma(A_0)$, then $\sigma_2(g) = \sigma_2(f)$. By Lemma 2.1, $g$ satisfies (2.1). Set $G(z) = \varphi^{(k)} + A_{k-1}\varphi^{(k-1)} + \dots + A_0\varphi$. If $G \equiv 0$, then by [5] we have $\sigma_2(\varphi) = \sigma(A_0)$, a contradiction; thus $G \not\equiv 0$. Now, since $\sigma_2(g) = \sigma_2(f) = \sigma(A_0) > \max\{\sigma_2(G), \sigma_2(A_j)\}$, then the assumption of Lemma 2.14 is hold for $n = 2$, and then we have $\overline{\lambda_2}(g) = \lambda_2(g) = \sigma_2(g)$. Then, we conclude that $\overline{\lambda_2}(f - \varphi) = \lambda_2(f - \varphi) = \sigma_2(f) = \sigma_M(A_0)$. Now we prove (1.3) for $i \ge 1$. Set $g_i = f^{(i)} - \varphi$. Since $\sigma_2(f^{(i)}) = \sigma_2(f) = \sigma(A_0)$ and $\sigma_2(\varphi) < \sigma(A_0)$, then we have $\sigma_2(g_i) = \sigma_2(f) = \sigma(A_0)$. By Lemma 2.2, $g_i$ satisfies (2.2). Set $G_i = \varphi^{(k)} + U_{k-1}^i\varphi^{(k-1)} + \dots + U_0^i\varphi$. If $G_i \equiv 0$, by Lemma 2.9 and Lemma 2.10, we obtain $\sigma_2(\varphi) \ge \sigma(A_0)$, a contradiction with $\sigma_2(\varphi) < \sigma(A_0)$; so $G_i \not\equiv 0$. Now, by Lemma 2.14, for $n = 2$, we obtain $\overline{\lambda_2}(g_i) = \lambda_2(g_i) = \sigma_2(g_i)$ i.e. $\overline{\lambda_2}(f^{(i)} - \varphi) = \lambda_2(f^{(i)} - \varphi) = \sigma_2(f) = \sigma_M(A_0)$.

**Case (2).** $0 < \sigma(A_0) < \infty$, $\sigma(A_j) = \sigma(A_0)$ for all $j \in \{1, \dots, k-1\}$ and $\max\{\tau_{M,n}(A_j) : j \ne 0\} < \tau_{M,n}(A_0)$. Assume that $f \not\equiv 0$ is a solution of (1.2) and $\varphi(z) \not\equiv 0$ is an analytic function in the unit disc $D$ satisfying $\sigma_2(\varphi) < \sigma(A_0)$. As above, we start to prove (1.3) for $i = 0$, i.e. $\overline{\lambda_2}(f - \varphi) = \lambda_2(f - \varphi) = \sigma_2(f) = \sigma_M(A_0)$. From Lemma 2.15, we have $\sigma_2(f) = \sigma_M(A_0)$. Set $g = f - \varphi$. We have $\sigma_2(g) = \sigma_2(f)$. As above, $g$ satisfies (2.1). If $G \equiv 0$, then by Lemma 2.15 we have $\sigma_2(\varphi) = \sigma(A_0)$, a contradiction; thus $G \not\equiv 0$. Now by Lemma 2.14, we obtain $\overline{\lambda_2}(g) = \lambda_2(g) = \sigma_2(g)$. So, we conclude that $\overline{\lambda_2}(f-\varphi) = \lambda_2(f-\varphi) = \sigma_2(f) = \sigma_M(A_0)$. Now we prove (1.3) for $i \geq 1$. Set $g_i = f^{(i)} - \varphi$. We have $\sigma_2(g_i) = \sigma_2(f)$, and $g_i$ satisfies (2.2). If $G_i \equiv 0$, by Lemma 2.7 and Lemma 2.8, we obtain $\sigma_2(\varphi) \geq \sigma(A_0)$, a contradiction with $\sigma_2(\varphi) < \sigma(A_0)$; so $G_i \not\equiv 0$. As above, by Lemma 2.14, for $n = 2$, we obtain $\overline{\lambda_2}(g_i) = \lambda_2(g_i) = \sigma_2(g_i)$ i.e. $\overline{\lambda_2}(f^{(i)} - \varphi) = \lambda_2(f^{(i)} - \varphi) = \sigma_2(f) = \sigma_M(A_0)$. □

*Proof of Theorem 1.8.* The inequality $\sigma_2(f) \leq \alpha_M$ follows by [12, Theorem 5.1]. From (1.2) we obtain

$$m(r, A_0) \leq m\big(r, \frac{f^{(k)}}{f}\big) + m\big(r, \frac{f^{(k-1)}}{f}\big) + \cdots + m\big(r, \frac{f'}{f}\big) + \sum_{j=1}^{k-1} m(r, A_j) + \log(k+1),$$

and then

$$m(r, A_0) \leq c\big(\log^+ T(r,f) + \log\frac{1}{1-r}\big) + \sum_{j=1}^{k-1} m(r, A_j), \quad r \notin E. \tag{3.1}$$

Set $\sum_{j\in J} \tau(A_j) = \tau^*$, $\max\{\sigma(A_j) : j \in J'\} = \sigma^*$, $\tau(A_0) = \tau$ and $\sigma(A_0) = \sigma$. By (3.1), the assumptions of Theorem 1.8 and Lemma 2.6, there exists a set $F \subset (0,1)$ of infinite logarithmic measure such that for all $r \in F - E$, we have

$$\frac{\tau - \varepsilon}{(1-r)^{\sigma}} \leq c\big(\log^+ T(r,f) + \log\frac{1}{1-r}\big) + \frac{k-1}{(1-r)^{\sigma^*+\varepsilon}} + \frac{\tau^* + \varepsilon}{(1-r)^{\sigma}}, \tag{3.2}$$

where $0 < 2\varepsilon < \min(\tau - \tau^*, \sigma - \sigma^*)$. From (3.2), we obtain that $\sigma(A_0) \leq \sigma_2(f)$. □

*Proof of Theorem 1.10.* Suppose that $f \not\equiv 0$ is a solution of (1.2) and $\varphi(z) \not\equiv 0$ is an analytic function in the unit disc $D$ of finite order. By [11], we have $\sigma(f) = \infty$. If $G \equiv 0$, then by [11] we have $\sigma(\varphi) = \infty$, a contradiction; thus $G \not\equiv 0$; and by Lemma 2.14 we obtain the result (1). Now for $i \geq 1$, if $G_i \equiv 0$, then by taking into account that if $A_j(z)$ $j = 1, 2, \dots, k-1$ are an $\mathcal{H}$-functions and $A_0(z)$ is analytic not being an $\mathcal{H}$-function then $U_j^i$ $(j = 1, \dots, k)$ are non admissible functions while $U_0^i$ are admissible, $(i \in \mathbb{N})$; so we obtain that $\sigma(\varphi) = \infty$, a contradiction; thus $G_i \not\equiv 0$; and by Lemma 2.14 we obtain the result (2). □

*Proof of Theorem 1.11.* Suppose that $f \not\equiv 0$ is a meromorphic solution of (1.2) and $\varphi(z) \not\equiv 0$ is a meromorphic function in the unit disc $D$ satisfying $\sigma_2(\varphi) < \sigma(A_0)$. By Lemma 2.12, we have $\sigma_2(f) \geq \sigma(A_0)$. If $G \equiv 0$, then by Lemma 2.12 we have $\sigma_2(\varphi) \geq \sigma(A_0)$, a contradiction; thus $G \not\equiv 0$; and by Lemma 2.14 we obtain the result (1.5) for $i = 0$. Now for $i \geq 1$, if $G_i \equiv 0$, then by Lemma 2.12 and Lemma 2.13 we have $\sigma_2(\varphi) \geq \sigma(A_0)$, a contradiction; thus $G_i \not\equiv 0$; and by Lemma 2.14 we obtain (1.5). □

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