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HÖLDER CONTINUITY OF BOUNDED WEAK SOLUTIONS TO GENERALIZED PARABOLIC p-LAPLACIAN EQUATIONS I: DEGENERATE CASE

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Abstract. Here we generalize quasilinear parabolic p-Laplacian type equations to obtain the prototype equation

$$u_t - \operatorname{div}\Big(\frac{g(|Du|)}{|Du|}Du\Big) = 0,$$

where g is a nonnegative, increasing, and continuous function trapped in between two power functions $|Du|^{g_0-1}$ and $|Du|^{g_1-1}$ with $2 \le g_0 \le g_1 < \infty$. Through this generalization in the setting from Orlicz spaces, we provide a proof for the Hölder continuity of such solutions which has much in common with that proof of Hölder continuity of solutions of singular equations.

1. Introduction

In 1957, DeGiorgi [5] showed that bounded weak solutions of linear elliptic partial differential equations are Hölder continuous, and his method was used by Ladyzhenskaya and Ural'tseva in [15] to show that bounded weak solutions of the quasilinear elliptic equation

$$\operatorname{div}\mathcal{A}(x,u,Du) = 0$$

are Hölder continuous if there are positive constants $p > 1$, C_0, and C_1 such that

$$\mathcal{A}(x,u,\xi)\cdot\xi \ge C_0|\xi|^p, \quad |\mathcal{A}(x,u,\xi)| \le C_1|\xi|^{p-1}$$

for all $\xi \in \mathbb{R}^N$, where N is the number of space dimensions. (The theorem of De Giorgi is really just the case $p = 2$ here.) For parabolic equations

$$u_t - \operatorname{div}\mathcal{A}(x,t,u,Du) = 0, \tag{1.1}$$

Ladyzhenskaya and Ural'tseva followed De Giorgi's method with some modifications but they were only able to prove Hölder continuity under the structure conditions

$$\mathcal{A}(x,t,u,\xi)\cdot\xi \ge C_0|\xi|^p, \quad |\mathcal{A}(x,u,\xi)| \le C_1|\xi|^{p-1} \tag{1.2}$$

when $p = 2$.

There was little progress on the Hölder continuity of solutions when $p \ne 2$ until 1986, when DiBenedetto [7] proved the Hölder continuity result for $p > 2$. A key

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new step was his introduction of the concept of intrinsic scaling (first introduced in a simpler setting in [6]), which has since become an important aspect in the theory and which is discussed at great length in [23]. It took several more years until the joint work of Chen and DiBenedetto [2, 3] showed that bounded weak solutions are Hölder continuous also for $p < 2$. Unfortunately, these proofs are quite technical and their exposition (for example, [8, Chapters III and IV]) is quite long. More recently, Gianazza, Surnachev, and Vespri [11] developed a more geometric approach to the Hölder continuity of solutions to equations when $p > 2$; their proof is simpler and more natural than the original one, but several issues from that proof still remain that we address here.

The more important ones are related to the distinction between the cases $p > 2$ and $p < 2$. All previously published proofs of Hölder continuity have treated these cases separately because of different qualitative behavior of solutions in the two cases. For example, any nonnegative solution of (1.1) which vanishes at a point (x_0, t_0) also vanishes in any cylinder with top center point (x_0, t_0) if $p \geq 2$; however, when $p < 2$, nonnegative solutions generally become zero in finite time. (We refer the reader to [8, Sections VI.2, VII.2, and VII.3] for a more complete discussion of these phenomena.) Such behavior must be accounted for, but our proof points out some significant common elements. A further issue is that the newer proofs (see the Remark on [11, page 278] for the case $p > 2$ and [10, Section 4] for a related result in case $p < 2$) give a Hölder exponent which degenerates as p approaches 2; in both cases, the proof must be further modified for p close to 2 if the Hölder exponent is to remain positive near $p = 2$ even though the original proof of Hölder continuity for degenerate equations in [7] allowed a stable Hölder exponent in this case.

In this paper and its companion [13], we take a more general approach to the problem: We study (1.1) when there is an increasing function g such that

$$\mathcal{A}(x, t, u, \xi) \cdot \xi \geq C_0 G(|\xi|), \tag{1.3a}$$

$$|\mathcal{A}(x, t, u, \xi)| \leq C_1 g(|\xi|) \tag{1.3b}$$

for some positive constants C_0 and C_1, where G is defined by

$$G(\sigma) = \int_0^\sigma g(s)\, ds,$$

and we assume that there are constants g_0 and g_1 satisfying $1 < g_0 \leq g_1 < \infty$ such that

$$g_0 G(\sigma) \leq \sigma g(\sigma) \leq g_1 G(\sigma) \tag{1.4}$$

for all $\sigma > 0$. (The two inequalities in (1.4) are essentially the Δ_2 and ∇_2 conditions in Orlicz space theory as in [14, Sections I.3 and I.4] and in [22, Section 2.3]. The precise connection between (1.4) and these conditions is the topic of [20].) Our primary concern here is with the case $g_0 \geq 2$ and the companion paper [13] is concerned with the case $1 < g_0 \leq g_1 \leq 2$, but since many of our intermediate results are true (with the same proof) for the full range of g_0 and g_1, we shall pay attention to these two possibilities carefully. The case $g_0 \geq 2$ is known as the degenerate case, and the case $1 < g_0 \leq g_1 \leq 2$ is known as the singular case. In [13], we analyze the singular case, taking advantage of the results from the degenerate case that are still relevant to that one. The structure (1.2) is contained in this model as the special case $g(s) = s^{p-1}$ for $p \geq 2$, in which case we may take $g_0 = g_1 = p$, and $p = 2$ will fit into the degenerate structure studied here as well as the singular structure from [13]. In addition, our structure allows consideration of more general equations; as shown in [18, pages 313 and 314], for any α and β with $1 < \alpha < \beta < \infty$, we can find a function g satisfying (1.4) such that

$$\limsup_{s\to\infty} \frac{g(s)}{s^\beta} > 0, \quad \liminf_{s\to\infty} \frac{g(s)}{s^\alpha} < \infty,$$

so we consider a class of structure functions g much wider than that of just power functions. In this way, we obtain a uniform proof of Hölder continuity (with appropriate uniformity of constants) for all $p \in [2, \infty)$ at once under the structure condition (1.2) as well as a proof of Hölder continuity under more general structure conditions. We note especially that our main estimates do not depend on g_0 (although the condition $g_0 \geq 2$ will be critical to the results), so our result in this case is stable as p approaches 2. We also point out that if we replace G and g by suitable multiples of these functions (and appropriately modifying C_0 and C_1), we can achieve any number of normalizations: for example, $G(1) = 1$, $g(1) = 1$, $C_0 = 1$, or $C_1 = 1$. It is interesting to note that our estimates are independent of the normalization.

The motivation for considering (1.4) comes from [18] in which corresponding results for elliptic equations were proved. The extension of the methods used in [18] for proving Hölder continuity of weak solutions to parabolic equations is not straightforward; this paper and [13] present the only such extensions known to the authors.

We could consider the full quasilinear equation

$$u_t - \operatorname{div} \mathcal{A}(x, t, u, Du) = B(x, t, u, Du),$$

satisfying the structure conditions

$$\begin{gathered}\mathcal{A}(x, t, u, \xi) \cdot \xi \geq C_0 G(|\xi|) - \varphi_0(x, t),\\ |\mathcal{A}(x, t, u, \xi)| \leq C_1 g(|\xi|) + \varphi_1(x, t),\\ |B(x, t, u, \xi)| \leq C_2 G(|\xi|) + \varphi_2(x, t)\end{gathered}$$

for constants $C_0 > 0$, $C_1 \geq C_0$, $C_2 \geq 0$ and suitable nonnegative functions φ_0, φ_1, and φ_2. In the special case $G(\sigma) = \sigma^p$, the choice of function spaces (for φ_0, φ_1 and φ_2) is easily determined from the parabolic Sobolev imbedding theorem. We refer the reader to the introduction of [7] for a detailed description of these spaces. In our more general case, the extension is complicated by the lack of a general parabolic Orlicz-Sobolev imbedding theorem (which would, presumably, be based on the work of Cianchi [4] for general Orlicz-Sobolev imbedding theorems). Alternatively, one can use the approach in [18] and use much smaller Lebesgue spaces. In the first case, the description of the nonhomogeneous terms involves an unnecessary technical complication, and, in the second case, the nonhomogeneous terms are handled by a straightforward but messy adaption of the arguments in [7] and [8]. Hence, we omit this extension.

For our investigation, we also need a suitable definition of weak solution, which we present here. For an arbitrary open set $\Omega \subset \mathbb{R}^{n+1}$, we introduce the generalized Sobolev space $W^{1,G}(\Omega)$, which consists of all functions u defined on Ω with weak derivative Du satisfying

$$\iint_\Omega G(|Du|)\, dx\, dt < \infty.$$

We say that $u \in C_{\rm loc}(\Omega) \cap W^{1,G}(\Omega)$ is a *weak supersolution* of (1.1) if

$$0 \le -\iint_\Omega u\varphi_t \,dx\,dt + \iint_\Omega \mathcal{A}(x,t,u,Du)\cdot D\varphi \,dx\,dt$$

for all $\varphi \in C^1(\bar\Omega)$ which vanish on the parabolic boundary of Ω, which we also denote by $\partial_P\Omega$; a *weak subsolution* is defined by reversing the inequality. A *weak solution* is then a function which is both a weak supersolution and a weak subsolution. In fact, a standard approximation shows that we need only assume that $\varphi \in L^\infty(\Omega) \cap W^{1,G}(\Omega)$ with $\varphi_t \in L^1(\Omega)$.

We mention here the paper [11] of Gianazza, Surnachev, and Vespri, which gave a different proof for the Hölder continuity from that in [1, 7]. One of the key ideas in [11] is to use a more geometric approach in place of alternative based on the size of the set on which $|u|$ is close to its maximum. The geometric approach is very useful in the study of Harnack estimates, which we do not discuss here: however, the geometry in [11] will reappear in [13]. Our main justification for avoiding the geometric simplicity of [11] is that, unlike the alternative approach used here, the geometric approach must look separately at the cases p large and p close to 2.

We begin by discussing the two alternatives, which refer to nonnegative weak supersolutions u of (1.1) in a scaled cylinder. The first alternative states that, if u is large on most of one subcylinder in a suitable family of subcylinders of the original cylinder, then u is bounded from zero on all of a subcylinder with the same center-top point as the original cylinder. The second alternative states that, if u is large on a fixed fraction of every subcylinder in this family of subcylinders, then u is bounded from zero on all of a subcylinder with the same center-top point as the original cylinder. Eventually, we shall see the precise quantitative description of these results.

In Section 2, we provide some preliminary results, mostly involving notation for our geometric setting. In Section 3, we use the two alternatives to show that if the oscillation of a solution u of (1.1) over a cylinder is less than or equal to a number ω appropriately connected to the cylinder, then the oscillation over a smaller subcylinder is less than or equal to $\sigma\omega$ with $\sigma \in (0,1)$; this oscillation control is then used to prove Hölder continuity. We also prove an oscillation estimate near the initial surface in this section, and discuss briefly oscillation estimates near the lateral surface. The reason for discussing initial regularity in detail here is that it can be proved rather simply while regularity near the lateral boundary is proved (as shown in, for example, [7, Sections 7 and 8]) via a simple but tedious modification of the interior regularity results. Moreover, as we shall see in [13], our proof of initial regularity applies also to singular problems with an interesting twist. The same integral estimate is used in the degenerate and singular cases, proved here as Proposition 4.5 for the full range $1 < g_0 \le g_1$, but different geometric and algebraic considerations are used for the two cases. We prove the alternatives in Section 4, after first developing some estimates for nonnegative weak supersolutions. With one exception, these supersolution estimates will reappear in [13]. Finally, Section 5 presents some integral inequalities which are used to prove our other results. For the most part, the results in Section 5 are standard results, but some of them are interesting variations of standard results. We provide proofs for the variations but refer the reader to other sources for the standard results.

2. Preliminaries

2.1. **Notation.** (1) The parameters g_0, g_1, N, C_0, and C_1 are the data. When we make the additional assumption that $g_0 \geq 2$, we use the word "data" to denote the constants g_1, N, C_0, and C_1.

(2) Let K_ρ^y denote the $N-$dimensional cube centered at $y \in \mathbb{R}^N$ with the side length 2ρ, i.e.,

$$K_\rho^y := \{x \in \mathbb{R}^N : \max_{1\leq i\leq N} |x^i - y^i| < \rho\}.$$

(Here, we use superscripts to denote the coordinates of x; we'll use subscripts to indicate different points.) For simpler notation, let $K_\rho := K_\rho^0$. We also define the spatial distance $|\cdot|_\infty$ by

$$|x-y|_\infty = \max_{1\leq i\leq N} |x^i - y^i|.$$

In fact, all of our work can be recast with the ball

$$B_\rho^y = \{x \in \mathbb{R}^N : |x-y| < \rho\},$$

where $|x-y|$ is the usual Euclidean distance, in place of K_ρ^y with only slight notational changes. This observation is more important to the singular case in [13], and we shall comment on it there in more detail.

(3) For given $(x_0, t_0) \in \mathbb{R}^{N+1}$, and given positive constants θ, ρ and k, we say

$$\begin{gathered}
T_{k,\rho}(\theta) := \theta k^2 G\big(\frac{k}{\rho}\big)^{-1}, \\
Q_{k,\rho}^{x_0,t_0}(\theta) := K_\rho^{x_0} \times [t_0 - T_{k,\rho}, t_0], \\
Q_{k,\rho}(\theta) := Q_{k,\rho}^{0,0}(\theta).
\end{gathered}$$

The point (x_0, t_0) is called the *top-center point of* $Q_{k,\rho}^{x_0,t_0}(\theta)$. We also abbreviate

$$T_{k,\rho} = T_{k,\rho}(1), \quad Q_{k,\rho}^{x_0,t_0} = Q_{k,\rho}^{x_0,t_0}(1), \quad Q_{k\rho} = Q_{k,\rho}(1).$$

2.2. **Geometry.** The local energy estimate (5.2) plays a crucial role in this paper which is nonhomogeneous unless $g_0 = g_1 = 2$. By controlling the length of the time axis, we make two competing terms in (5.2) equivalent; that is, find $T_{k,\rho}$ from

$$G^{r-1}\big(\frac{\omega}{\rho}\big)\omega^{s+2}\frac{1}{T_{k,\rho}} \sim G^r\big(\frac{\omega}{\rho}\big)\omega^s,$$

for some constants r and s which directly leads to our definition of $T_{k,\rho}$.

This choice of the time scale is called intrinsic scaling. It was introduced by DiBenedetto [6] (but see also [8, 23]); roughly speaking, a weak solution of parabolic p-Laplacian type equation behaves like a solution of the heat equation in an intrinsically scaled cylinder.

The parameter θ is introduced to simplify some arguments. It should be noted that the arguments in [7, 8] also introduce various similar constants.

Now, suppose that u is a function defined in some open subset Ω of $\mathbb{R}^{N+1}$, let $(x_0, t_0) \in \Omega$, and let ω and θ be positive constants. Since Ω is open, there are positive constants r and s such that $K_r^{x_0} \times (t_0 - s, t_0) \subset \Omega$. If we set

$$R = \frac{1}{4}\min\big\{r, \frac{\omega}{G^{-1}(\theta\omega^2 s^{-1})}\big\},$$

we conclude that $Q_{\omega,4R}^{x_0,t_0}(\theta) \subset \Omega$.

Without loss of generality, we let $(x_0, t_0) = (0, 0)$. Then for any θ and ω, we can fit the cylinder $Q_{\omega,4R}(\theta)$ in Ω by selecting R properly. Basically, we are going to work with the cylinder $Q_{\omega,4R}(\theta)$ to find a proper subcylinder where a solution has less oscillation eventually leading to Hölder continuity.

2.3. **Useful inequalities.** Because of the generalized functions g and G, we are not able to apply Hölder's inequality or the typical Young's inequality. Here we deliver essential inequalities which will be used through out the paper.

Lemma 2.1. *For a nonnegative and nondecreasing function $g \in C[0, \infty)$, let G be the antiderivative of g. Suppose that g and G satisfies* (1.4)*. Then for all nonnegative real numbers σ, σ_1, and σ_2, we have*

(a) *$G(\sigma)/\sigma$ is a monotone increasing function of σ.*
(b) *For $\beta > 1$, $\beta^{g_0} G(\sigma) \le G(\beta\sigma) \le \beta^{g_1} G(\sigma)$.*
(c) *For $0 < \beta < 1$, $\beta^{g_1} G(\sigma) \le G(\beta\sigma) \le \beta^{g_0} G(\sigma)$.*
(d) *$\sigma_1 g(\sigma_2) \le \sigma_1 g(\sigma_1) + \sigma_2 g(\sigma_2)$.*
(e) *(Young's inequality) For any $\epsilon \in (0, 1)$,*

$$\sigma_1 g(\sigma_2) \le \epsilon^{1-g_1} g_1 G(\sigma_1) + \epsilon g_1 G(\sigma_2).$$

Proof. This lemma is essentially [18, Lemma 1.1]. We include a proof for the reader's convenience.

(a) For $\sigma > 0$, due to the left hand side inequality of (1.4), we easily obtain

$$\frac{d}{d\sigma}\Big(\frac{G(\sigma)}{\sigma}\Big) = \frac{\sigma g(\sigma) - G(\sigma)}{\sigma^2} \ge (g_0 - 1)\frac{G(\sigma)}{\sigma^2} > 0$$

because $g_0 > 1$.

(b) The left inequality of (1.4) gives

$$\frac{g_0}{\xi} \le \frac{g(\xi)}{G(\xi)} \quad \text{for } \xi \in (0, \infty).$$

By taking the integral from σ to $\beta\sigma$, we obtain

$$g_0 \log \frac{\beta\sigma}{\sigma} \le \log \frac{G(\beta\sigma)}{G(\sigma)}$$

which implies

$$\beta^{g_0} G(\sigma) \le G(\beta\sigma).$$

A similar argument with the right hand side of (1.4) completes the proof.

(c) Like the proof for (b), but take integrals over the interval $[\beta\sigma, \sigma]$.

(d) It is clear because g is nondecreasing function, so either

$$\sigma_1 g(\sigma_2) \le \sigma_1 g(\sigma_1) \quad \text{or} \quad \sigma_1 g(\sigma_2) \le \sigma_2 g(\sigma_2).$$

(e) For any $0 < \epsilon < 1$, because of (d) we obtain

$$\sigma_1 g(\sigma_2) = \epsilon \frac{\sigma_1}{\epsilon} g(\sigma_2) \le \epsilon\Big[\frac{\sigma_1}{\epsilon} g\Big(\frac{\sigma_1}{\epsilon}\Big) + \sigma_2 g(\sigma_2)\Big].$$

Applying the right inequality of (1.4) and (b) leads to

$$\sigma_1 g(\sigma_2) \le \epsilon\Big[g_1 G\Big(\frac{\sigma_1}{\epsilon}\Big) + g_1 G(\sigma_2)\Big] \le \epsilon g_1 \epsilon^{-g_1} G(\sigma_1) + \epsilon g_1 G(\sigma_2).$$

□

The next inequalities will be used to derive the logarithmic energy estimate (5.11) which plays a crucial role in Proposition 4.2.

Lemma 2.2. *For any $\sigma > 0$, let*

$$h(\sigma) = \frac{1}{\sigma}\int_0^\sigma g(s)\,ds, \quad H(\sigma) = \int_0^\sigma h(s)\,ds.$$

Then we have

$$g_0 h(\sigma) \le g(\sigma) \le g_1 h(\sigma), \tag{2.1a}$$
$$g_0 H(\sigma) \le G(\sigma) \le g_1 H(\sigma), \tag{2.1b}$$
$$(g_0 - 1)h(\sigma) \le \sigma h'(\sigma) \le (g_1 - 1)h(\sigma), \tag{2.1c}$$
$$\frac{1}{g_1}\sigma h(\sigma) \le H(\sigma) \le \frac{1}{g_0}\sigma h(\sigma), \tag{2.1d}$$
$$\beta^{g_0} H(\sigma) \le H(\beta\sigma) \le \beta^{g_1} H(\sigma) \tag{2.1e}$$

for any $\beta > 1$.

Proof. Here we note that h acts like g and H acts like G. Dividing (1.4) by σ gives (2.1a), and integrating (2.1a) gives (2.1b). Since

$$h'(\sigma) = \frac{g(\sigma)}{\sigma} - \frac{G(\sigma)}{\sigma^2},$$

we infer (2.1c) by applying (1.4).

We infer (2.1d) from (2.1b) since $G(\sigma) = \sigma h(\sigma)$, and the proof of (2.1e) is similar to the proof of Lemma 2.1(b). □

In fact, because of this lemma, we could have assumed initially that $g \in C^1$ and that g satisfies the inequalities

$$g_0 - 1 \le \frac{sg'(s)}{g(s)} \le g_1 - 1$$

for all $s > 0$. Our choice for using g and G to describe the structure conditions is more consistent with the published literature although [18] uses the stronger hypothesis $g \in C^1$.

3. The two alternatives and the proof of Hölder continuity

In this section, we prove the Hölder continuity of solutions of (1.1) for degenerate equations (that is, equations with $g_0 \ge 2$). Our proof is based on some estimates for nonnegative supersolutions of the equation, and these estimates will be proved in the next section. These estimates are usually described as the first alternative and the second alternative.

For notational convenience, we take ν_0 to be the constant from Proposition 4.4 corresponding to $\theta = 1$ and, with ω and R given positive constants, we set

$$\Delta = \Big(\frac{\omega}{2}\Big)^2 G\Big(\frac{\omega}{4R}\Big)^{-1}. \tag{3.1}$$

Our first alternative is that, if u is a positive subsolution u of a degenerate equation which stays close to its maximum on most of one suitable small subcylinder, then u is bounded away from zero on a suitable subcylinder.

Lemma 3.1 (The first alternative)**.** *Let $\theta_0 > 1$ be a given constant and suppose u is a nonnegative supersolution of* (1.1) *in*

$$Q = K_{2R} \times (-\theta_0\Delta, 0) \tag{3.2}$$

with $g_0 \geq 2$*. If there is a constant* $T_0 \in [-\theta_0\Delta, -\Delta]$ *such that*

$$\Big|K_{2R} \times (T_0, T_0 + \Delta) \cap \{u \leq \frac{\omega}{2}\}\Big| \leq \nu_0 |K_{2R}|\Delta,$$

then there is a constant $\delta_1 \in (0,1)$ *determined only by* θ_0 *and data such that*

$$\operatorname{ess\,inf}_{\mathcal{Q}} u \geq \delta_1 \omega$$

with

$$\mathcal{Q} = Q_{\omega/4, R/2}. \tag{3.3}$$

The proof of this lemma will be given in the next section. Our second alternative states that if u is a positive subsolution u of a degenerate equation which stays close to its maximum on a suitable fraction of all suitable small subcylinders, then u is bounded away from zero on a suitable subcylinder.

Lemma 3.2 (The second alternative)**.** *There are constants* $\theta_0 > 1$ *and* $\delta_2 \in (0,1)$ *(determined only by data) such that, if u is a nonnegative supersolution of* (1.1) *in* Q *(given by* (3.2)*) with* $g_0 \geq 2$ *and*

$$\Big|K_{2R} \times (T_0, T_0 + \Delta) \cap \{u \leq \frac{\omega}{2}\}\Big| \leq (1 - \nu_0)|K_{2R}|\Delta$$

for all $T_0 \in [-\theta_0\Delta, -\Delta]$*, then there is a constant* $\delta_2 \in (0,1)$*, determined only by data, such that*

$$\operatorname{ess\,inf}_{\mathcal{Q}} u \geq \delta_2 \omega$$

with $\mathcal{Q}$ *given by* (3.3)*.*

Also, we prove this lemma in the next section. From these lemmata, we infer a decay estimate for the oscillation of a bounded solution of (1.1). It is interesting to note that, unlike the usual proofs of Hölder continuity, we do not estimate the oscillation of a bounded over a cylinder in terms of its oscillation over a larger cylinder. Instead, we estimate the oscillation over the smaller cylinder in terms of a quantity larger than the oscillation over the larger cylinder.

Lemma 3.3. *Let* C_0*,* C_1*,* g_0*,* g_1*,* ρ*, and* ω *be positive constants with* $C_0 \leq C_1$ *and* $2 \leq g_0 \leq g_1$*. Suppose also that u is a bounded weak solution of* (1.1) *in* $Q_{\omega,\rho}$ *with*

$$\operatorname{ess\,osc}_{Q_{\omega,\rho}} u \leq \omega.$$

Then there are positive constants σ *and* λ*, both less than one and determined only by data such that*

$$\operatorname{ess\,osc}_{Q_{\sigma\omega,\lambda\rho}} u \leq \sigma\omega.$$

Proof. We begin by introducing some constants. First, we take ν_0 from Proposition 4.4 (corresponding to $\theta = 1$) and then δ_2 and θ_0 from Lemma 3.2. With this θ_0, we also take δ_1 from Lemma 3.1. Next, we set

$$\theta_1 = 2\big(1 + \big(\frac{\theta_0}{4}\big)^{1/g_0}\big)$$

and $R = \rho/\theta_1$. Then, we define Δ by (3.1), and we set $\lambda = 1/(2\theta_1)$ and $\sigma = 1 - \min\{\delta_1, \delta_2\}$. Moreover, we define

$$u_1 = u - \operatorname{ess\,inf}_{Q_{\omega,\rho}} u, \quad u_2 = \omega - u_1.$$

To begin the proof, we observe that $\theta_1 \geq 2$ and $\theta_1 \geq 2(\theta_0/4)^{1/g_1}$ and hence Q, defined by (3.2), is a subset of $Q_{\omega,\rho}$.

If there is a $T_0 \in (-\theta_0\Delta, -\Delta)$ such that

$$\Big|K_\rho \times (T_0, T_0+\Delta) \cap \{u_1 \le \frac{\omega}{2}\}\Big| \le \nu_0 |K_\rho|\Delta,$$

then Lemma 3.1 applied to u_1 implies that

$$\operatorname{ess\,inf}_{\mathcal{Q}} u_1 \ge \delta_1\omega$$

with $\mathcal{Q}$ defined by (3.3). and hence

$$\operatorname{ess\,osc}_{\mathcal{Q}} u \le \sigma\omega. \tag{3.4}$$

On the other hand, if

$$\Big|K_\rho \times (T_0, T_0+\Delta) \cap \{u_1 \le \frac{\omega}{2}\}\Big| \ge \nu_0 |K_\rho|\Delta$$

for all $T_0 \in (-\theta_0\Delta, -\Delta)$, it follows that

$$\Big|K_\rho \times (T_0, T_0+\Delta) \cap \{u_2 \le \frac{\omega}{2}\}\Big| \le (1-\nu_0) |K_\rho|\Delta,$$

so Lemma 3.2 applied to u_2 gives $\operatorname{ess\,inf}_{\mathcal{Q}} u_2 \ge \delta_2\omega$, which implies (3.4) in this case.

The proof will be complete once we show that $Q_{\sigma\omega,\lambda\rho} \subset \mathcal{Q}$. To prove this inclusion, we first note that

$$\lambda\rho \le \frac{R}{2}.$$

Then we use the definition of λ to infer from Lemma 2.1 that

$$G\big(\frac{\omega}{2R}\big) = G\big(\frac{\omega}{4\lambda\rho}\big) \le (4\sigma)^{-g_0} G\big(\frac{\sigma\omega}{\lambda\rho}\big).$$

It follows that

$$(\sigma\omega)^2 G\left(\frac{\sigma\omega}{\lambda\rho}\right)^{-1} \le (4\sigma)^{2-g_0}\left(\frac{\omega}{4}\right)^2 G\left(\frac{\omega}{2R}\right)^{-1}$$

Since $\sigma \ge 1/2$, it follows that $(4\sigma)^{2-g_0} \le 1$ and $Q_{\sigma\omega,\lambda\rho} \subset \mathcal{Q}$. $\square$

As we shall see in [13], this lemma is also valid for singular equations although the proof in that case is quite different.

For our Hölder continuity estimates, we define a time scale in terms of the function G, the function u and the set Ω on which u is defined. We shall now include u and Ω in the notation for simplicity. Specifically, for any real number τ, we define

$$|\tau|_G = \frac{U}{G^{-1}(U^2/|\tau|)},$$

where $U = \operatorname{ess\,osc}_\Omega u$.

With this time scale, we define the parabolic distance between two sets such $\mathcal{K}_1$ and $\mathcal{K}_2$ by

$$\operatorname{dist}_P(\mathcal{K}_1;\mathcal{K}_2) := \inf_{\substack{(x,t)\in\mathcal{K}_1\\ (y,s)\in\mathcal{K}_2,\, s\le t}} \max\{|x-y|_\infty, |t-s|_G\}$$

with $|\cdot|_\infty$ as defined in Section 2. (Note that, strictly speaking, this quantity is not a distance because it is not symmetric with respect to the order in which we write the sets. Nonetheless, the terminology of distance is useful as a suggestion of the technically correct situation.)

Because of the generalized function G, it is natural to obtain a modulus of continuity in terms of G. We are also able to derive a Hölder estimate written in terms of exact powers.

Theorem 3.4. *Let u be a bounded weak solution of* (1.1) *with* (1.3) *in* Ω*, and suppose $g_0 \geq 2$. Then u is locally continuous. Moreover, there exist positive constants $\alpha < 1$ and γ, depending only upon the data, such that, for any two distinct points (x_1, t_1) and (x_2, t_2) in any subset Ω' of Ω with* $\operatorname{dist}_P(\Omega'; \partial_p\Omega)$ *positive, we have*

$$|u(x_1,t_1) - u(x_2,t_2)| \leq \gamma U \Big(\frac{|x_1 - x_2|_\infty + |t_1 - t_2|_G}{\operatorname{dist}_P(\Omega'; \partial_P\Omega)}\Big)^\alpha. \tag{3.5}$$

In addition (with the same constants),

$$|u(x_1,t_1) - u(x_2,t_2)| \leq \gamma U \Big(\frac{|x_1 - x_2|_\infty + |1|_G \max\{|t_1 - t_2|^{1/g_0}, |t_1 - t_2|^{1/g_1}\}}{\operatorname{dist}_P(\Omega'; \partial_P\Omega)}\Big)^\alpha. \tag{3.6}$$

Proof. If $U = 0$, then this result is true for any choice of γ and α, so we assume that $U > 0$ and set $\omega_0 = U$. We also set

$$\rho_0 = \operatorname{dist}_P(\{(x_0, t_0)\}, \partial_P\Omega).$$

We then define $\varepsilon = \min\{\lambda, \frac{1}{2}\sigma^{(2-g_0)/g_0}\}$ (where λ and σ are the constants from Lemma 3.3),

$$\rho_n = \varepsilon^n \rho_0, \quad \omega_n = \sigma^n \omega_0,$$

and define a sequence of cylinders (Q_n) by

$$Q_n = Q^{x_1,t_1}_{\sigma_n,\rho_n}.$$

It is easy to check that $Q_0 \subset \Omega$ and that $Q_{n+1} \subset Q_n$ for any n. Combining Lemma 3.3 with an easy induction, we find that $\operatorname{ess\,osc}_{Q_n} \leq \omega_n$ for any n. If $(x_2, t_2) \in Q_0$ with $x_1 \neq x_2$ and $t_1 \neq t_2$, then there are nonnegative integer n and m such that

$$\rho_{n+1} < |x_1 - x_2| \leq \rho_n, \tag{3.7a}$$

$$\omega_{m+1}^2 G\left(\frac{\omega_{m+1}}{\rho_{m+1}}\right)^{-1} < |t_1 - t_2| \leq \omega_m^2 G\left(\frac{\omega_m}{\rho_m}\right)^{-1}. \tag{3.7b}$$

As a result, we obtain that

$$|u(x_1,t_1) - u(x_2,t_2)| \leq \max\{\omega_n, \omega_m\}.$$

From the first inequality of (3.7a), we derive

$$\frac{|x_1 - x_2|}{\rho_0} > \varepsilon^{n+1} = \big(\sigma^{\log_\sigma \varepsilon}\big)^{n+1}$$

which implies

$$\omega_n = \sigma^n \omega_0 < \sigma^{-1}\omega_0 \Big(\frac{|x_1 - x_2|}{\rho_0}\Big)^{\alpha_1}$$

for $\alpha_1 = \log_\varepsilon \sigma$.

On the other hand, the first inequality of (3.7b) implies that

$$|t_1 - t_2|_G \geq \frac{U}{G^{-1}\big(U^2 \omega_{m+1}^{-2} G(\frac{\omega_{m+1}}{\rho_{m+1}})\big)}.$$

We now estimate the expression in the denominator of this fraction:

$$\begin{aligned} U^2 \omega_{m+1}^{-2} G\big(\frac{\omega_{m+1}}{\rho_{m+1}}\big) &= \sigma^{-2(m+1)} G\big(\frac{\omega_{m+1}}{\rho_{m+1}}\big) \\ &\geq G\big(\sigma^{-2(m+1)/g_0} (\frac{\sigma}{\varepsilon})^{m+1} \frac{\omega_0}{\rho_0}\big). \end{aligned}$$

To proceed, we now define $\beta = \varepsilon\sigma^{(2-g_0)/g_0}$ and note that the choice of ε implies that $\beta < 1$. We then have

$$|t_1 - t_2|_G \geq \frac{U}{\beta^{-(m+1)\frac{\omega_0}{\rho_0}}} = \beta^{m+1}\rho_0.$$

Hence by letting $\alpha_2 = \log_\beta \sigma$, we have

$$\omega_m \leq \beta^{\alpha_2 m}\omega_0 \leq \Big(\frac{|t_1 - t_2|_G}{\beta\rho_0}\Big)^{\alpha_2}\omega_0.$$

Therefore, for some $\gamma > 0$,

$$|u(x_1, t_1) - u(x_2, t_2)| \leq \gamma U\Big[\Big(\frac{|x_1 - x_2|}{\rho_0}\Big)^{\alpha_1} + \Big(\frac{|t_1 - t_2|_G}{\rho_0}\Big)^{\alpha_2}\Big].$$

This inequality implies (3.5) with $\alpha = \min\{\alpha_1, \alpha_2\}$ because $\rho_0 \geq \operatorname{dist}_P(\Omega'; \partial_P\Omega)$. If $x_1 = x_2$ or if $t_1 = t_2$, then a similar (but simpler) argument yields the result.

If $(x_2, t_2) \notin Q_0$, then $|x_1 - x_2| + |t_1 - t_2|_G \geq \rho_0$, so (3.5) follows, for any α, by taking $\gamma \geq 1$.

To prove (3.6), we consider two cases. First, if $|t_1 - t_2| \leq 1$, then

$$G\Big(\frac{G^{-1}(U^2)}{|t_1 - t_2|^{1/g_0}}\Big) \geq \frac{1}{|t_1 - t_2|}U^2,$$

so

$$|t_1 - t_2|_G \leq \frac{U}{G^{-1}(U^2)}|t_1 - t_2|^{1/g_0}.$$

Second, if $|t_1 - t_2| > 1$, then

$$G\Big(\frac{G^{-1}(U^2)}{|t_1 - t_2|^{1/g_1}}\Big) \geq \frac{1}{|t_1 - t_2|}U^2,$$

so

$$|t_1 - t_2|_G \leq \frac{U}{G^{-1}(U^2)}|t_1 - t_2|^{1/g_1}.$$

Combining these inequalities with (3.5) and the observation that $U/G^{-1}(U^2) = |1|_G$ then gives (3.6). $\square$

For initial regularity, we have the following variant of Lemma 3.3. To simplify notation, we define the following cylinders:

$$Q^{+,x_0t_0}_{k,R}(\theta) = K^{x_0}_R \times \Big(t_0, t_0 + \theta k^2 G\big(\frac{k}{R}\big)^{-1}\Big), \quad Q^+_{k,R}(\theta) = Q^{+,0,0}_{k,R}(\theta),$$

and we set $Q^+_{k,R} = Q^+_{k,R}(1)$. With ν_0 the constant from Proposition 4.5 and U a given constant, we also define $Q_R(U)$ to be the cylinder $Q^+_{U,R}(\nu_0/9)$.

Our shrinking lemma, analogous to Lemma 3.3, takes the following form. Note that the result of this lemma is essentially the same as [7, (6.10)], but the proof is much simpler.

Lemma 3.5. *Let C_0, C_1, g_0, g_1, ρ, U, and ω be positive constants with $C_0 \leq C_1$, $2 \leq g_0 \leq g_1$, and $\omega \leq U$. Suppose also that u is a bounded weak solution of* (1.1) *in $Q_{2R}(U)$ with $\operatorname{ess\,osc}_{Q_{2R}(U)} u \leq \omega$. Then there is a constant $\lambda \in (0, 1)$, determined only by data, such that*

$$\operatorname{ess\,osc}_{Q_R(U)} u \leq \max\Big\{\frac{5}{6}\omega, 3\operatorname{ess\,osc}_{K_{2R}\times\{0\}} u\Big\}.$$

Proof. We begin by setting

$$\omega^* = \operatorname{ess\,osc}_{K_{2R}\times\{0\}} u.$$

If $\omega < 3\omega^*$, the result is immediate, so we suppose that $\omega \ge 3\omega*$. We also set

$$\theta = \frac{(\nu_0/9)U^2 G\left(\frac{U}{2R}\right)^{-1}}{\left(\frac{\omega}{3}\right)^2 G\left(\frac{\omega/3}{2R}\right)^{-1}}$$

and note that $Q_R(U) = Q^+_{\omega/3,2R}(\theta)$. Since $g_0 \ge 2$, it follows that

$$U^2 G\Big(\frac{U}{2R}\Big)^{-1} \le \omega^2 G\left(\frac{\omega}{2R}\right)^{-1}.$$

Moreover G is increasing so $G(\omega/(2R))^{-1} \le G((\omega/3)/(2R))$, and therefore $\theta \le \nu_0$.

We now set $k = \omega/3$ and we consider two cases. First, if

$$\operatorname{ess\,inf}_{K_{2R}\times\{0\}} u_1 \ge k, \tag{3.8}$$

we apply Proposition 4.5 to u_1 in $Q^+_{k,2R}(\theta)$ to infer that

$$\operatorname{ess\,inf}_{Q_R(U)} u_1 \ge \frac{k}{2}.$$

It follows that

$$\operatorname{ess\,osc}_{Q_R(U)} u \le \omega - \frac{k}{2} = \frac{5}{6}\omega. \tag{3.9}$$

If (3.8) does not hold, then some straightforward algebra shows that

$$\operatorname{ess\,inf}_{K_{2R}\times\{0\}} u_2 \ge k,$$

so we can apply Proposition 4.5 to u_2, again obtaining (3.9). □

From this lemma, we infer a continuity estimate near the initial surface. When Ω has the form $\mathcal{O}\times(t_1,t_2)$ for some bounded open subset $\mathcal{O}$ of $\mathbb{R}^N$ and numbers $t_1 < t_2$, the initial surface is just $\mathcal{O}\times\{t_1\}$, but we wish to provide a result that applies also to non-cylindrical domains Ω, so we introduce some additional terminology and notation.

First,we recall from [19] that $B\Omega$ is the set of all $(x_0,t_0) \in \partial_P\Omega$ such that, for some positive numbers r and s, the cylinder

$$K_r^{x_0} \times (t_0, t_0+s)$$

is a subset of Ω. Here we consider the following subset of $B\Omega$, which we call an *initial surface* for Ω and which we denote by $B'\Omega$. Specifically, $B'\Omega$ is the set of all $(x_0,t_0) \in B\Omega$ such that $K_r\times\{t_0\} \subset \partial_P\Omega$ for some $r' > 0$. When $\Omega = \mathcal{O}\times(t_1,t_2)$, we have that $B\Omega = B'\Omega = \mathcal{O}\times\{t_1\}$, but the sets $B\Omega$ and $B'\Omega$ are usually different for non-cylindrical domains. For example, if $\Omega = \{(x,t) \in \mathbb{R}^{N+1} : |x|^4 - |x|^2 < t < 1\}$, then $B\Omega = \{(0,0)\}$ but $B'\Omega$ is empty.

We also define a slightly different distance function. Let $(x_0,t_0) \in B'\Omega$ and let $\omega > 0$. Then we write $\operatorname{dist}_B(x_0,t_0)$ for the supremum of the set of all numbers r such that $Q^{+,x_0,t_0}_{\omega,r} \subset \Omega$ and $K_r^{x_0,t_0} \times \{0\} \subset \partial_P\Omega$.

Theorem 3.6. *Let u be a bounded weak solution of* (1.1) *with* (1.3) *in Ω, and suppose $1 < g_0 \le g_1$. Suppose also that the restriction of u to $B'\Omega$ is continuous at* *some $(x_0,t_0)\in B'\Omega$. Then u is locally continuous up to (x_0,t_0). Specifically, if there is a continuous increasing function $\tilde\omega$ defined on $[0,\operatorname{dist}_B(x_0,t_0))$ with $\tilde\omega(0)=0$,*

$$\frac{5}{6}\tilde\omega(2r)\le\tilde\omega(r) \tag{3.10}$$

for all $r\in(0,\operatorname{dist}_B(x_0,t_0)/2)$, and with

$$|u(x_0,t_0)-u(x_1,t_0)|\le\tilde\omega(|x_0-x_1|)$$

for all x_1 with $|x_0-x_1|<\operatorname{dist}_B(x,t_0)$, then there exist constants γ and $\alpha\in(0,1)$ depending only upon the data such that, for any $(x,t)\in\Omega$ with $t\ge t_0$, we have

$$\begin{aligned}&|u(x_0,t_0)-u(x,t)|\\ &\le\gamma U\Big(\frac{|x_0-x|+|t_0-t|_G}{\operatorname{dist}_B(x_0,t_0)}\Big)^\alpha+3\tilde\omega\Big(2|x_0-x|_\infty+\frac{18}{\nu_0}|t_0-t|_G\Big).\end{aligned}$$

Proof. We start by taking $\omega_0=U$ and $\rho_0=\operatorname{dist}_B(x_0,t_0)$. If $(x,t)\notin Q^{+,x_0,t_0}_{\omega_0,\rho_0}$, then the result is immediate for any α as long as $\gamma\ge1$.

If $(x,t)\in Q^{+,x_0,t_0}_{\omega_0,\rho_0}$, then we define $\rho_n=\lambda^n\rho_0$ and $Q_n=Q_{\rho_n}(U)$. We also define ω_n' for $n>0$ inductively as $\omega_{n+1}'=\max\{\frac56\omega_n',3\omega^*(\rho_n)\}$. It follows from Lemma 3.5 that $\operatorname{ess\,osc}_{Q_n}u\le\omega_n'$, but this estimate must be improved. To this end, we set

$$\omega_n=\max\big\{\big(\frac56\big)^n\omega_0,3\tilde\omega(\rho_{n-1})\big\},$$

and we claim that $\omega_n'\le\omega_n$ for $n>0$. The claim is immediate for $n=1$, and if it holds for n equal to some positive integer m, then

$$\begin{aligned}\omega_{m+1}'&=\max\big\{\big(\frac56\big)\omega_m',3\tilde\omega(\rho_m)\big\}\\ &\le\max\big\{\big(\frac56\big)^{m+1}\omega_0,3\big(\frac56\big)\tilde\omega(\rho_{m-1}),3\tilde\omega(\rho_m)\big\}\end{aligned}$$

and the claim follows for $n=m+1$ from this inequality by using (3.10) with $r=\rho_m$. Hence the claim is true for all n and we infer that

$$\operatorname{ess\,osc}_{Q_n}u\le\omega_n.$$

As before, we assume that $x\ne x_0$ and $t\ne t_0$, so there are nonnegative integers n and m such that

$$\rho_{n+1}\le|x_0-x|_\infty<\rho_n,$$

and

$$U^2G\Big(\frac{U}{\rho_{m+1}}\Big)^{-1}\le|t_0-t|<U^2G\Big(\frac{U}{\rho_m}\Big)^{-1}.$$

With $\alpha=\log_{1/2}(5/6)$, it follows that

$$\big(\frac56\big)^n\le\Big(\frac{2|x_0-x|_\infty}{\rho_0}\Big)^\alpha,\quad\tilde\omega(\rho_n)\le\tilde\omega(2|x_0-x|_\infty),$$

and that

$$\big(\frac56\big)^m\le\Big(\frac{2|(9/\nu_0)(t_0-t)|_G}{\rho_0}\Big)^\alpha,\quad\tilde\omega(\rho_m)\le\tilde\omega\big(\frac{18}{\nu_0}|t_0-t|_\infty\big).$$

We now observe that, for any $\beta>1$ and $\tau>0$, if we set $\sigma=G^{-1}(U^2/\tau)$, we have

$$\frac{U^2}{\beta\tau}=\beta^{-1}G(\sigma)\ge G(\beta^{-1/g_0}\sigma).$$

Since

$$\frac{U^2}{\beta\tau} = G^{-1}\Big(\frac{U^2}{\beta\tau}\Big),$$

it follows that

$$\frac{U}{G^{-1}(\frac{U^2}{\beta\tau})} \le \frac{U}{G^{-1}(\frac{U^2}{\tau})}.$$

In particular, for $\tau = |t_0 - t|$ and $\beta = 9/\nu_0$, we infer that

$$\Big|\frac{9}{\nu_0}(t_0 - t)\Big|_G \le \big(\frac{9}{\nu_0}\big)^{1/g_0}|t - t_0|_G.$$

Easy algebra now completes the proof. □

At this point, we remark that condition (3.10), which seems quite unnatural, is just a technical restriction on the description of the modulus of continuity of the restriction of u to $B'\Omega$. There is no loss of generality in assuming that there is a concave, increasing function ω' such that

$$|u(x_0, t_0) - u(x_1, t_0)| \le \omega'(|x_0 - x|)$$

for all x_1 with $|x_0 - x_1| < \operatorname{dist}_B(x_0, t_0)$ (see [21, Section 5] for a detailed explanation of this statement), and then we can take

$$\tilde\omega(r) = \omega'\left((\operatorname{dist}_B(x_0, t_0))^{1-\alpha} r^\alpha\right)$$

(with $\alpha = \log_{1/2}(5/6)$) to obtain a function satisfying the hypotheses of the theorem.

We shall not discuss regularity near the lateral boundary in any detail. We just note that the proof of interior regularity can be modified along the lines of [7, Theorems 3 and 4] to give such results.

4. Proof of the two alternatives

Throughout this section, let u be a bounded nonnegative weak supersolution of (1.1) in a suitably scaled cylinder with (1.3). The proof of the two alternatives rests on some estimates which show that, if u is bounded away from zero on some set (with a suitable lower bound), then it is bounded away from zero (with a different but related lower bound) on a different set. Then Proposition 4.1 implies that a spatial cube at some fixed time level can be found on which u is away from its minimum (zero value) on arbitrary fraction of the spatial cube. From the spatial cube, positive information spreads to later time and over the space variables with time limitations (Propositions 4.2 and 4.3). Controlling the positive quantity $\theta > 0$ on $T_{k,\rho}$ is key to overcoming those time restrictions. Once we have a subcylinder centered at $(0, 0)$ in $Q_{\omega,4R}$ with arbitrary fraction of the subcylinder, we finally apply modified De Giorgi iteration (Proposition 4.4) to obtain strictly positive infimum of u in a smaller cylinder with top-center point $(0, 0)$.

4.1. **Basic results.** Our first proposition shows that if a nonnegative function is large on part of a cylinder, then it is large on part of a suitable time slice. Except for some minor variation in notation, our result is [7, Lemma 4.1] or [8, Lemma III.7.1]; we include a proof for completeness.

Proposition 4.1. *Let k, ρ, and T be positive constants. If u is a measurable nonnegative function defined on $Q = K_\rho \times (-T, 0)$ and if there is a constant $\nu_1 \in [0, 1)$ such that*
$$|Q \cap \{u \le k\}| \le (1 - \nu_1)|Q|,$$
then there is a number
$$\tau_1 \in \big(-T, -\frac{\nu_1}{2-\nu_1}T\big)$$
for which
$$|\{x \in K_\rho : u(x, \tau_1) \le k\}| \le \big(1 - \frac{\nu_1}{2}\big)|K_\rho|.$$

Proof. To simplify the notation, we set $\tau = \frac{\nu_1}{2-\nu_1}T$. If there were no such τ_1, then we would have
$$\begin{aligned}
|Q \cap \{u \le k\}| &= \int_{-T}^{0} |\{x \in K_\rho : u(x,t) \le k\}|\,dt \\
&\ge \int_{-T}^{-\tau} |\{x \in K_\rho : u(x,t) \le k\}|\,dt \\
&> \big(1 - \frac{\nu_1}{2}\big)\big(1 - \frac{\nu_1}{2-\nu_1}\big)|K_\rho|T \\
&= (1 - \nu_1)|Q|.
\end{aligned}$$
□

Our next proposition is similar to [7, Lemma 4.2] and [8, Lemmata III.4.1, III.7.2, IV.10.2]. If $g_0 > 2$, then the next proposition can be replaced by [11, Corollary 3.4] which does not involve the logarithmic energy estimate.

Proposition 4.2. *Let ν, k, ρ, and θ be given positive constants with $\nu \le 1$. Then, for any $\epsilon \in (0,1)$, there exists a constant $\delta = \delta(\nu, \epsilon, \theta, data)$ such that, if u is a nonnegative supersolution of (1.1) in $K_\rho \times (-\tau, 0)$ with $g_0 \ge 2$ and*
$$|\{x \in K_\rho : u(x, -\tau) \le k\}| \le (1 - \nu)\,|K_\rho| \tag{4.1}$$
for some
$$\tau \le \theta k^2 G\big(\frac{k}{\rho}\big)^{-1}, \tag{4.2}$$
then
$$|\{x \in K_\rho : u(x,t) \le \delta k\}| \le (1 - (1-\epsilon)\nu)\,|K_\rho|$$
for any $t \in (-\tau, 0]$.

Proof. Here we apply the logarithmic energy estimate (5.11) in a parabolic cylinder $K_\rho \times [-\tau, -s]$ for any $-s \in [-\tau, 0)$. For some $\sigma \in (0,1)$ to be determined later, we introduce a piecewise linear cutoff function independent of the time variable; that is,
$$\zeta = \begin{cases} 1 & \text{inside } K_{(1-\sigma)\rho} \times [-\tau, -s] \\ 0 & \text{on the lateral boundary of } \ K_\rho \times [-\tau, -s]. \end{cases}$$
satisfying
$$|D\zeta| \le \frac{1}{\sigma\rho}, \quad \zeta_t = 0.$$

From (5.11) by letting $q = g_1$, it follows that

$$\begin{aligned}
&\int_{K_\rho\times\{-s\}} H(\Psi^2)\zeta^{g_1}\,dx \\
&\le \int_{K_\rho\times\{-\tau\}} H(\Psi^2)\zeta^{g_1}\,dx \\
&\quad + 2C_0^{1-g_1}C_1^{g_1}g_1^{2g_1}\int_{-\tau}^{-s}\int_{K_\rho} h(\Psi^2)|\Psi||\Psi'|^2 G\big(\frac{|D\zeta|}{\Psi'}\big)\,dx\,dt,
\end{aligned}\tag{4.3}$$

where h and H are defined in Lemma 2.2. Let $\delta = 2^{-j}$ where j is to be chosen large enough. We recall

$$\Psi = \ln^+\big[\frac{k}{(1+\delta)k-(u-k)_-}\big], \quad \Psi' = \frac{1}{(u-k)_- - (1+\delta)k},$$

and that $\Psi = \Psi' = 0$ when $u \ge (1-\delta)k$. Since $0 \le (u-k)_- \le k$, we also have

$$\Psi \le \ln^+\delta^{-1} = j\ln 2, \quad \frac{1}{(1+\delta)k} \le |\Psi'| \le \frac{1}{\delta k}.$$

The first integral term on the right hand side of (4.3) is bounded by

$$\begin{aligned}
\int_{K_\rho\times\{-\tau\}} H(\Psi^2)\zeta^{g_1}\,dx &\le H\left(j^2(\ln 2)^2\right)|\{x\in K_\rho : u(x,-\tau)\le(1-\delta)k\}| \\
&\le (1-\nu)H\left(j^2(\ln 2)^2\right)|K_\rho|
\end{aligned}$$

because of the assumption (4.1).

Now to handle the second integral on the right hand side of (4.3), we make observations of upper bounds of the quantity

$$|\Psi'|^2 G\Big(\frac{|D\zeta|}{\Psi'}\Big).$$

We use the inequalities $1 \le (1+\delta)k|\Psi'|$ and $\delta < 1$. Hence we derive from (4.2) that

$$\begin{aligned}
|\Psi'|^2 G\Big(\frac{|D\zeta|}{\Psi'}\Big) &\le \left((1+\delta)k|\Psi'|\right)^{2-g_0}\left((1+\delta)k\right)^{-2}G\left((1+\delta)k|D\zeta|\right) \\
&\le 2^{g_1}\sigma^{-g_1}k^{-2}G\big(\frac{k}{\rho}\big)
\end{aligned}$$

because $2-g_0 \le 0$ and $1 < 1+\delta < 2$. Therefore, for any $-s \in (-\tau, 0]$ (which implies that $|\tau - s| \le \tau$), we have

$$\begin{aligned}
&\int_{-\tau}^{-s}\int_{K_\rho} h(\Psi^2)|\Psi||\Psi'|^2 G\big(\frac{|D\zeta|}{|\Psi'|}\big)\,dx\,dt \\
&\le 2^{g_1}\theta h\left(j^2(\ln 2)^2\right)(j\ln 2)\sigma^{-g_1}|K_\rho| \\
&\le 2^{g_1}g_1\theta\frac{H(j^2(\ln 2)^2)}{j\ln 2}\sigma^{-g_1}|K_\rho|.
\end{aligned}\tag{4.4}$$

To obtain the lower bound of the left hand side of (4.3), we integrate over the smaller set $\{u \le \delta k\} = \{u \le 2^{-j}k\}$. Note that

$$\Psi \ge \ln^+(2\delta)^{-1} = (j-1)\ln 2$$

on this set. Therefore the left hand side of the inequality (4.3) is lower bounded by

$$\int_{K_\rho\times\{-s\}} H(\Psi^2)\zeta^{g_1}\,dx \ge H\left((j-1)^2(\ln 2)^2\right)\left|\left\{x\in K_{(1-\sigma)\rho} : u(x,-s)\le \delta k\right\}\right|.$$

It follows that

$$\begin{aligned}
&|\{x\in K_\rho : u(x,-s)\le \delta k\}| \\
&\le \left|\left\{x\in K_{(1-\sigma)\rho} : u(x,t)\le \delta k\right\}\right| + \left|K_\rho\setminus K_{(1-\sigma)\rho}\right| \\
&\le \Big[(1-\nu)\frac{H\left(j^2(\ln 2)^2\right)}{H\left((j-1)^2(\ln 2)^2\right)} + \frac{C\theta H\left(j^2(\ln 2)^2\right)}{j\sigma^{g_1}H\left((j-1)^2(\ln 2)^2\right)} + N\sigma\Big]|K_\rho|
\end{aligned}$$

upon combining upper bounds of (4.3), where C depends on C_0, C_1, and g_1. For brevity, set

$$H_0 = \frac{H\left(j^2(\ln 2)^2\right)}{H\left((j-1)^2(\ln 2)^2\right)}.$$

For any given $\epsilon\in(0,1)$, we choose an integer j large enough and $\sigma\in(0,1)$ small enough so that the following three inequalities hold:

$$H_0 \le 1+\epsilon\nu, \tag{4.5a}$$

$$\frac{C\theta H_0}{j\sigma^{g_1}} \le \frac{\epsilon\nu^2}{2}, \tag{4.5b}$$

$$N\sigma \le \frac{\epsilon\nu^2}{2}. \tag{4.5c}$$

Then inequalities (4.5) yield our conclusion.

Now we complete the proof by going back to (4.5) and finding j and σ. From (4.5c), first fix

$$\sigma = \frac{\epsilon\nu^2}{2N}.$$

Then assuming (4.5a), the inequality (4.5b) holds if

$$j \ge \frac{C\theta(1+\epsilon\nu)}{2\sigma^{g_1}\epsilon\nu^2},$$

which gives

$$j \ge \frac{C(1+\epsilon\nu)(4N)^{g_1}}{2\epsilon^{1+g_1}\nu^{2(1+g_1)}}.$$

It is sufficient to choose

$$j \ge C(C_0, C_1, g_1, N)\left(\epsilon\nu^2\right)^{-1-g_1}.$$

Finally, (4.5a) is satisfied if j is so large that

$$\Big(\frac{j}{j-1}\Big)^{g_1} \le 1+\epsilon\nu,$$

which is equivalent to

$$j > \frac{(1+\epsilon\nu)^{1/g_1}}{(1+\epsilon\nu)^{1/g_1}-1}.$$

So the proof is completed by taking j to be any integer greater than

$$\max\Big\{C(C_0, C_1, g_1, N)\theta(\epsilon\nu^2)^{-1-g_1}, \frac{(1+\epsilon\nu)^{1/g_1}}{(1+\epsilon\nu)^{1/g_1}-1}\Big\}.$$

$\square$

The following proposition demonstrates the spreading of positivity over space. When we have some portion of positive data along all the time, then a mixture of Poincaré's inequality and a local energy estimate generates arbitrary fractional control over that cylinder. Especially when $g_1 > 2$, somewhat large length of the time interval for the initially given positive data collected place is required to spread positivity properly. Proposition 4.3 is analogous to [7, Lemma 4.4], [11, Lemma 3.5], and [9, Proposition 6.1].

Proposition 4.3. *Suppose $g_0 \geq 2$. Let k and ρ be positive numbers and suppose u is a nonnegative supersolution of* (1.1) *in $K_{2\rho} \times (-2\tau, 0)$ for some $\tau > 0$. Then for any ν and α in $(0,1)$ and any $\theta > 0$, there exists a constant $\delta^* = \delta^*(\alpha, \nu, \min\{1,\theta\}, data) \in (0,1)$ such that, if*

$$\tau \geq \theta(\delta^* k)^2 G\Big(\frac{\delta^* k}{\rho}\Big)^{-1} \tag{4.6}$$

and if

$$|\{x \in K_{2\rho} : u(x,t) \leq k\}| \leq (1-\alpha)|K_{2\rho}| \tag{4.7}$$

for all $t \in (-2\tau, 0]$, then we have

$$|\{(x,t) \in K_\rho \times [-\tau, 0] : u(x,t) \leq \delta^* k\}| \leq \nu\, |K_\rho \times [-\tau, 0]|\,. \tag{4.8}$$

Proof. Let $k_j = 2^{-j}k$ for $j = 0, 1, 2, \dots, j^*$ with j^* to be determined later. Denote $\delta^* = 2^{-j^*}$. For simplicity, denote

$$A_j = \{(x,t) \in K_\rho \times [-\tau, 0] : u(x,t) \leq k_j\}\,.$$

We work with a piecewise linear cutoff function

$$\zeta = \begin{cases} 1 & \text{inside of } \; K_\rho \times [-\tau, 0] \\ 0 & \text{on the parabolic boundary of } \; K_{2\rho} \times [-2\tau, 0] \end{cases}$$

with

$$|D\zeta| \leq \frac{1}{\rho}, \quad \zeta_t \leq \frac{1}{\tau}.$$

The local energy estimate (5.2) (by ignoring the first term on the left hand side) provides

$$\begin{aligned} &\int_{-2\tau}^0 \int_{K_{2\rho}} G(|D(u-k_j)_-|) G^{r-1}\Big(\frac{\zeta(u-k_j)_-}{\rho}\Big)(u-k_j)_-^s \zeta^q \,dx\,dt \\ &\leq \gamma_1 \int_{-2\tau}^0 \int_{K_{2\rho}} G^{r-1}\Big(\frac{\zeta(u-k_j)_-}{\rho}\Big)(u-k_j)_-^{s+2} \zeta^{q-1}\zeta_t \,dx\,dt \\ &\quad + \gamma_2 \int_{-2\tau}^0 \int_{K_{2\rho}} G^{r}\Big(\frac{\zeta(u-k_j)_-}{\rho}\Big)(u-k_j)_-^{s} \zeta^{q-1-2g_1} \,dx\,dt. \end{aligned} \tag{4.9}$$

Here note that for $j = 0, \dots, j^*$

$$k_j^2 \zeta_t \leq \frac{1}{\theta} G\big(\frac{k_j}{\rho}\big)$$

because (4.6) implies that, for any $j = 0, \dots, j^*$,

$$\tau \geq k_j^2 G\big(\frac{k_j}{\rho}\big)^{-1}.$$

The integral estimate (4.9) simplifies to

$$\int_{-\tau}^{0}\int_{K_\rho} G\left(|D(u-k_j)_-|\right)\,dx\,dt \le \gamma G\big(\frac{k_j}{\rho}\big)\,|K_{2\rho}\times[-2\tau,0]|\,. \tag{4.10}$$

Owing to the assumption (4.7), we may apply the Poincaré type inequality, Lemma 5.3. For any $t\in[-\tau,0]$, it follows that

$$\begin{aligned}&(k_j-k_{j+1})\,|\{x\in K_\rho: u(x,t)<k_{j+1}\}|\\&\le \frac{\gamma\rho^{N+1}}{\theta\alpha\rho^N}\int_{K_\rho\cap\{k_{j+1}\le u<k_j\}}|D(u-k_j)_-|\,dx.\end{aligned}$$

Note $k_j-k_{j+1}=k_{j+1}$. After integrating over the time variable from $-\tau$ to 0, we obtain

$$\frac{k_{j+1}}{\rho}\,|A_{j+1}|\le\frac{\gamma}{\theta\alpha}\iint_{A_j\setminus A_{j+1}}|D(u-k_j)_-|\,dx\,dt. \tag{4.11}$$

After dividing (4.11) by $|A_j\setminus A_{j+1}|$ and assuming (without loss of generality) that the constant γ in this inequality is at least 1, we apply Jensen's inequality and Lemma 2.1(b) to infer that

$$G\Big(\frac{|A_{j+1}|}{|A_j\setminus A_{j+1}|}\frac{k_{j+1}}{\rho}\Big)\le\frac{\gamma^*}{|A_j\setminus A_{j+1}|}\iint_{A_j\setminus A_{j+1}}G\left(|D(u-k_j)_-|\right)\,dx\,dt \tag{4.12}$$

with

$$\gamma^*=\Big(\frac{\gamma}{\min\{1,\theta\}\alpha}\Big)^{g_1}.$$

Because of (4.10), the inequality (4.12) generates

$$G\Big(\frac{|A_{j+1}|}{|A_j\setminus A_{j+1}|}\frac{k_{j+1}}{\rho}\Big)\le\gamma 2^{N+1}\gamma^*\frac{|K_\rho\times[-\tau,0]|}{|A_j\setminus A_{j+1}|}G\big(\frac{k_j}{\rho}\big). \tag{4.13}$$

Denote $\Omega_\tau:=K_\rho\times[-\tau,0]$. There are two cases to consider for any j: either

$$|A_{j+1}|>|A_j\setminus A_{j+1}|,$$

or

$$|A_{j+1}|\le|A_j\setminus A_{j+1}|.$$

First, if $|A_{j+1}|>|A_j\setminus A_{j+1}|$, then we have

$$\Big(\frac{|A_{j+1}|}{|A_j\setminus A_{j+1}|}\Big)^{g_0}2^{-g_1}G\big(\frac{k_j}{\rho}\big)\le G\Big(\frac{|A_{j+1}|}{|A_j\setminus A_{j+1}|}\frac{k_{j+1}}{\rho}\Big).$$

Therefore, (4.13) generates

$$\Big(\frac{|A_{j+1}|}{|\Omega_\tau|}\Big)^{\frac{g_0}{g_0-1}}\le\gamma(\gamma^*)^{\frac{1}{g_0-1}}\frac{|A_j\setminus A_{j+1}|}{|\Omega_\tau|}. \tag{4.14}$$

Second, if $|A_{j+1}|\le|A_j\setminus A_{j+1}|$, then we observe that

$$\Big(\frac{|A_{j+1}|}{|A_j\setminus A_{j+1}|}\Big)^{g_1}2^{-g_1}G\big(\frac{k_j}{\rho}\big)\le G\Big(\frac{|A_{j+1}|}{|A_j\setminus A_{j+1}|}\frac{k_{j+1}}{\rho}\Big).$$

The inequality (4.13) gives

$$\Big(\frac{|A_{j+1}|}{|A_j\setminus A_{j+1}|}\Big)^{g_1}2^{-g_1}\le\gamma\gamma^*\frac{|\Omega_\tau|}{|A_j\setminus A_{j+1}|},$$

and hence

$$\Big(\frac{|A_{j+1}|}{|\Omega_\tau|}\Big)^{\frac{g_1}{g_1-1}} \le \gamma(\gamma^*)^{\frac{1}{1-g_1}}\frac{|A_j \setminus A_{j+1}|}{|\Omega_\tau|}.$$

Since $|A_{j+1}|/\Omega_\tau| \le 1$ and $g_1/(g_1-1) \le g_0/(g_0-1)$, it follows that

$$\Big(\frac{|A_{j+1}|}{|\Omega_\tau|}\Big)^{\frac{g_0}{g_0-1}} \le \Big(\frac{|A_{j+1}|}{|\Omega_\tau|}\Big)^{\frac{g_1}{g_1-1}}.$$

In addition, since $\gamma^* \ge 1$ and $1/(g_1-1) \le 1/(g_0-1)$, it follows that

$$(\gamma^*)^{1/(g_1-1)} \le (\gamma^*)^{1/(g_0-1)}.$$

Therefore, (4.14) is valid for all $j \in \{0,\dots,j^*-1\}$.

Next we take the sum for $j = 0,\dots,j^*-1$ of the inequality (4.14). Noting that $|A_{j^*}| \le |A_{j+1}|$ for $j = 0,\dots,j^*-1$, we conclude that

$$j^*\left(\frac{|A_{j^*}|}{|\Omega_\tau|}\right)^{g_0/(g_0-1)} \le \gamma(\gamma^*)^{1/(g_0-1)}.$$

We now reach our conclusion (4.8) by choosing j^* such that

$$j^* \ge \frac{1}{\gamma}\nu^{g_0/(1-g_0)}(\gamma^*)^{1(1-g_0)}.$$

□

The following proposition is modified DeGiorgi iteration with generalized structure conditions (1.3). Basically, our Proposition 4.4 is equivalent to [7, Lemma 3.1] and [8, Lemmata III.4.1 and IV.4.1]. (We shall have more to say about [8, Lemma IV.4.1] in [13].)

Proposition 4.4. *For a given positive constant θ, there exists $\nu_0 = \nu_0(\theta, data) \in (0,1)$ such that, if u is a nonnegative supersolution of* (1.1) *in $Q_{k,2\rho}(\theta)$ with*

$$|Q_{k,2\rho}(\theta) \cap \{u \le k\}| \le \nu_0|Q_{k,2\rho}(\theta)|$$

for some positive constants k and ρ, then

$$\operatorname{ess\,inf}_{Q_{k,\rho}(\theta)} u(x,t) \ge \frac{k}{2}.$$

Proof. First, we construct two sequences $\{\rho_n\}_{n=0}^\infty$ and $\{k_n\}_{n=0}^\infty$ such that

$$\rho_n = \rho + \frac{\rho}{2^n}, \quad k_n = \frac{k}{2} + \frac{k}{2^{n+1}} \quad \text{for } n = 0,1,\dots.$$

Because G is increasing, the sequence $\{Q_n\}_{n=0}^\infty$, given by

$$Q_n = K_{\rho_n} \times [-T_{k,\rho_n}(\theta), 0],$$

is a nested and shrinking sequence of cylinders. Let us take a sequence of piecewise linear cutoff functions $\{\zeta_n\}_{n=0}^\infty$ such that

$$\zeta_n = \begin{cases} 1 & \text{inside of } \ Q_{n+1} \\ 0 & \text{on the parabolic boundary of } Q_n, \end{cases}$$

satisfying

$$|D\zeta_n| \le \frac{2^{n+1}}{\rho},$$

$$0 \le (\zeta_n)_t \le \frac{1}{\theta k^2 (G(\frac{k}{\rho_n})^{-1} - G(\frac{k}{\rho_{n+1}})^{-1})}$$

Let us note that

$$\frac{2^{n+1}}{\rho} \ge \frac{2^{n+2}}{\rho_n}$$

because $\rho_n \le 2\rho$. We also need a different upper bound for $(\zeta_n)_t$. As a first step, we write

$$G\big(\frac{k}{\rho_n}\big)^{-1} - G\big(\frac{k}{\rho_{n+1}}\big)^{-1} = \int_{\rho_{n+1}}^{\rho_n} \frac{k}{s^2} g(\frac{k}{s}) G(\frac{k}{s})^{-2}\, ds.$$

By using the first inequality in (1.4) and Lemma 2.1(b), we conclude that

$$\frac{k}{s^2} g(\frac{k}{s}) G(\frac{k}{s})^{-2} \ge \frac{g_0}{s} G(\frac{k}{s})^{-1} \ge \frac{g_0}{\rho_n} G\big(\frac{k}{\rho_n}\big)^{-1}$$

for any $s \in (\rho_{n+1}, \rho_n)$ and hence

$$\begin{aligned}
(\zeta_n)_t &\le \frac{1}{\theta k^2 g_0} G\big(\frac{k}{\rho_n}\big) \frac{\rho_n}{\rho_n - \rho_{n+1}} \\
&= \frac{2^n + 2}{g_0 \theta} k^{-2} G\big(\frac{k}{\rho_n}\big) \\
&\le \frac{2^{n+1}}{g_0 \theta} k^{-2} G\big(\frac{k}{\rho_n}\big).
\end{aligned}$$

Note that

$$G(|D\zeta_n| \zeta_n (u-k_n)_-) \le 2^{(n+1)g_1} G\Big(\frac{\zeta_n (u-k_n)_-}{\rho_n}\Big).$$

Therefore, the local energy estimate (5.2) yields, for some constants γ_0 and γ_1, that

$$\begin{aligned}
&\sup_t \int_{K_{\rho_n}} G^{r-1}\Big(\frac{\zeta_n (u-k_n)_-}{\rho_n}\Big)(u-k_n)_-^{s+2} \zeta_n^q \, dx \\
&+ \iint_{Q_n} G\left(|D(u-k_n)_-|\right) G^{r-1}\Big(\frac{\zeta_n (u-k_n)_-}{\rho_n}\Big)(u-k_n)_-^{s} \zeta_n^q \, dx\, dt \\
&\le \gamma_0 \iint_{Q_n} G^{r-1}\Big(\frac{\zeta_n (u-k_n)_-}{\rho_n}\Big)(u-k_n)_-^{s+2} \zeta_n^{q-1} (\zeta_n)_t \, dx\, dt \\
&\quad + \gamma_1 2^{(n+1)g_1} \iint_{Q_n} G^{r}\Big(\frac{\zeta_n (u-k_n)_-}{\rho_n}\Big)(u-k_n)_-^{s} \, dx\, dt.
\end{aligned} \tag{4.15}$$

We now observe that

$$(u-k_n)_- = \max\{0, k_n - u\} \le k_n \le k,$$

and that $G^{r-1}(\sigma)\sigma^{s+2}$ and $G^r(\sigma)\sigma^s$ are increasing with respect to σ. Since $q \ge 1$, we conclude that the right hand side of (4.15) is bounded by

$$RHS \le \big\{\gamma_0 \frac{2^{n+1}}{g_0 \theta} + \gamma_1 2^{(n+2)g_1}\big\} G^r\big(\frac{k}{\rho_n}\big) k^s \, |A_n| \,,$$

where $A_n = Q_n \cap \{u \le k_n\}$.

Using $u_n := (u-k_n)_-$ for simpler notation, we obtain

$$\begin{aligned} &2^{-2(n+2)}k^2G\big(\frac{k}{\rho_n}\big)^{-1}\sup_t\int_{K_{\rho_n}} G^r\big(\frac{\zeta_n u_n}{\rho_n}\big)u_n^s\zeta_n^q\,dx \\ &+\iint_{Q_n} G\left(|Du_n|\right)G^{r-1}\big(\frac{\zeta_n u_n}{\rho_n}\big)u_n^s\zeta_n^q\,dx\,dt \\ &\le \gamma 2^{ng_1}\big(1+\frac{1}{\theta}\big)G^r\big(\frac{k}{\rho_n}\big)k^s\,|A_n|\,. \end{aligned}\tag{4.16}$$

We now consider the function

$$v = G^r\big(\frac{\zeta_n u_n}{2\rho_n}\big)u_n^s\zeta_n^q.$$

After differentiating v and applying Lemma 2.1, we derive, for some constants c_0 and c_1,

$$|Dv| \le \frac{c_0}{\rho_n}G(|Du_n|)G^{r-1}\left(\frac{u_n}{2\rho_n}\right)u_n^s + \frac{c_1 2^n}{\rho_n}v.$$

It follows from this inequality and (4.16) that

$$\sup_t\int_{K_{\rho_n}} v\,dx \le \gamma\big(1+\frac{1}{\theta}\big)2^{n(g_1+2)}k^{s-2}G^{r+1}\big(\frac{k}{\rho_n}\big)|A_n|$$

and that

$$\iint_{Q_n}|Dv|\,dx\,dt \le \gamma\big(1+\frac{1}{\theta}\big)\frac{1}{\rho_n}2^{ng_1}k^sG^r\big(\frac{k}{\rho_n}\big)|A_n|.$$

Hence, from Theorem 5.4 (and recalling that $\rho/2\le\rho_n\le\rho$), we conclude that

$$\begin{aligned} &\iint_{Q_n} G^r\big(\frac{\zeta_n u_n}{\rho_n}\big)u_n^s\zeta_n^q\,dx\,dt \\ &\le \gamma\big(1+\frac{1}{\theta}\big)2^{n(g_1+2)}k^{s-2/(N+1)}\rho^{-N/(N+1)} \\ &\quad\times G^{r+1/(N+1)}\big(\frac{k}{\rho}\big)|A_n|^{(N+2)/(N+1)}. \end{aligned}\tag{4.17}$$

To find a lower bound for the left hand side of (4.17), we observe that in the set $\{u<k_{n+1}\}$, we have

$$u_n = \max\{0,k_n-u\} \ge k_n-k_{n+1} = \frac{k}{2^{n+2}},$$

It follows that, in A_{n+1}, we have

$$G^r\Big(\frac{\zeta_n u_n}{\rho_n}\Big)u_n^s\zeta_n^q \ge G^r\Big(\frac{k}{2^{n+2}\rho_n}\Big)k^s2^{-s(n+2)}$$

because $\zeta_n=1$ in Q_{n+1}. Since G^r is increasing, we infer that

$$G^r\Big(\frac{\zeta_n u_n}{\rho_n}\Big)u_n^s\zeta_n^q \ge 2^{-(s+g_1)(n+2)-g_1}k^sG^r\big(\frac{k}{\rho}\big)$$

in A_{n+1}, and therefore it follows that

$$\begin{aligned} |A_{n+1}| &\le \gamma\big(1+\frac{1}{\theta}\big)^{N/(N+1)}2^{n(2g_1+s+2)} \\ &\quad\times k^{-2/(N+1)}\rho_n^{-N/(N+1)}G^{1/(N+1)}\big(\frac{k}{\rho}\big)|A_n|^{(N+2)/(N+1)}. \end{aligned}$$

Hence (5.15) is satisfied with

$$Y_n = |A_n|, \quad C = \gamma\big(1+\frac{1}{\theta}\big)^{N/(N+1)} k^{-\frac{2}{N+1}} \rho_n^{-\frac{N}{N+1}} G^{\frac{1}{N+1}}\big(\frac{k}{\rho}\big),$$
$$b = 2^{2g_1+s+2}, \quad \alpha = \frac{1}{N+1}.$$

Applying Lemma 5.5 completes the proof because

$$C^{-1/\alpha} = \gamma\big(1+\frac{1}{\theta}\big)^{-N-1} k^2\rho^N G\big(\frac{k}{\rho}\big)^{-1} = \gamma\frac{\theta^N}{(1+\theta)^{N+1}}|Q_{k,\rho}(\theta)|.$$

□

Note that ν_0 has the form $\nu_1\theta^N(1+\theta)^{-N-1}$ with ν_1 determined only by the data. A variant form of this proposition will also be useful in our study of degenerate equations. This variant is analogous to [7, Lemma 3.3] and [8, Lemma III.6.1]. We do point out, however, that [7, Lemma 3.3] does not explicitly mention the dependence of the integer s on the parameter η. Moreover, the proof of [8, Lemma III.6.1] needs some clarification to see that the constants are stable as $p \nearrow 2$: the inequality $2^{-p}(2^{s_1}/A)^{p-2} \geq 1$ means that $s_1 \to \infty$ as $p \to 2$. Fortunately the choice $s_1 \geq \log_2 A$ guarantees that $2^{-p}(2^{s_1}/A)^{p-2} \geq 1/4$ and this weaker inequality suffices for the proof.

Proposition 4.5. *There exists $\nu^* \in (0,1)$, determined only by the data, such that, if u is a nonnegative supersolution of (1.1) in $Q_{k,2\rho}(\theta)$ with*

$$|\{(x,t) \in Q_{k,2\rho}(\theta) : u(x,t) < k\}| < \frac{\nu^*}{\theta}\,|Q_{k,2\rho}(\theta)| \tag{4.18a}$$

for some positive constants k, ρ, and θ and if

$$u(x, -T_{k,2\rho}(\theta)) \geq k \tag{4.18b}$$

for all $x \in K_{2\rho}$, then

$$\operatorname{ess\,inf}_{K_\rho\times(-T_{k,2\rho}(\theta),0)} u \geq \frac{k}{2}.$$

Proof. With ρ_n and k_n as in the proof of Proposition 4.4, we set

$$Q_n = K_{\rho_n} \times (-T_{k,2\rho}(\theta), 0),$$

and we take ζ_n to be a time-independent cut-off function. In other words,

$$\zeta_n = \begin{cases} 1 & \text{inside } Q_{n+1}, \\ 0 & \text{on the lateral boundary of } Q_n \end{cases}$$

with $\zeta_{n,t} = 0$ and $|D\zeta_n| \leq 2^{n+1}/\rho_n$.

In place of (4.15), we now have

$$\begin{aligned}
&\sup_t \int_{K_{\rho_n}} G^{r-1}\Big(\frac{\zeta_n(u-k_n)_-}{\rho_n}\Big)(u-k_n)_-^{s+2}\zeta_n^q\,dx \\
&+ \iint_{Q_n} G\left(|D(u-k_n)_-|\right) G^{r-1}\Big(\frac{\zeta_n(u-k_n)_-}{\rho_n}\Big)(u-k_n)_-^s\zeta_n^q\,dx\,dt \\
&\leq \gamma_1 2^{(n+1)g_1}\iint_{Q_n} G^r\Big(\frac{\zeta_n(u-k_n)_-}{\rho_n}\Big)(u-k_n)_-^s\,dx\,dt.
\end{aligned}$$

Arguing as in the proof of Proposition 4.4, we now infer that (5.15) is satisfied with

$$Y_n = |A_n|, \quad C = \gamma k^{-\frac{2}{N+1}} \rho_n^{-\frac{N}{N+1}} G^{\frac{1}{N+1}}\big(\frac{k}{\rho}\big), \quad b = 2^{2g_1+s+2}, \quad \alpha = \frac{1}{N+1}.$$

The proof is completed by noting that

$$C^{-1/\alpha} = \frac{\gamma}{\theta}|Q_{k,\rho}(\theta)| \ge \frac{\gamma}{\theta}|Q_{k,2\rho}(\theta)|.$$

□

Let us emphasize that Propositions 4.4 and 4.5 are valid for the full range $1 < g_0 \le g_1 < \infty$.

4.2. **Proof of the first alternative.**

Proof. First, with Δ as in Section 3, we set

$$T_1 = T_0 + \Delta - \big(\frac{\omega}{2}\big)^2 G\big(\frac{\omega}{2R}\big)^{-1}$$

and we use Proposition 4.4with $\theta = 1$, $k = \omega/2$ and $\rho = R$ to infer that

$$u \ge \frac{\omega}{4} \text{ on } K_R \times \{T_1\}.$$

It then follows from Proposition 4.2 with $\rho = R$, $\nu = 1$, $\tau = -T_1$, and $k = \omega/4$ that, for any $\epsilon \in (0,1)$, there is a constant $\delta \in (0,1)$ determined only by data, ϵ, and θ_0 such that

$$|\{x \in K_R : u(x,t) \le \frac{\delta\omega}{4}\}| \le \epsilon |K_R|$$

for all $t \in [T_1, 0)$. We now choose $\epsilon = \nu^*/\theta_0$, with ν^* from Proposition 4.5 and set

$$\theta = \frac{-T_1}{(\delta\omega/4)^2 G(\delta\omega/(8R))^{-1}}.$$

We first observe that

$$G\big(\frac{\delta\omega}{8R}\big) \le \big(\frac{\delta}{4}\big)^{g_0} G\big(\frac{\omega}{2R}\big)$$

and that $-T_1 \le -T_0$, so

$$\theta \le \frac{\theta_0 \big(\frac{\omega}{2}\big)^2 G\left(\frac{\omega}{2R}\right)^{-1}}{\left(\frac{\delta\omega}{4}\right)^2 \left(\frac{\delta}{4}\right)^{-g_0} G\left(\frac{\omega}{2R}\right)^{-1}} = \frac{1}{4}\delta^{g_0-2}\theta_0 \le \theta_0.$$

For $k = \delta\omega/4$, and $\rho = R/2$, we now have that (4.18) holds. The proof is completed by applying Proposition 4.5 and noting that

$$T_1 \le -\big(\frac{\omega}{2}\big)^2 G\left(\frac{\omega}{2R}\right)^{-1} \le -\left(\frac{\omega}{4}\right)^2 G\left(\frac{\omega/4}{R/2}\right)^{-1}.$$

□

4.3. **Proof of the second alternative.**

Proof. First, we set

$$\alpha = \frac{\nu_0}{2-\nu_0},$$

and we apply Proposition 4.1 with $\theta = 1$, $\nu_1 = \nu_0$, $k = \omega/2$, $\rho = 2R$, and $T = \Delta$ to infer that, for each $T_0 \in (-\theta_0\Delta, -\Delta)$, there is a number $\tau_1 \in (T_0, T_0 + \alpha\Delta)$ such that

$$\Big|\Big\{x \in K_{2R} : u(x,\tau_1) \le \frac{\omega}{2}\Big\}\Big| \le \Big(1 - \frac{\nu_0}{2}\Big)\,|K_{2R}|.$$

We momentarily fix $T_0 \in (-\theta_0\Delta, -\Delta)$. It follows from Proposition 4.2 with $\rho = 2R$, $\nu = \nu_0/2$, $\epsilon = \frac{1}{2}$, and $\theta = 1$ that there is a constant $\delta \in (0,1)$, determined only by data, such that

$$\big|\big\{x \in K_{2R} : u(x,t) \le \frac{\delta\omega}{2}\big\}\big| \le \big(1 - \frac{\nu_0}{4}\big)|K_{2R}| \tag{4.19}$$

for all $t \in (\tau_1, \min\{\tau_1 + \Delta, 0\})$. Since T_0 is arbitrary, we conclude that (4.19) holds for all $t \in ((-\theta_0 + 1)\Delta, 0)$.

For our next step, we take δ^* to be the constant from Proposition 4.3 corresponding to $\alpha = \nu_0/4$, $\nu = \nu_0$ and $\theta = 1/2$. We also choose

$$\theta_0 = 1 + (\delta^*\delta)^{2-g_1}.$$

Since

$$(\theta_0 - 1)\Delta \ge (\delta^*\delta\omega)^2 G\Big(\frac{\delta^*\delta\omega}{2R}\Big)^{-1},$$

we infer from Proposition 4.3 with $\tau = \frac{1}{2}(\theta_0 - 1)\Delta$ (which is easily seen to satisfy (4.6)) that

$$\big|Q_{\delta^*\delta\omega/2,R} \cap \big\{u \le \frac{\delta^*\delta\omega}{2}\big\}\big| \le \nu_0|Q_{\delta^*\delta\omega,R}|.$$

We then use Proposition 4.4 with $\theta = 1$, $k = \delta^*\delta\omega/2$, and $\rho = R$ to infer that

$$\operatorname{ess\,inf}_{Q_{\delta^*\delta\omega/2,R/2}} u \ge \frac{1}{4}\delta^*\delta\omega.$$

We now observe that $\delta^* \le 1$ and $\delta \le 1/2$, and hence

$$\begin{aligned}
\big(\frac{\delta^*\delta\omega}{2}\big)^2 G\big(\frac{\delta^*\delta\omega}{R}\big)^{-1} &\ge \big(\frac{\delta^*\delta\omega}{2}\big)^2 (2\delta^*\delta)^{-g_0} G\left(\frac{\omega}{2R}\right)^{-1} \\
&= (2\delta^*\delta)^{2-g_0}\left(\frac{\omega}{4}\right)^2 G\left(\frac{\omega}{2R}\right)^{-1}.
\end{aligned}$$

The proof is now complete because this inequality implies that $\mathcal{Q} \subset Q_{\delta^*\delta\omega/2,R/2}$. □

5. Proof of auxiliary theorems

We now present the basic results used in the previous sections of the paper. Some are proved here because their proofs are slightly different from the corresponding results for the parabolic p-Laplacian equation, but the others, which are already known in the form we need, are just quoted.

5.1. **Local energy estimate.** This estimate is a fundamental inequality that plays an important roles in several proofs, especially those of Propositions 4.1, 4.2, and 4.3. The inequality is essentially equivalent to [7, (1.17)], [8, Proposition II.3.1], and [23, Proposition 2.4] if $g_0 = g_1 = p$. Some techniques come from Section 3 in [18]. We point out here that Propositions 5.1 and 5.2 as well as their proofs are valid for the full range $1 < g_0 \le g_1$.

Proposition 5.1. *Let G satisfy structure conditions* (1.3) *in a cylinder $Q_\rho := K_\rho \times [t_0, t_1]$, and let ζ be a cutoff function on the cylinder Q_ρ, vanishing on the parabolic boundary of Q_ρ with $0 \le \zeta \le 1$. Define constants r, s, and q by*

$$r = 1 - \frac{1}{g_1}, \quad s = \frac{g_0}{g_1}, \quad q = 2g_1. \tag{5.1}$$

(a) If u is a locally bounded weak supersolution of (1.1)*, then there exist constants c_0, c_1, and c_2 depending on data such that*

$$\begin{aligned} &\int_{K_\rho \times \{t_1\}} G^{r-1}\Big(\frac{\zeta(u-k)_-}{\rho}\Big)(u-k)_-^{s+2}\zeta^q\,dx \\ &+ c_0 \iint_{Q_\rho} G\left(|D(u-k)_-|\right) G^{r-1}\Big(\frac{\zeta(u-k)_-}{\rho}\Big)(u-k)_-^s\zeta^q\,dx\,dt \\ &\le c_1 \iint_{Q_\rho} G^{r-1}\Big(\frac{\zeta(u-k)_-}{\rho}\Big)(u-k)_-^{s+2}\zeta^{q-1}\,|\zeta_t|\,dx\,dt \\ &\quad + c_2 \iint_{Q_\rho} G\left(|D\zeta|\zeta(u-k)_-\right) G^{r-1}\Big(\frac{\zeta(u-k)_-}{\rho}\Big)(u-k)_-^s\,dx\,dt \end{aligned} \tag{5.2}$$

for any constant k.

(b) If u is a locally bounded weak subsolution of (1.1)*, then there exist constants c_0, c_1, and c_2 depending on data such that*

$$\begin{aligned} &\int_{K_\rho \times \{t_1\}} G^{r-1}\Big(\frac{\zeta(u-k)_+}{\rho}\Big)(u-k)_+^{s+2}\zeta^q\,dx \\ &+ c_0 \iint_{Q_\rho} G\left(|D(u-k)_+|\right) G^{r-1}\Big(\frac{\zeta(u-k)_+}{\rho}\Big)(u-k)_+^s\zeta^q\,dx\,dt \\ &\le c_1 \iint_{Q_\rho} G^{r-1}\Big(\frac{\zeta(u-k)_+}{\rho}\Big)(u-k)_+^{s+2}\zeta^{q-1}\,|\zeta_t|\,dx\,dt \\ &\quad + c_2 \iint_{Q_\rho} G\left(|D\zeta|\zeta(u-k)_+\right) G^{r-1}\Big(\frac{\zeta(u-k)_+}{\rho}\Big)(u-k)_+^s\,dx\,dt \end{aligned} \tag{5.3}$$

for any constant k.

Proof. To prove (a), we assume that u is differentiable in terms of the time variable. Such an assumption is removed by applying Steklov average. The choices (5.1) are made to satisfy

$$(r-1)g_1 + (s+1) > 0, \tag{5.4a}$$

$$(r-1)g_0 + s \le 0, \tag{5.4b}$$

$$(r-1)g_1 + q \ge 0, \tag{5.4c}$$

Inequality (5.4a) implies that $G^{r-1}(\sigma)\sigma^{s+1}$ is increasing with respect to σ, and inequality (5.4b) implies that $G^{r-1}(\sigma)\sigma^s$ is nonincreasing with respect to σ.

We use the test function

$$\varphi(x,t) = G^{r-1}\Big(\frac{\zeta(u-k)_-}{\rho}\Big)(u-k)_-^{s+1}\zeta^q,$$

in the integral inequality

$$\iint_{Q_\rho} u_t\varphi\,dx\,dt + \iint_{Q_\rho} D\varphi\cdot\mathcal{A}\,dx,dt \ge 0. \tag{5.5}$$

For simple notation, let $\bar u := (u-k)_-$. Then we have

$$\begin{aligned} D\varphi &= \big\{(r-1)\frac{\zeta\bar u}{\rho}g\big(\frac{\zeta\bar u}{\rho}\big) + (s+1)G\big(\frac{\zeta\bar u}{\rho}\big)\big\}G^{r-2}\big(\frac{\zeta\bar u}{\rho}\big)\bar u^s\zeta^q D\bar u \\ &\quad + \big\{(r-1)\frac{\zeta\bar u}{\rho}g\big(\frac{\zeta\bar u}{\rho}\big) + qG\big(\frac{\zeta\bar u}{\rho}\big)\big\}G^{r-2}\big(\frac{\zeta\bar u}{\rho}\big)\bar u^{s+1}\zeta^{q-1}D\zeta. \end{aligned}$$

From the second inequality of (1.4) and the definition of r, it follows that

$$\begin{aligned} (r-1)\frac{\zeta\bar u}{\rho}g\big(\frac{\zeta\bar u}{\rho}\big) + (s+1)G\big(\frac{\zeta\bar u}{\rho}\big) &\ge [(r-1)g_1 + (s+1)]G\big(\frac{\zeta\bar u}{\rho}\big) \\ &= sG\big(\frac{\zeta\bar u}{\rho}\big). \end{aligned}$$

In addition, the second inequality of (1.4) and (5.4c) imply that

$$(r-1)\frac{\zeta\bar u}{\rho}g\big(\frac{\zeta\bar u}{\rho}\big) + qG\big(\frac{\zeta\bar u}{\rho}\big) \ge [(r-1)g_1 + q]G\big(\frac{\zeta\bar u}{\rho}\big) \ge 0.$$

It then follows from the first inequality of (1.4) that

$$\begin{aligned} \big|(r-1)\frac{\zeta\bar u}{\rho}g\big(\frac{\zeta\bar u}{\rho}\big) + qG\big(\frac{\zeta\bar u}{\rho}\big)\big| &= (r-1)\frac{\zeta\bar u}{\rho}g\big(\frac{\zeta\bar u}{\rho}\big) + qG\big(\frac{\zeta\bar u}{\rho}\big) \\ &\le [(r-1)g_0 + q]G\big(\frac{\zeta\bar u}{\rho}\big). \end{aligned}$$

Hence

$$\begin{aligned} &\iint_{Q_\rho} \mathcal{A}(x,t,u,Du)\cdot D\varphi\,dx\,dt \\ &\le -sC_0\iint_{Q_\rho} G(|Du|)G^{r-1}\big(\frac{\zeta\bar u}{\rho}\big)\bar u^s\zeta^q\,dx\,dt \\ &\quad + \{(r-1)g_0 + q\}C_1\iint_{Q_\rho} g(|Du|)|D\zeta|G^{r-1}\big(\frac{\zeta\bar u}{\rho}\big)\bar u^{s+1}\zeta^{q-1}\,dx\,dt. \end{aligned} \tag{5.6}$$

In Lemma 2.1(e), set $\sigma_1 = |D\zeta|\bar u/\zeta$ and $\sigma_2 = |Du|$ to obtain, for any $\epsilon_1 > 0$, that

$$\begin{aligned} G^{r-1}\big(\frac{\zeta\bar u}{\rho}\big)\bar u^s\zeta^q g(|Du|)\frac{|D\zeta\bar u|}{\zeta} &\le \epsilon_1 g_1 G(|Du|)G^{r-1}\big(\frac{\zeta\bar u}{\rho}\big)\bar u^s\zeta^q \\ &\quad + \epsilon_1^{1-g_1}g_1 G\left(\frac{|D\zeta|\bar u}{\zeta}\right)G^{r-1}\big(\frac{\zeta\bar u}{\rho}\big)\bar u^s\zeta^q. \end{aligned}$$

In particular, if we choose

$$\epsilon_1 = \frac{sC_0}{2g_1[(r-1)g_0 + q]C_1}$$

and if we use Lemma 2.1 to estimate

$$G\Big(\frac{|D\zeta|\bar u}{\zeta}\Big) \le \zeta^{-2g_1}G\left(|D\zeta|\zeta\bar u\right),$$

we infer that

$$\iint_{Q_\rho} D\varphi \cdot \mathcal{A}\, dx\, dt \le -\frac{1}{2} c_0 I_0 + \frac{1}{2} c_2 I_2, \tag{5.7a}$$

with

$$c_0 = sC_0, \tag{5.7b}$$

$$c_2 = 2\epsilon_1^{1-g_1} g_1 [(r-1)g_0 + q] C_1, \tag{5.7c}$$

$$I_0 = \iint_{Q_\rho} G(|D\bar{u}|) G^{r-1}\big(\frac{\zeta \bar{u}}{\rho}\big) \bar{u}^s \zeta^q \, dx\, dt, \tag{5.7d}$$

$$I_2 = \iint_{Q_\rho} G\left(|D\zeta| \zeta \bar{u}\right) G^{r-1}\big(\frac{\zeta \bar{u}}{\rho}\big) \bar{u}^s \zeta^{q-2g_1}\, dx\, dt \tag{5.7e}$$

Now, by setting

$$F = \int_0^{\bar{u}} G^{r-1}\big(\frac{\zeta \alpha}{\rho}\big) \alpha^{s+1}\, d\alpha,$$

we infer that

$$\iint_{Q_\rho} u_t \varphi(x,t)\, dx\, dt = -\int_{K_\rho \times \{t\}} F\zeta^q dx \Big|_{t_0}^{t_1} + q \iint_{Q_\rho} F \zeta^{q-1} \zeta_t \, dx\, dt. \tag{5.8}$$

We now note that $F \ge 0$ and that, because $G^{r-1}(\sigma)\sigma^s$ is increasing with respect to σ,

$$F \le G^{r-1}\left(\frac{\zeta \bar{u}}{\rho}\right) \bar{u}^{s+2}.$$

Hence

$$\iint_{Q_\rho} F\zeta^{q-1} \zeta_t \, dx\, dt \le \iint_{Q_\rho} G^{r-1}\big(\frac{\zeta \bar{u}}{\rho}\big) \bar{u}^{s+2} \zeta^{q-1} \left|\zeta_t\right|\, dx\, dt. \tag{5.9}$$

In addition, because $G^{r-1}(\sigma)\sigma^{s-1}$ is decreasing with respect to σ, we infer that

$$F \ge G^{r-1}\big(\frac{\zeta \bar{u}}{\rho}\big) \bar{u}^{s-1} \int_0^{\bar{u}} \alpha \, d\alpha = \frac{1}{2} G^{r-1}\big(\frac{\zeta \bar{u}}{\rho}\big) \bar{u}^{s+1},$$

Therefore,

$$\int_{K_\rho \times \{t\}} F\zeta^q dx \Big|_{t_0}^{t_1} \le -\frac{1}{2} \int_{K_\rho \times \{t_1\}} G^{r-1}\big(\frac{\zeta \bar{u}}{\rho}\big) \bar{u}^{s+1}\, dx. \tag{5.10}$$

The proof is complete, with c_0 and c_2 given by (5.7b) and (5.7c) and $c_1 = 2q$, by combining (5.5), (5.7a), (5.8), (5.9), and (5.10).

The proof of (b) is essentially the same with $(u-k)_+$ in place of $(u-k)_-$. $\square$

Note that, if we assume (5.4) and the inequality $q \ge 2g_1$ in place of (5.1), then (5.2) for supersolutions (or (5.3) for subsolutions) holds with the constants determined also by r, s, and q. We have made the choices in (5.1) for convenience only.

5.2. **Logarithmic energy estimate.** The logarithmic energy estimate (5.11), used to prove Proposition 4.2, is modified from [8, Proposition II.3.2] and similar to [7, (1.6)] and [23, Proposition 2.6]. The functions h and H are defined in Lemma 2.2.

Proposition 5.2. *Assume that G satisfies (1.3) in a cylinder $K_R\times[t_0,t_1]$. Let $q\geq g_1$ and $\delta\in(0,1)$ be constants, and let ζ be a cut-off function which is independent of the time variable.*

(a) Let u be a nonnegative weak supersolution of (1.1)and let k be a positive constant. Then

$$
\begin{aligned}
&\int_{K_R\times\{t_1\}} H(\Psi^2)\zeta^q\,dx + C_0(4g_0-2)\int_{t_0}^{t_1}\int_{K_R} G(|Du|)h(\Psi^2)(\Psi')^2\zeta^q\,dx\,dt \\
&\leq \int_{K_R\times\{t_0\}} H(\Psi^2)\zeta^q\,dx + C^*\int_{t_0}^{t_1}\int_{K_R} h(\Psi^2)\Psi(\Psi')^2 G\big(\frac{|D\zeta|}{|\Psi'|}\big)\zeta^{q-g_1}\,dx\,dt
\end{aligned}\tag{5.11}
$$

where

$$
C^*=\frac{C_0}{g_1}\Big(\frac{2qg_1C_1}{C_0}\Big)^{g_1},\quad \Psi(u)=\ln^+\big[\frac{k}{(1+\delta)k-(u-k)_-}\big].
$$

(b) If u is a nonpositive weak subsolution of (1.1) and k is a negative constant, then (5.11) holds with

$$
\Psi(u)=\ln^+\big[\frac{k}{(1+\delta)k+(u-k)_+}\big].
$$

Proof. As before, to prove (a), we assume that u is differentiable in terms of the time variable and later such an assumption is removed by applying the Steklov average. Define the test function $\varphi=2h(\Psi^2)\Psi\Psi'\zeta^q$, and note that

$$
\begin{gathered}
\Psi'(u)=\frac{-1}{(1+\delta)k-(u-k)_-},\\
\Psi''(u)=\frac{1}{\left[(1+\delta)k-(u-k)_-\right]^2}=(\Psi')^2.
\end{gathered}
$$

Since u is nonnegative, it follows that $0\leq(u-k)_-\leq k$, and therefore

$$
\frac{1}{(1+\delta)k}\leq|\Psi'|\leq\frac{1}{\delta k},\quad 0\leq\Psi\leq\ln^+\frac{1}{\delta}.
$$

Moreover, $\varphi\in L^\infty$ and $D\varphi\in L^\infty$, so φ is an admissible test function.

First, we have

$$
\begin{aligned}
\int_{t_0}^{t_1}\int_{K_R} u_t 2h(\Psi^2)\Psi\Psi'\zeta^q\,dx\,dt &= \int_{t_0}^{t_1}\int_{K_R}\big[\frac{d}{dt}H(\Psi^2)\big]\zeta^q\,dx\,dt\\
&= \int_{K_R\times\{t_1\}} H(\Psi^2)\zeta^q\,dx - \int_{K_R\times\{t_0\}} H(\Psi^2)\zeta^q\,dx.
\end{aligned}
$$

Second, we take the derivative of the test function

$$
\begin{aligned}
D\varphi &= [4h'(\Psi^2)(\Psi\Psi')^2+2h(\Psi^2)(\Psi')^2+2h(\Psi^2)\Psi\Psi'']\zeta^q Du\\
&\quad + 2qh(\Psi^2)\Psi\Psi'\zeta^{q-1}D\zeta.
\end{aligned}
$$

Using the inequality (2.1c) and the observation that $\Psi''=(\Psi')^2$, we estimate

$$
4h'(\Psi^2)(\Psi\Psi')^2+2h(\Psi^2)(\Psi')^2+2h(\Psi^2)\Psi\Psi''\geq[4g_0-2+2\Psi]h(\Psi^2)(\Psi')^2.
$$

It follows that

$$\begin{aligned}
&\int_{t_0}^{t_1}\int_{K_R}\mathcal{A}(x,t,u,Du)\cdot D\varphi\,dx\,dt\\
&\geq C_0\int_{t_0}^{t_1}\int_{K_R}G(|Du|)h(\Psi^2)[4g_0-2+\Psi](\Psi')^2\zeta^q\,dx\,dt\\
&\quad-2qC_1\int_{t_0}^{t_1}\int_{K_R}g(|Du|)h(\Psi^2)\Psi\,|\Psi'|\,\zeta^{q-1}|D\zeta|\,dx\,dt.
\end{aligned}$$

Using Lemma 2.1(e), we infer that

$$\begin{aligned}
&g(|Du|)h(\Psi^2)\Psi\,|\Psi'|\,\zeta^{q-1}\,|D\zeta|\\
&=h(\Psi^2)\Psi(\Psi')^2\zeta^q g(|Du|)\frac{|D\zeta|}{\zeta\,|\Psi'|}\\
&\leq h(\Psi^2)\Psi(\Psi')^2\zeta^q\big[\epsilon_2 g_1 G(|Du|)+\epsilon_2^{1-g_1}g_1 G\Big(\frac{|D\zeta|}{\zeta|\Psi'|}\Big)\,\big]
\end{aligned}$$

for any $\epsilon_2>0$. Choosing $\epsilon_2=C_0/(2qg_1C_1)$ leads to (5.11).

Again the proof of (b) is similar. □

5.3. **A Poincaré type inequality.** We shall use the following Poincarè inequality which is inequality [16, (5.5) Chapter 2] (see also [8, Lemma I.2.2]).

Lemma 5.3. *Let $v\in W^{1,1}\left(K_\rho^{x_0}\right)\cap C\left(K_\rho^{x_0}\right)$ for some $\rho>0$ and some $x_0\in\mathbb{R}^N$ and let k and l be any pair of real numbers such that $k<l$. Then there exists a constant γ depending only upon N,p and independent of k,l,v,x_0,ρ, such that*

$$(l-k)|K_\rho^{x_0}\cap\{v>l\}|\leq\gamma\frac{\rho^{N+1}}{|K_\rho^{x_0}\cap\{v\leq k\}|}\int_{K_\rho^{x_0}\cap\{:k<v<l\}}|Dv|\,dx.$$

5.4. **Embedding theorem.** Our next result is a variation on the Sobolev imbedding theorem.

Theorem 5.4. *For a nonnegative function $v\in W_0^{1,1}(Q)$ where $Q=K\times[t_0,t_1]$, $K\subset\mathbb{R}^N$, we have*

$$\begin{aligned}
\iint_Q v\,dx\,dt&\leq C(N)|Q\cap\{v>0\}|^{\frac{1}{N+1}}\\
&\quad\times\Big[\operatorname{ess\,sup}_{t_0\leq t\leq t_1}\int_K v\,dx\Big]^{\frac{1}{N+1}}\Big[\iint_Q|Dv|\,dx\,dt\Big]^{\frac{N}{N+1}}.
\end{aligned}\tag{5.12}$$

Proof. First, by Hölder's inequality, we obtain

$$\iint_Q v\,dx\,dt\leq|Q\cap\{v>0\}|^{\frac{1}{N+1}}\Big[\iint_Q v^{\frac{N+1}{N}}\,dx\,dt\Big]^{\frac{N}{N+1}}.\tag{5.13}$$

Second, by Hölder's inequality and Sobolev's inequality for $p=1$, we have

$$\begin{aligned}
\int_K v^{\frac{N+1}{N}}\,dx&\leq\Big[\int_K v^{\frac{N}{N-1}}\,dx\Big]^{\frac{N-1}{N}}\Big[\int_K v\,dx\Big]^{1/N}\\
&\leq C(N)\int_K|Dv|\,dx\Big[\int_K v\,dx\Big]^{1/N}.
\end{aligned}\tag{5.14}$$

Combining two inequalities (5.13) and (5.14) produces the inequality (5.12). □

5.5. **Iteration.** Finally, we recall [8, Lemma I.4.1], which is the same as [17, Lemma 2.4.7].

Lemma 5.5. *Let* $\{Y_n\}$*,* $n = 0, 1, 2, \dots$*, be a sequence of positive numbers, satisfying the recursive inequalities*

$$Y_{n+1} \le C b^n Y_n^{1+\alpha} \tag{5.15}$$

where $C, b > 1$ *and* $\alpha > 0$ *are given numbers. If*

$$Y_0 \le C^{-\frac{1}{\alpha}} b^{-\frac{1}{\alpha^2}},$$

then $\{Y_n\}$ *converges to zero as* $n \to \infty$*.*

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