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MULTIPLICITY OF POSITIVE SOLUTIONS FOR SECOND-ORDER DIFFERENTIAL INCLUSION SYSTEMS DEPENDING ON TWO PARAMETERS

ZIQING YUAN, LIHONG HUANG, CHUNYI ZENG

Abstract. We consider the two-point boundary-value system

$$\begin{gathered}-u_i''+u_i\in\lambda\partial_{u_i}F(u_1,\dots,u_n)+\mu\partial_{u_i}G(u_1,\dots,u_n),\\ u_i'(a)=u_i'(b)=0\quad u_i\geq 0,\quad 1\leq i\leq n.\end{gathered}$$

Applying a version of nonsmooth three critical points theorem, we show the existence of at least three positive solutions.

1. Introduction

In the previous decades, there has been a lot of interest in scalar periodic problems driven by the one dimension p-laplacian. Some results can be found in [3, 18, 25, 24] and references therein. We mention the works by Guo [12], Pino et al [23], Fabry and Fayad [9] and Dang and Oppenheimer [7]. The authors used degree theory and assumed that the right-hand side nonlinearity $f(t,\zeta)$ is jointly continuous in $t\in T=[a,b]$ and $\zeta\in\mathbb{R}$. Their conditions on f are also asymptotic and there is no interaction between the nonlinearity and the Fučik spectrum of the one-dimensional p-laplacian. Especially, Heidarkhani and Yu [13] considered the existence of at least three solutions for a class of two-point boundary-value systems of the form

$$\begin{gathered}-u_i''+u_i=\lambda\partial_{u_i}F(u_1,\dots,u_n)+\mu\partial_{u_i}G(u_1,\dots,u_n),\\ u_i'(a)=u_i'(b)=0\end{gathered}\tag{1.1}$$

for $1\leq i\leq n$, where $F,G:[a,b]\times\mathbb{R}^n\to\mathbb{R}$ are C^1-functionals with respect to $(u_1,\dots,u_n)\in\mathbb{R}^n$ for a.e. $x\in[a,b]$.

From the above results, a natural question arises: what will happen when the potential functions F and G are not differentiable in (1.1)? This is the main point of interest in our paper. Here, we extend the main results in [13] to a class of perturbed Motreanu-Panagiotopoulos functionals [19], which raises some essential difficulties. The presence of non-differentiable function probably leads to no solution of (1.1) in general. Therefore to overcome this difficulty, setting $f_i=\partial_{u_i}F(u_1,\dots,u_n)$ and $g_i=\partial_{u_i}G(u_1,\dots,u_n)$, we consider such functions f_i

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and g_i, which are locally essentially bounded measurable and we fill the discontinuity gaps of f_i and g_i, replacing f_i and g_i by intervals $[f_i^-(u_1,\dots,u_n), f_i^+(u_1,\dots,u_n)]$ and $[g_i^-(u_1,\dots,u_n), g_i^+(u_1,\dots,u_n)]$, where

$$\begin{aligned}
f_i^-(u_1,\dots,u_n) &= \lim_{\delta\to 0^+} \operatorname{ess\,inf}_{|u_i'-u_i|<\delta} \partial_{u_i} F(u_1,\dots,u_i',\dots,u_n),\\
f_i^+(u_1,\dots,u_n) &= \lim_{\delta\to 0^+} \operatorname{ess\,sup}_{|u_i'-u_i|<\delta} \partial_{u_i} F(u_1,\dots,u_i',\dots,u_n),\\
g_i^-(u_1,\dots,u_n) &= \lim_{\delta\to 0^+} \operatorname{ess\,inf}_{|u_i'-u_i|<\delta} \partial_{u_i} G(u_1,\dots,u_i',\dots,u_n),\\
g_i^+(u_1,\dots,u_n) &= \lim_{\delta\to 0^+} \operatorname{ess\,sup}_{|u_i'-u_i|<\delta} \partial_{u_i} G(u_1,\dots,u_i',\dots,u_n).
\end{aligned}$$

Then $f_i^-(u_1,\dots,u_n)$, $g_i^-(u_1,\dots,u_n)$ are lower semi-continuous, and $f_i^+(u_1,\dots,u_n)$, $g_i^+(u_1,\dots,u_n)$ are upper semi-continuous.

So instead of (1.1) we consider the following second-order Neumann inclusion systems on a bounded interval $[a,b]$ in $\mathbb{R}$ $(a<b)$ with nonsmooth potentials (hemivariational inequality):

$$\begin{gathered}
-u_i'' + u_i \in \lambda \partial_{u_i} F(u_1,\dots,u_n) + \mu \partial_{u_i} G(u_1,\dots,u_n),\\
u_i'(a) = u_i'(b) = 0\ , u_i \ge 0 \quad \text{for } 1 \le i \le n,
\end{gathered} \tag{1.2}$$

where λ,μ are two positive parameters, $F,G:\mathbb{R}^n\to\mathbb{R}$ are measurable and locally Lipschitz functions. We denote by $\partial_{u_i}F(u_1,\dots,u_n)$ $(1\le i\le n)$ the partial generalized gradient of $F(u_1,\dots,u_n)$ with respect to u_i $(1\le i\le n)$, and by $\partial_{u_i}G(u_1,\dots,u_n)$ $(1\le i\le n)$ the partial generalized gradient of $G(u_1,\dots,u_n)$ to u_i $(1\le i\le n)$. Hemivariational inequality is a new type of variational expressions which arises in problems of engineering and mechanics, when one deals with nonsmooth and nonconvex energy functionals. For several concrete applications, we refer the reader to the monographs of [11, 19, 20, 21, 22] and references [8, 15, 10] and therein. More precisely, Iannizzoto [14] established a nonsmooth three critical points theorem and give two applications for variational-hemivariational inequalities depending on two parameters. Marano and Motreanu [17] obtained a nonsmooth version of Ricceri's theorem and used the theorem to discuss the existence of solutions to the following discontinuous variational-hemivariational inequality problem

$$\begin{aligned}
&-\int_\Omega \nabla u(x)\cdot\nabla(v(x)-u(x))\,\mathrm{d}x\\
&\le \lambda\int_\Omega [J^\circ(x,u(x);v(x)-u(x)) + (\mu K)^\circ(x,u(x);v(x)-u(x))]\,\mathrm{d}x.
\end{aligned}$$

Kristály et al [16] generalized a result of Ricceri concerning the existence of three critical points of certain nonsmooth functional, also gave two applications, both in the theory of differential inclusions. Our approach is based on the nonsmooth critical point for non-differential functions due to Chang [5] and the nonsmooth three critical points which was proved by Iannizzotto [14]. Compared with the results in [14, 16, 17], our framework presents new nontrivial difficulties. In particular, the presence of set-valued reaction terms ∂G and ∂F require completely different devices in order to verify the appropriate conditions.

We say that $u = (u_1, \dots, u_n) \in (W^{1,2}([a,b]))^n$ is a weak solution of (1.2) if the following conditions are satisfied

$$\begin{aligned} &\int_a^b u_i'(x)v_i'(x)\,\mathrm{d}x + \int_a^b u_i(x)v_i(x)\,\mathrm{d}x \\ &+ \lambda \int_a^b F^0_{u_i}(u_1(x), \dots, u_n(x); -v_i(x))\,\mathrm{d}x \\ &+ \mu \int_a^b G^0_{u_i}(u_1(x), \dots, u_n(x); -v_i(x))\,\mathrm{d}x \geq 0 \end{aligned} \tag{1.3}$$

for $1 \leq i \leq n$ and all $v = (v_1, \dots, v_n) \in (W^{1,2}([a,b]))^n$. Moreover we assume that the nonsmooth potential functions F and G satisfy the following assumptions:

(A1) F and G are regular on $\mathbb{R}^n$ (in the sense of Clarke [6]);

(A2) There exists $k_1 > 0$ and $a_1 > 0$ such that $|\omega_1| + \dots + |\omega_n| \leq k_1(|u_1| + \dots + |u_n|) + a_1$ for all $(u_1, \dots, u_n) \in \mathbb{R}^n$ and all $\omega_i \in \partial_{u_i} F(u_1, \dots, u_n)$ $(1 \leq i \leq n)$;

(A3) There exists $k_2 > 0$ and $a_2 > 0$ such that $|\xi_1| + \dots + |\xi_n| \leq k_2(|u_1| + \dots + |u_n|) + a_2$ for all $(u_1, \dots, u_n) \in \mathbb{R}^n$ and all $\xi_i \in \partial_{u_i} G(u_1, \dots, u_n)$ $(1 \leq i \leq n)$.

Our main results are the following:

Theorem 1.1. *Assume that* (A1)–(A3) *are satisfied and there exist* $2n+3$ *positive constants* $d, e, r\eta_i, \gamma_i$, *for* $1 \leq i \leq n$, *such that* $d + e < b - a$, $0 < k_1 < \frac{M_1}{2nr}$ *and*

$$\sum_{i=1}^n \eta_i^2 \leq c \sum_{i=1}^n \gamma_i^2,$$

where $c = de(b - a - \frac{4}{5}(d+e)) + \frac{4}{3}(d+e)$, *and*

(A4) $F(\zeta_1, \dots, \zeta_n) \geq 0$ *for each* $\zeta_i \in [0, de\gamma_i]$ $(1 \leq i \leq n)$ *and* $F(0, \dots, 0) = 0$;

(A5)
$$\begin{aligned} M_1 = \frac{\sum_{i=1}^n \eta_i^2}{c\sum_{i=1}^n \gamma_i^2}&(b - a - (d+e))F(de\gamma_1, \dots, de\gamma_n) \\ &- (b-a) \max_{(\zeta_1,\dots,\zeta_n) \in A_1} F(\zeta_1, \dots, \zeta_n) > 0, \end{aligned}$$

where
$$A_1 = \{(\zeta_1, \dots, \zeta_n) | \sum_{i=1}^n \zeta_i^2 \leq \frac{2de}{b-a} \sum_{i=1}^n \eta_i^2\};$$

then, there exist $\lambda', \lambda'' \in (0, \nu]$, $0 < \nu < \frac{1}{2nk_1}$, $\lambda' < \lambda''$, $\mu_1 > 0$ *and* $\sigma_1 > 0$ *such that for every* $\lambda \in [\lambda', \lambda'']$ *and* $\mu \in (0, \mu_1)$, *system* (1.2) *has at least three positive solutions in* $(W^{1,2}([a,b]))^n$ *whose norms are less than* σ_1.

If $n = 1$, then system (1.2) turns into

$$\begin{gathered} -u'' + u \in \lambda \partial_u F(u) + \mu \partial_u G(u), \\ u'(a) = u'(b) = 0, \quad u \geq 0. \end{gathered} \tag{1.4}$$

From Theorem 1.1, we have the following result.

Corollary 1.2. *Assume that the following conditions are satisfied:*

(A1') F *and* G *are regular on* $\mathbb{R}$;

(A2') *There exists* $k_3 > 0$ *and* $a_3 > 0$ *such that* $|\omega| \leq k_3|u| + a_3$ *for all* $u \in \mathbb{R}$ *and all* $\omega \in \partial_u F(u)$;

(A3') *There exists* $k_4 > 0$ *and* $a_4 > 0$ *such that* $|\xi| \leq k_4|u| + a_4$ *for all* $u \in \mathbb{R}$ *and all* $\xi \in \partial_u G(u)$;

and there exist five positive constants d, e, r, η, γ *such that* $d + e < b - a$, $k_3 < \frac{M_2}{2r}$ *and* $\eta^2 \leq c\gamma^2$, *where* $c = de(b - a - \frac{4}{5}(d + e)) + \frac{4}{3}(d + e)$, *and*

(A4') $F(u) \geq 0$ *for all* $u \in [0, de\gamma]$ *and* $F(0) = 0$;

(A5') $M_2 = \frac{\eta^2}{c\gamma^2}(b - a - (d + e))F(de\gamma) - (b - a)\max_{u \in A_2} F(u) > 0$, *where* $A_2 = \{u| - \eta(\frac{2de}{b-a})^{1/2} \leq u \leq \eta(\frac{2de}{b-a})^{1/2}\}$;

then, there exist $\lambda', \lambda'' \in (0, \nu]$, $0 < \nu < \frac{1}{2nk_3}$, $\lambda' < \lambda''$, $\mu_1 > 0$ *and* $\sigma_1 > 0$ *such that for every* $\lambda \in [\lambda', \lambda'']$ *and* $\mu \in (0, \mu_1)$, *system* (1.4) *has at least three positive solutions in* $W^{1,2}([a, b])$ *whose norms are less than* σ_1.

Next, we give an example that illustrate Theorem 1.1 (and Corollary 1.2). Set

$$F(u) = \begin{cases} 10^{199}e^{-9900}ue^{-u^3} & u \geq 10, \\ u^{200}e^{-u^4} & 0 < u < 10, \\ 0 & u \leq 0, \end{cases} \quad G(u) = \begin{cases} u & u \geq 1, \\ u^2 & 0 < u < 1, \\ 0 & u \leq 0, \end{cases}$$

and choose $a = 0$, $b = 1$, $d = 0.25$, $e = 0.5$, $\eta = 0.5$, $\gamma = 16$. Then it is easy to check that $F(u)$ and $G(u)$ satisfy the assumptions in Theorem 1.1.

2. Preliminaries

In this section we state some definitions and lemmas, which will be used in this article. First of all, we give some definitions: $(X, \|\cdot\|)$ denotes a (real) Banach space and $(X^*, \|\cdot\|_*)$ its topological dual. While $x_n \to x$ (respectively, $x_n \rightharpoonup x$) in X means the sequence $\{x_n\}$ converges strongly (respectively, weakly) in X.

Definition 2.1. A function φ: $X \to \mathbb{R}$ is locally Lipschitz if for every $u \in X$ there exist a neighborhood U of u and $L > 0$ such that for every $\nu, \omega \in U$,

$$|\varphi(\nu) - \varphi(\omega)| \leq L\|\nu - \omega\|.$$

If φ is locally Lipschitz on bounded sets, then clearly it is locally Lipschitz.

Definition 2.2. Let $\varphi : X \to \mathbb{R}$ be a locally Lipschitz functional and $u, \nu \in X$, the generalized derivative of φ in u along the direction ν, is

$$\varphi^0(u; \nu) = \limsup_{\omega \to u, \tau \to 0^+} \frac{\varphi(\omega + \tau\nu) - \varphi(\omega)}{\tau}.$$

It is easy to see that the function $\nu \to \varphi^0(u; \nu)$ is sublinear, continuous and so is the support function of a nonempty, convex and w^*- compact set $\partial\varphi(u) \subset X^*$,

$$\partial\varphi(u) = \{u^* \in X^* : \langle u^*, \nu\rangle_X \leq \varphi^0(u; \nu)\}.$$

If $\varphi \in C^1(X)$, then $\partial\varphi(u) = \{\varphi'(u)\}$.

Clearly, these definitions extend those of the Gâteaux directional derivative and gradient.

Definition 2.3. A mapping $A : X \to X^*$ is of type $(S)_+$, for every sequence $\{u_n\}$ such that $u_n \rightharpoonup u \in X$ and

$$\limsup_n \langle A(u_n), u_n - u\rangle \leq 0,$$

one has $u_n \to u$.

Definition 2.4. Let $\varphi : X \to \mathbb{R}$ be a locally Lipschitz functional and $\mathcal{X} : X \to \mathbb{R} \cup \{+\infty\}$ be a proper, convex, lower semicontinuous (l.s.c.) functional whose restriction to the set

$$\operatorname{dom}(\mathcal{X}) = \{u \in X : \mathcal{X}(u) < +\infty\}$$

is continuous, then $\varphi + \mathcal{X}$ is a Motreanu-Panagiotopouls functional.

In most applications, C is a nonempty, closed, convex subset of X; the indicator of C is the function $\mathcal{X}_C : X \to \mathbb{R} \cup \{+\infty\}$ defined by

$$\mathcal{X}_C = \begin{cases} 0 & \text{if } u \in C, \\ +\infty & \text{if } u \notin C. \end{cases}$$

It is easy to see that $\mathcal{X}_C$ is proper, convex and l.s.c., while its restriction to $\operatorname{dom}(\mathcal{X}) = C$ is the constant 0.

Definition 2.5. Let $\varphi + \mathcal{X}$ be a Motreanu-Panagiotopouls functional, $u \in X$. Then, u is a critical point of $\varphi + \mathcal{X}$ if for every $v \in X$

$$\varphi^\circ(u; v - u) + \mathcal{X}(v) - \mathcal{X}(u) \geq 0.$$

The next propositions will be used later.

Proposition 2.6 ([6])**.** *Let $h : X \to \mathbb{R}$ be locally Lipschitz on X. Then*

(i) $h^\circ(u; v) = \max\{\langle \omega, v\rangle_X : \omega \in \partial h(u)\}$ *for all* $u, v \in X$,

(ii) *(Lebourg's mean value theorem) Let u and v be two points in X, then there exists a point ζ in the open segment between u and v and $\omega_\zeta \in \partial h(\zeta)$ such that*

$$h(u) - h(v) = \langle \omega_\zeta, u - v\rangle_X.$$

We say that h is regular at $u \in X$ (in the sense of Clarke [6]) if for all $z \in X$ the usual one-sided directional derivative

$$h'(u; z) = \lim_{t \to 0^+} \frac{h(u + tz) - h(u)}{t}$$

exists and $h'(u; z) = h^\circ(u; z)$. Moreover, we say that h is regular on X, if it is regular in every point $u \in X$.

Proposition 2.7. *Let $h : X \to \mathbb{R}$ be a locally Lipschitz function which is regular at $(u_1, \dots, u_n) \in X$, then*

(i)

$$\partial h(u_1, \dots, u_n) \subset \partial_{u_1} h(u_1, \dots, u_n) \times \dots \times \partial_{u_n} h(u_1, \dots, u_n),$$

where $\partial_{u_i} h(u_1, \dots, u_n)$ denotes the partial generalized gradient of $h(u_1, \dots, u_i, \dots, u_n)$ to u_i for $1 \leq i \leq n$.

(ii) $h^\circ(u_1, \dots, u_n; v_1, \dots, v_n) \leq h_1^\circ(u_1, \dots, u_n; v_1) + \dots + h_n^\circ(u_1, \dots, u_n; v_n)$ *for all* $(v_1, \dots, v_n) \in X$.

Proof. For the proof of (i), see [6, Proposition 2.3.15]. From Proposition 2.6 (i), it follows that there exists a $\omega \in \partial h(u,v)$ such that $h^\circ(u;v) = \langle \omega, v\rangle_X$. From (i) we have $\omega = (\omega_1,\dots,\omega_n)$, where $\omega_i \in \partial_{u_i} h(u_1,\dots,u_n)$ $(1 \le i \le n)$, and using the definition of the generalized gradient, we derive $h^\circ(u;v) = \langle \omega_1, v_1\rangle_{W^{1,2}([a,b])} + \dots + \langle \omega_n, v_n\rangle_{W^{1,2}([a,b])} \le h_1^\circ(u_1,\dots,u_n;v_1) + \dots + h_n^\circ(u_1,\dots,u_n;v_n)$. □

The following theorems are the main tools for proving our main results.

Theorem 2.8 (see [14]). *Let $(X,\|\cdot\|)$ be a reflexive Banach space, $\Lambda \subset \mathbb{R}$ an interval, C a nonempty, closed, convex subset of X, $\mathcal{N} \in C^1(X,\mathbb{R})$ a sequentially weakly l.s.c. functional, bounded on any bounded subset of X, such that $\mathcal{N}'$ is of type $(S)_+$, $\Gamma : X \to \mathbb{R}$ is a locally Lipschitz functional with compact gradient, and $\rho_1 \in \mathbb{R}$. Assume also that the following conditions hold:*

- (i) $\sup_{\lambda\in\Lambda}\inf_{u\in C}[\mathcal{N}(u)+\lambda(\rho_1-\Gamma(u))] < \inf_{u\in C}\sup_{\lambda\in\Lambda}[\mathcal{N}(u)+\lambda(\rho_1-\Gamma(u))]$;
- (ii) $\lim_{\|u\|\to+\infty}[\mathcal{N}(u)-\lambda\Gamma(u)] = +\infty$ *for every* $\lambda \in \Lambda$.

Then there exist λ', $\lambda'' \in \Lambda$ $(\lambda' < \lambda'')$ and $\sigma_1 > 0$ such that for every $\lambda \in [\lambda',\lambda'']$ and every locally Lipschitz functional $\mathcal{G} : X \to \mathbb{R}$ with compact gradient, there exists $\mu_1 > 0$ such that for every $\mu \in (0,\mu_1)$, the functional $\mathcal{N} - \lambda\Gamma - \mu\mathcal{G} + \mathcal{X}_C$ has at least three critical points whose norms are less than σ_1.

Theorem 2.9 ([2]). *Let X be a nonempty set and Φ, Ψ two real functions on X. Assume that $\Phi(u) \ge 0$ for every $u \in X$ and there exists $u_0 \in X$ such that $\Phi(u_0) = \Psi(u_0) = 0$. Further, assume that there exist $u_1 \in X$, $r > 0$ such that $\Phi(u_1) > r$ and*

$$\sup_{\Phi(u)<r}\Psi(u) < r\frac{\Psi(u_1)}{\Phi(u_1)}.$$

Then, for every $h > 1$ and for every $\rho \in \mathbb{R}$ satisfying

$$\sup_{\Phi(u)<r}\Psi(u) + \frac{r\frac{\Psi(u_1)}{\Phi(u_1)} - \sup_{\Phi(u)<r}\Psi(u)}{h} < \rho < r\frac{\Psi(u_1)}{\Phi(u_1)},$$

one has

$$\sup_{\lambda\in\mathbb{R}}\inf_{u\in X}[\Phi(u)+\lambda(-\Psi(u)+\rho)] < \inf_{u\in X}\sup_{\lambda\in[0,\nu]}[\Phi(u)+\lambda(-\Psi(u)+\rho))],$$

where

$$\nu = \frac{hr}{r\frac{\Psi(u_1)}{\Phi(u_1)} - \sup_{\Phi(u)<r}(\Phi(u))}.$$

3. Proof of main results

Let $X = (W^{1,2}([a,b]))^n$ equipped with the norm

$$\|(u_1,\dots,u_n)\| = \Big(\sum_{i=1}^n \|u_i\|^2\Big)^{1/2},$$

where $\|u_i\| = (\int_a^b(|u_i'(x)|^2 + |u_i(x)|^2)\,\mathrm{d}x)^{1/2}$ for $1 \le i \le n$, and we introduce the functionals $\Phi, \Psi : X \to \mathbb{R}$ for each $u = (u_1,\dots,u_n) \in X$, as follows

$$\Phi(u) = \sum_{i=1}^n \frac{1}{2}\|u_i\|^2, \quad \Psi(u) = \int_a^b F(u_1(x),\dots,u_n(x))\,\mathrm{d}x$$

and

$$J(u)=\int_a^b G(u_1(x),\dots,u_n(x))\,\mathrm{d}x.$$

Let $C=\{u\in X: u(x)\geq 0 \text{ for every } x\in[a,b]\}$, then for all $\lambda,\mu>0$ and $u\in X$,

$$\varphi(u)=\Phi(u)-\lambda\Psi(u)-\mu J(u)+\mathcal{X}_C(u).$$

The next lemma displays some properties of Φ.

Lemma 3.1. $\Phi\in C^1(X,\mathbb{R})$ *and its gradient, defined for* $u,v\in X$ *by*

$$\langle\Phi'(u),v\rangle=\int_a^b \nabla u(x)\nabla v(x)\,\mathrm{d}x$$

is of type $(S)_+$*, where* $u=(u_1,\dots,u_n)$ *and* $v=(v_1,\dots,v_n)$*.*

The proof of the above lemma is similar to the one in Chabrowski [4, Section 2.2]. We omit it here. Next we consider some properties of Ψ.

Lemma 3.2. *If* (A1)–(A2) *are satisfied, then* $\Psi(u):X\to\mathbb{R}$ *is a locally Lipschitz function with compact gradient. Moreover,*

$$\Psi^\circ(u_1,\dots,u_n;v_1,\dots,v_n)\leq\int_a^b F^\circ(u_1,\dots,u_n;v_1,\dots,v_n)\,\mathrm{d}x, \tag{3.1}$$

for all $(u_1,\dots,u_n),(v_1,\dots,v_n)\in X$*.*

Proof. First, let $u=(u_1,\dots,u_n)$, $v=(v_1,\dots,v_n)\in X$ be fixed elements. Using the regularity of F and Lebourg's mean value theorem (see Proposition 2.6) we derive a $\omega\in\partial F(\zeta_1,\dots,\zeta_n)$ such that

$$F(u)-F(v)=\langle\omega,u-v\rangle,$$

where $(\zeta_1,\dots,\zeta_n)$ is in the open line segment between $(u_1,\dots,u_n)$ and $(v_1,\dots,v_n)$. Using Proposition 2.7, there exist $\omega_i\in\partial_{\zeta_i}F(\zeta_1,\dots,\zeta_n)$ $(1\leq i\leq n)$, such that

$$F(u_1,\dots,u_n)-F(v_1,\dots,v_n)=\omega_1(u_1-v_1)+\dots+\omega_n(u_n-v_n). \tag{3.2}$$

From (A1) and (3.2), we obtain

$$\begin{aligned}
&|F(u_1,\dots,u_n)-F(v_1,\dots,v_n)|\\
&\leq(|\omega_1|+\dots+|\omega_n|)(|u_1-v_1|+\dots+|u_n-v_n|)\\
&\leq[k_1(|u_1|+\dots+|u_n|+|v_1|+\dots+|v_n|)+a_1](|u_1-v_1|+\dots+|u_n-v_n|).
\end{aligned} \tag{3.3}$$

Using (3.3), Hölder's inequality and the fact the embedding $W^{1,2}([a,b])\hookrightarrow L^2([a,b])$ is continuous, we derive

$$\begin{aligned}
&|\Psi(u_1,\dots,u_n)-\Psi(v_1,\dots,v_n)|\\
&\leq nk_1\sum_{i=1}^n(\|u_i\|_2+\|v_i\|_2+m_1)(\|u_1-v_1\|_2+\dots+\|u_n-v_n\|_2)\\
&\leq c_1\sum_{i=1}^n(\|u_i\|+\|v_i\|+m_1)(\|u_1-v_1\|+\dots+\|u_n-v_n\|)
\end{aligned}$$

for some $c_1>0$, $m_1>0$ and $\|\cdot\|_2$ denotes the L^2-norm. From this relation it follows that Ψ is locally Lipschitz on X.

Now choose $u = (u_1, \dots, u_n)$, $h = (h_1, \dots, h_n) \in X$, since $F(u_1, \dots, u_n)$ is continuous, $F^\circ(u_1, \dots, u_n; h_1, \dots, h_n)$ can be expressed as the upper limit of

$$\frac{F(u_1^0 + th_1, \dots, u_n^0 + th_n) - F(u_1^0, \dots, u_n^0)}{t},$$

where $t \to 0^+$ taking rational values and $(u_1^0, \dots, u_n^0) \to (u_1, \dots, u_n)$ taking values in a countable dense subset of X. Therefore, the map

$$x \mapsto F^\circ(u_1(x), \dots, u_n(x); h_1(x), \dots, h_n(x))$$

is also measurable. By (A2), the map $x \mapsto F^\circ(u_1(x), \dots, u_n(x); h_1(x), \dots, h_n(x))$ belongs to $L^1([a,b])$. Since X is separable, there exist functions $(u_1^k, \dots, u_n^k) \in X$ and numbers $t_k \to 0^+$ such that $(u_1^k, \dots, u_n^k) \to (u_1, \dots, u_n)$ in X and

$$\Psi^\circ(u_1, \dots, u_n; h_1, \dots, h_n) = \lim_{k \to +\infty} \frac{\Psi(u_1^k + t_k h_1, \dots, u_n^k + t_k h_n) - \Psi(u_1^k, \dots, u_n^k)}{t_k}.$$

We define $g_n : [a,b] \to \mathbb{R} \cup \{+\infty\}$ by

$$\begin{aligned} g_k(x) = &-\frac{F(u_1^k + t_k h_1, \dots, u_n^k + t_k h_n) - F(u_1^k, \dots, u_n^k)}{t_k} + k_1(|h_1| + \dots + |h_n|) \\ &\times \Big(|u_1^k| + \dots + |u_n^k| + |u_1^k + t_k h_1| + \dots + |u_n^k + t_k h_n| + \frac{a_1}{k_1}\Big). \end{aligned}$$

Then the function g_k is measurable and nonnegative (see (3.3)). From Fatou's Lemma, we have

$$I = \limsup_{k \to +\infty} \int_a^b [-g_k(x)]\,\mathrm{d}x \le \int_a^b \limsup_{k \to +\infty} [-g_k(x)]\,\mathrm{d}x = H.$$

Let $L_k = B_k + g_k$, where

$$B_k(x) = \frac{F(u_1^k + t_k h_1, \dots, u_n^k + t_k h_n) - F(u_1^k, \dots, u_n^k)}{t_k}.$$

From the Lebesgue dominated convergence theorem, we obtain

$$\begin{aligned} &\limsup_{k \to +\infty} \int_a^b L_k(x)\,\mathrm{d}x \\ &= 2k_1 \int_a^b (|h_1(x)| + \dots + |h_n(x)|)\Big(|u_1(x)| + \dots + |u_n(x)| + \frac{a_1}{2k_1}\Big)\,\mathrm{d}x. \end{aligned}$$

Hence, we derive

$$\begin{aligned} I &= \limsup_{k \to +\infty} \frac{\Psi(u_1^k + t_k h_1, \dots, u_n^k + t_k h_n) - \Psi(u_1^k, \dots, u_n^k)}{t_k} - \lim_{k \to +\infty} \int_a^b L_k\,\mathrm{d}x \\ &= \Psi^\circ(u_1, \dots, u_n; h_1, \dots, h_n) - 2k_1 \int_a^b (|h_1(x)| + \dots + |h_n(x)|) \\ &\quad \times \Big(|u_1(x)| + \dots + |u_n(x)| + \frac{a_1}{2k_1}\Big)\,\mathrm{d}x. \end{aligned}$$

Now, we obtain the estimates $H \le H_B - H_L$, where $H_B = \int_a^b \limsup_{k \to +\infty} B_k(x)\,\mathrm{d}x$ and $H_L = \int_a^b \liminf_{k \to +\infty} L_k(x)\,\mathrm{d}x$. Since $(u_1^k(x), \dots, u_n^k(x)) \to (u_1(x), \dots, u_n(x))$ a.e. in $[a,b]$ and $t_k \to 0^+$, we derive

$$H_L = 2k_1 \int_a^b (|h_1(x)| + \dots + |h_n(x)|)\Big(|u_1(x)| + \dots + |u_n(x)| + \frac{a_1}{2k_1}\Big)\,\mathrm{d}x.$$

On the other hand,

$$
\begin{aligned}
H_B &= \int_a^b \limsup_{k\to+\infty} \frac{F(u_1^k + t_k h_1, \dots, u_n^k + t_k h_n) - F(u_1^k, \dots, u_n^k)}{t_k}\,\mathrm{d}x \\
&\le \int_a^b \limsup_{(u_1^0,\dots,u_n^0)\to(u_1,\dots,u_n),\, t\to 0^+} \frac{F(u_1^0 + t h_1, \dots, u_n^0 + t h_n) - F(u_1^0, \dots, u_n^0)}{t}\,\mathrm{d}x \\
&= \int_a^b F^\circ(u_1,\dots,u_n; h_1,\dots,h_n)\,\mathrm{d}x,
\end{aligned}
$$

which implies (3.1).

At last, we prove that $\partial\Psi$ is compact. Let $\{u^k\}_{k\ge1}$ be a sequence in X, where $u^k = (u_1^k, \dots, u_n^k)$, such that $\|u^k\| \le M$ and choose $\omega^k \in \partial\Psi(u^k)$, where $\omega^k = (\omega_1^k, \dots, \omega_n^k)$, $k \ge 1$, $k \in \mathbb{N}$ and $M > 0$. From (A1), for every $v = (v_1, \dots, v_n) \in X$, we obtain

$$
\begin{aligned}
&\langle \omega^k, v\rangle \\
&\le \int_a^b |\omega^k(x)||v(x)|\,\mathrm{d}x \le \int_a^b k_1\Big(|u_1^k| + \dots + |u_n^k| + \frac{a_1}{k_1}\Big)(|v_1| + \dots + |v_n|)\,\mathrm{d}x \\
&\le k_1\Big[\Big(\int_a^b (|u_1^k| + \dots + |u_n^k|)^2\,\mathrm{d}x\Big)^{1/2} + \frac{a_1}{k_1}(b-a)^{1/2}\Big] \\
&\quad \times \Big(\int_a^b (|v_1| + \dots + |v_n|)^2\,\mathrm{d}x\Big)^{1/2} \\
&= k_1\Big[\Big(\int_a^b (|u_1^k|^2 + \dots + |u_n^k|^2 + 2|u_1^k||u_2^k| + \dots + 2|u_{n-1}^k||u_n^k|)\,\mathrm{d}x\Big)^{1/2} \\
&\quad + \frac{a_1}{k_1}(b-a)^{1/2}\Big]\Big(\int_a^b (|v_1|^2 + \dots + |h_n|^2 + 2|v_1||v_2| + \dots + 2|v_{n-1}||v_n|)\,\mathrm{d}x\Big)^{1/2} \\
&\le k_1\Big[n\Big(\int_a^b (|u_1^k|^2 + \dots + |u_n^k|^2)\,\mathrm{d}x\Big)^{1/2} + \frac{a_1}{k_1}(b-a)^{1/2}\Big] \\
&\quad \times \Big(\int_a^b (|v_1|^2 + \dots + |v_n|^2)\,\mathrm{d}x\Big)^{1/2} \\
&\le \Big(c^1\|u^k\| + \frac{a_1}{k_1}(b-a)^{1/2}\Big)\|v\| \\
&\le \Big(c^1 M + \frac{a_1}{k_1}(b-a)^{1/2}\Big)\|v\|,
\end{aligned}
$$

where c^1 is a positive constant. Hence

$$
\|\omega^k\|_* \le c^1 M + \frac{a_1}{k_1}(b-a)^{1/2} = c^2\,.
$$

This means that $\{\omega^k\}$ is bounded. Passing to a subsequence $\omega^k \rightharpoonup \omega \in X^*$, where $\omega = (\omega_1, \dots, \omega_n)$. We need to prove that the convergence is strong.

We proceed by contradiction. Suppose that there exists $\varepsilon > 0$, such that for every $k \in \mathbb{N}$

$$
\|\omega^k - \omega\|_* > \varepsilon,
$$

where $\omega^k = (\omega_1^k, \dots, \omega_n^k)$. That is for all $k \in \mathbb{N}$, there exists a $v^k = (v_1^k, \dots, v_n^k) \in B(0,1) \times \dots \times B(0,1)$ such that

$$
\langle \omega^k - \omega, v^k\rangle > \varepsilon\,. \tag{3.4}
$$

Since $\{v^k\}_{k\geq 1}$ is bounded, passing to a subsequence, $v_n \rightharpoonup v = (v_1^0, \dots, v_n^0) \in X$ and $\|v^k - v\| \to 0$, so for k big enough,

$$|\langle \omega^k - \omega, v\rangle| < \frac{\varepsilon}{3}, \quad |\langle \omega, v^k - v\rangle| < \frac{\varepsilon}{3}, \quad \|v^k - v\| < \frac{\varepsilon}{3c^2},$$

this implies

$$\begin{aligned}
\langle \omega^k - \omega, v^k\rangle &= \langle \omega^k - \omega, v\rangle + \langle \omega^k, v^k - v\rangle - \langle \omega, v^k - v\rangle \\
&\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + c^2\|v^k - v\| \\
&< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon,
\end{aligned}$$

which contradicts (3.4). The proof is complete. □

Analogously, we deduce the properties of the function J.

Lemma 3.3. *If* (A1) *and* (A3) *are satisfied, then* $J : X \to \mathbb{R}$ *is a locally Lipschitz function with compact gradient and*

$$J^\circ(u_1, \dots, u_n; v_1, \dots, v_n) \leq \int_a^b G^\circ(u_1, \dots, u_n; v_1, \dots, v_n)\,\mathrm{d}x$$

for all $(u_1, \dots, u_n)$, $(v_1, \dots, v_n) \in X$.

Now, we are in a position to establish the following proposition.

Lemma 3.4. *If* (A1)–(A3) *are satisfied, then, for every* $\lambda, \mu > 0$, $\varphi : X \to \mathbb{R} \cup \{+\infty\}$ *is a Motreanu-Panagiotopoulos function and the critical points* $(u_1, \dots, u_n)$ *belong to* X *of* φ *is a weak solution of* (1.1).

Proof. From Lemmas 3.1–3.3 the function $I = \Phi - \lambda\Psi - \mu J$ is locally Lipschitz; furthermore, C is a closed convex subset of X and $C \neq \emptyset$; thus φ is a Motreanu-Panagiotopoulos function. Since $(u_1, \dots, u_n) \in X$ is a critical point of φ, then $u \in C$ and for all $v = (v_1, \dots, v_n) \in C$ we have

$$\begin{aligned}
0 &\leq I^\circ(u_1, \dots, u_n; v_1, \dots, v_n) \\
&= \int_a^b \sum_{i=1}^n (u_i'v_i' + u_iv_i)\,\mathrm{d}x + \lambda(-\Psi)^\circ(u_1, \dots, u_n; v_1, \dots, v_n) \\
&\quad + \mu(-J)^\circ(u_1, \dots, u_n; v_1, \dots, v_n) \\
&\leq \int_a^b \sum_{i=1}^n (u_i'v_i' + u_iv_i)\,\mathrm{d}x + \lambda \int_a^b F^\circ(u_1, \dots, u_n; -v_1, \dots, -v_n)\,\mathrm{d}x \\
&\quad + \mu \int_a^b G^\circ(u_1, \dots, u_n; -v_1, \dots, -v_n)\,\mathrm{d}x.
\end{aligned}$$

From Proposition 2.7 (ii), we have

$$\begin{aligned}
0 &\leq \int_a^b \sum_{i=1}^n (u_i'v_i' + u_iv_i)\,\mathrm{d}x + \lambda \int_a^b F_{u_1}^\circ(u_1, \dots, u_n; -v_1)\,\mathrm{d}x + \dots \\
&\quad + \lambda \int_a^b F_{u_n}^\circ(u_1, \dots, u_n; -v_n)\,\mathrm{d}x \\
&\quad + \mu \int_a^b G_{u_1}^\circ(u_1, \dots, u_n; -v_1)\,\mathrm{d}x + \dots + \mu \int_a^b G_{u_n}^\circ(u_1, \dots, u_n; -v_n)\,\mathrm{d}x.
\end{aligned}$$

Taking $v_1 = \dots = v_{i-1} = v_{i+1} = \dots = v_n = 0$ in the above inequality for $1 \le i \le n$, then we lead to (1.3), i.e., $(u_1, \dots, u_n)$ is a weak solution of (1.2). $\square$

Proof of Theorem 1.1. We apply Theorem 2.8 to prove this theorem. For this purpose, it is easy to see that X is a reflexive Banach space. We put $\Lambda = (0, \nu]$, where $0 < \nu < \frac{1}{2nk_1}$. The functional $\Phi \in C^1(X, \mathbb{R})$ is continuous and convex, hence weakly l.s.c. and obviously bounded on any bounded subset of X. Moreover, Φ' is of type (S_+) (Lemma 3.1) and Ψ is a locally Lipschitz function with compact gradient (Lemma 3.2). We only need to test conditions (i) and (ii) in Theorem 2.8.

We first check condition (i). Let $v(x) = (v_1(x), \dots, v_n(x))$ such that for $1 \le i \le n$,

$$v_i(x) = \begin{cases} \frac{e\gamma_i}{d}(x-a)^2 & \text{if } a \le x < a+d, \\ de\gamma_i & \text{if } a+d \le x \le b-e, \\ \frac{d\gamma_i}{e}(b-x)^2 & \text{if } b-e \le x \le b. \end{cases} \tag{3.5}$$

It is obvious that $v \in X$. From a simple computation, we have

$$\int_a^b (|v_i'(x)|^2 + |v_i(x)|^2)\,\mathrm{d}x = \Big(d^2e^2\Big(b-a-\frac{4(d+e)}{5}\Big) + \frac{4}{3}de(d+e)\Big)\gamma_i^2. \tag{3.6}$$

Set $r = \frac{de}{2}\sum_{i=1}^n \eta_i^2$, since $\sum_{i=1}^n \eta_i^2 \le c\sum_{i=1}^n \gamma_i^2$, from (3.6), we obtain

$$\Phi(v_1, \dots, v_n) = \frac{dec}{2}\sum_{i=1}^n \gamma_i^2 > \frac{de}{2}\sum_{i=1}^n \eta_i^2 = r.$$

Let $u_0 = (0, \dots, 0)$, $u_1 = (v_1, \dots, v_n)$. From (A4), we have $\Psi(u_0) = 0$, and $\Phi(u_0) = 0$. From [1], we obtain

$$\max_{x \in [a,b]} |u_i(x)| \le \Big(\frac{2}{b-a}\Big)^{1/2} \|u_i\|$$

for all $u_i \in W^{1,2}([a,b])$, $1 \le i \le n$. Hence

$$\sup_{x \in [a,b]} \sum_{i=1}^n \frac{|u_i(x)|^2}{2} \le \frac{2}{b-a}\sum_{i=1}^n \frac{\|u_i(x)\|^2}{2} \tag{3.7}$$

for all $u = (u_1, \dots, u_n) \in X$. From (3.7), for each $r > 0$,

$$\begin{aligned} &\Phi^{-1}((-\infty, r)) \\ &= \{u = (u_1, \dots, u_n) \in X : \Phi(u) < r\} \\ &= \Big\{u = (u_1, \dots, u_n) \in X : \sum_{i=1}^n \frac{\|u_i\|^2}{2} < r\Big\} \\ &\subset \Big\{u = (u_1, \dots, u_n) \in X : \sum_{i=1}^n |u_i|^2 < \frac{4r}{b-a} \text{ for all } x \in [a,b]\Big\}. \end{aligned} \tag{3.8}$$

By (A5), we have

$$\frac{\sum_{i=1}^n \eta_i^2}{c\sum_{i=1}^n \gamma_i^2}(b-a-(d+e))F(de\gamma_1, \dots, de\gamma_n) > (b-a)\max_{(\zeta_1, \dots, \zeta_n) \in A_1} F(\zeta_1, \dots, \zeta_n). \tag{3.9}$$

From (A4), (A5), (3.5), (3.8) and (3.9), for all $u=(u_1,\dots,u_n)\in X$, we have

$$
\begin{aligned}
\sup_{u\in\Phi^{-1}((-\infty,r))}\Psi(u) &\le \sup_{\sum_{i=1}^n|u_i(x)|^2\le\frac{4r}{b-a}}\int_a^b F(u_1(x),\dots,u_n(x))\,\mathrm{d}x\\
&\le \int_a^b \sup_{(\zeta_1,\dots,\zeta_n)\in A_1} F(\zeta_1,\dots,\zeta_n)\,\mathrm{d}x\\
&< \frac{\sum_{i=1}^n\eta_i^2}{c\sum_{i=1}^n\gamma_i^2}(b-a-(d+e))F(de\gamma_1,\dots,de\gamma_n)\\
&\le \frac{r\int_{a+d}^{b-e}F(de\gamma_1,\dots,de\gamma_n)\,\mathrm{d}x}{\Phi(v_1,\dots,v_n)}\\
&\le \frac{r\int_a^b F(v_1,\dots,v_n)\,\mathrm{d}x}{\Phi(v_1,\dots,v_n)}=\frac{r\Psi(v)}{\Phi(v)}.
\end{aligned}\tag{3.10}
$$

Note that $0<k_1<\frac{M_1}{2nr}$. Fix $1<h<\frac{M_1}{2nk_1r}$ and ρ such that

$$
\sup_{\Phi(u)<r}\Phi(u)+\frac{r\frac{\Psi(v)}{\Phi(v)}-\sup_{\Phi(u)<r}\Psi(u)}{h}<\rho<r\frac{\Psi(v)}{\Phi(v)}.
$$

From Theorem 2.9, we obtain

$$
\sup_{\lambda\in\mathbb{R}}\inf_{u\in X}[\Phi(u)+\lambda(\rho-\Psi(u))]<\inf_{u\in X}\sup_{\lambda\in\Lambda}[\Phi(u)+\lambda(\rho-\Psi(u))].
$$

Next, we test condition (ii). From (A2), Lebourg's mean value theorem and Proposition 2.7, there exist $\omega_i\in\partial_{\zeta_i}F(\zeta_1,\dots,\zeta_n)$, where ζ_i is between the segment u_i and 0 for $1\le i\le n$, such that

$$
\begin{aligned}
F(u_1,\dots,u_n)&=F(u_1,\dots,u_n)-F(0,\dots,0)=\omega_1u_1+\dots+\omega_nu_n\\
&\le(|\omega_1|+\dots+|\omega_n|)(|u_1|+\dots+|u_n|)\\
&\le k_1(|u_1|+\dots+|u_n|)^2+a_1(|u_1|+\dots+|u_n|)\\
&=k_1(|u_1|^2+\dots+|u_n|^2+2|u_1||u_2|+\dots+2|u_{n-1}||u_n|)\\
&\quad+a_1(|u_1|+\dots+|u_n|)\\
&\le nk_1(|u_1|^2+\dots+|u_n|^2)+a_1(|u_1|+\dots+|u_n|).
\end{aligned}\tag{3.11}
$$

By (3.11), we can find two positive constants $\mathcal{K}$ and τ satisfying $\mathcal{K}\le nk_1$ and

$$
F(\zeta_1,\dots,\zeta_n)\le\mathcal{K}\sum_{i=1}^n\zeta_i^2+\tau
$$

for all $(\zeta_1,\dots,\zeta_n)\in\mathbb{R}^n$. Let $u=(u_1,\dots,u_n)\in X$, then it is easy to see that

$$
F(u_1,\dots,u_n)\le\mathcal{K}\sum_{i=1}^n|u_i(x)|^2+\tau\quad\text{a.e. } x\in[a,b].\tag{3.12}
$$

Choosing $\lambda\in(0,\nu]$, from (3.12), we obtain

$$
\begin{aligned}
\Phi(u)-\lambda\Psi(u)&=\sum_{i=1}^n\frac{\|u_i\|^2}{2}-\lambda\int_a^b F(u_1(x),\dots,u_n(x))\,\mathrm{d}x\\
&\ge\sum_{i=1}^n\frac{\|u_i\|^2}{2}-\lambda\mathcal{K}\sum_{i=1}^n\int_a^b|u_i(x)|^2\,\mathrm{d}x-\lambda(b-a)\tau
\end{aligned}
$$

$$
\begin{aligned}
&\geq \sum_{i=1}^{n} \frac{\|u_i\|^2}{2} - \lambda \mathcal{K} \sum_{i=1}^{n} \int_a^b |u_i(x)|^2 \, \mathrm{d}x - \frac{b-a}{2nk_1}\tau \\
&\geq \Big(\frac{1}{2} - \lambda \mathcal{K}\Big) \sum_{i=1}^{n} \|u_i\|^2 - \frac{b-a}{2nk_1}\tau.
\end{aligned}
$$

Since $0 < \mathcal{K} \leq nk_1$ and $0 < \nu < \frac{1}{2nk_1}$, we have

$$
\lim_{\|u\| \to +\infty} (\Phi(u) - \lambda \Psi(u)) = +\infty,
$$

then we have tested condition (ii). The proof is complete. □

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Ziqing Yuan

College of Mathematics and Econometrics, Hunan University, Changsha, Hunan 410082, China

E-mail address: junjyuan@sina.com

Lihong Huang (corresponding author)

College of Mathematics and Econometrics, Hunan University, Changsha, Hunan 410082, China

E-mail address: lhhuang@hnu.edu.cn

Chunyi Zeng

Department of Foundational Education, Southwest University for Nationalities, Chengdu, Sichuan 610000, China

E-mail address: ykbzcy@163.com