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EXISTENCE OF SOLUTIONS FOR QUASILINEAR ELLIPTIC EQUATIONS INVOLVING A NONLOCAL TERM

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Abstract. This article establishes the existence of solutions for a partial differential equation involving a quasilinear elliptic operator and a nonlocal term. The proofs of the main results are based on Schauder's fixed point theorem combined with variational arguments.

1. Introduction

Let $\Omega \subset \mathbb{R}^N$ denote a bounded domain with smooth boundary $\partial\Omega$, and ν denote the outward unit normal to $\partial\Omega$. We consider the problem

$$\begin{gathered} -a\Big(\int_\Omega u(x)\,dx\Big)\operatorname{div}(a(u(x))\nabla u(x)) + u(x) = 0, \quad x \in \Omega, \\ a(u(x))\frac{\partial u}{\partial \nu}(x) = g(x), \quad x \in \partial\Omega, \end{gathered} \tag{1.1}$$

where $a : \mathbb{R} \to \mathbb{R}$ is a continuous function for which there exist two constants $a_1, a_2 \in (0, \infty)$ such that

$$0 < a_1 \le a(t) \le a_2 < \infty, \quad \forall t \in \mathbb{R}, \tag{1.2}$$

and $g : \partial\Omega \to \mathbb{R}$ is a function satisfying

$$g \in L^2(\partial\Omega). \tag{1.3}$$

According to [6, p. 160], a physical motivation for studying equations of type (1.1) comes from the fact that the diffusion of the temperature in a material has a velocity given by the *Fourier law*

$$\vec{v} = -a\nabla u\,,$$

where a is a constant proper to each material. Imagining a material for which the constant is not the same for temperatures between 0^o and 200^o, i.e. it depends on the temperature of the material itself, a more realistic Fourier law can be written as

$$\vec{v} = -a(u)\nabla u\,.$$

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This last relation can lead to equations of type (1.1) since the expression above appears in the divergence operator from (1.1). For more physical motivations concerning problems of type (1.1) the reader may also consult [5, Chapter 1].

On the other hand, note that the presence of function a depending on u in the divergence form from equation (1.1) represents the main difficulty in analysing the existence of solutions for this problem since it does not enable one to associate to the problem a so called *energy functional* whose critical points would offer weak solutions to our equation. For that reason even if our treatment is in part variational we have to combine it with a fixed point argument offered by Schauder's fixed point theorem. Moreover, the presence of the nonlocal term $a(\int_\Omega u(x)\,dx)$ allows us to be able to control the number of solutions for our equation. Actually, as we will see in the next section, in the particular case when $\int_{\partial\Omega} g\,d\sigma(x) = 1$ the number of fixed points of function a gives the number of solutions of equation (1.1), and, thus, we may have a unique solution of the equation, a finite number of solutions or infinitely many solutions. Moreover, if $\int_{\partial\Omega} g\,d\sigma(x) = 1$ we can prescribe the number of solutions of our equation just by prescribing the number of fixed points of a. Finally, note that condition (1.2) on function a was first introduced in the pioneering paper by Arcoya & Boccardo [2] but it was no longer assumed in subsequent papers by Filippucci [7, 8]. For other interesting results related to nonlocal problems we also refer to [9] and [10].

2. Main result

In this article we are interested in analyzing the existence and multiplicity of weak solutions for problems of type (1.1). We start by recalling the definition of a weak solution for problem (1.1).

Definition 2.1. We say that $u \in H^1(\Omega)$ is a *weak solution* of problem (1.1) if

$$a\Big(\int_\Omega u(x)\,dx\Big)\Big[\int_\Omega a(u(x))\nabla u\nabla\varphi\,dx - \int_{\partial\Omega} g\varphi\,d\sigma(x)\Big] + \int_\Omega u\varphi\,dx = 0, \tag{2.1}$$

for all $\varphi \in H^1(\Omega)$.

The main result of this article read as follows.

Theorem 2.2. *Assume conditions* (1.2) *and* (1.3) *are fulfilled. If* $\int_{\partial\Omega} g(x)\,d\sigma(x) \neq 0$ *then* (1.1) *has as many weak solutions as equation* $a(\mu)\int_{\partial\Omega} g\,d\sigma(x) = \mu$ *while if* $\int_{\partial\Omega} g(x)\,d\sigma(x) = 0$ *then* (1.1) *has at least a weak solution.*

To prove Theorem 2.2 we establish first the following result whose proof will be given in the next section.

Theorem 2.3. *Assume conditions* (1.2) *and* (1.3) *are fulfilled. Then, for each* $\mu \in \mathbb{R}$ *fixed, the problem*

$$\begin{gathered} -a(\mu)\,\mathrm{div}(a(u(x))\nabla u(x)) + u(x) = 0, \quad x \in \Omega, \\ a(u(x))\frac{\partial u}{\partial \nu}(x) = g(x), \quad x \in \partial\Omega \end{gathered} \tag{2.2}$$

has a weak solution $u_\mu \in H^1(\Omega)$*, that is* $u_\mu \in H^1(\Omega)$ *satisfies*

$$a(\mu)\Big[\int_\Omega a(u_\mu(x))\nabla u_\mu\nabla\varphi\,dx - \int_{\partial\Omega} g\varphi\,d\sigma(x)\Big] + \int_\Omega u_\mu\varphi\,dx = 0,$$

for all $\varphi \in H^1(\Omega)$.

Now, we are ready to give the proof of Theorem 2.2.

Proof of Theorem 2.2. First, note that if $u \in H^1(\Omega)$ is a weak solution of (1.1), testing in (2.1) with $\varphi = 1$ we obtain

$$a\Big(\int_\Omega u(x)\,dx\Big)\int_{\partial\Omega} g\,d\sigma(x) = \int_\Omega u(x)\,dx\,. \tag{2.3}$$

On the one hand, if $\int_{\partial\Omega} g\,d\sigma(x) \neq 0$, then by (2.3) it follows that $\mu := \int_\Omega u(x)\,dx$ is a solution of the equation

$$a(\mu)\int_{\partial\Omega} g\,d\sigma(x) = \mu\,. \tag{2.4}$$

Next, let μ be a solution of the equation $a(\mu)\int_{\partial\Omega} g\,d\sigma(x) = \mu$. Then by Theorem 2.3 there exists $u_\mu \in H^1(\Omega)$ a weak solution of the problem

$$\begin{gathered} -a(\mu)\operatorname{div}(a(u(x))\nabla u(x)) + u(x) = 0, \quad x \in \Omega,\\ a(u(x))\frac{\partial u}{\partial \nu}(x) = g(x), \quad x \in \partial\Omega \end{gathered}$$

or

$$a(\mu)\Big[\int_\Omega a(u_\mu(x))\nabla u_\mu \nabla\varphi\,dx - \int_{\partial\Omega} g\varphi\,d\sigma(x)\Big] + \int_\Omega u_\mu\varphi\,dx = 0, \tag{2.5}$$

for all $\varphi \in H^1(\Omega)$. Testing in (2.5) with $\varphi = 1$ we obtain

$$a(\mu)\int_{\partial\Omega} g\,d\sigma(x) = \int_\Omega u_\mu\,dx\,.$$

Since μ is a solution of the equation $a(\mu)\int_{\partial\Omega} g\,d\sigma(x) = \mu$, we find that $\mu = \int_\Omega u_\mu\,dx$, and thus, u_μ is a solution of(1.1).

On the other hand, if $\int_{\partial\Omega} g\,d\sigma(x) = 0$ then relation (2.3) reads $\int_\Omega u(x)\,dx = 0$. Thus, in this situation we should seek solutions of problem (1.1) in

$$V := \big\{v \in H^1(\Omega);\ \int_\Omega v(x)\,dx = 0\big\}.$$

Recall that V is a closed and convex subspace of $H^1(\Omega)$ and $H^1(\Omega) = V \oplus \mathbb{R}$. Thus, in this particular case, $u \in V$ is a solution for problem (1.1) if u satisfies

$$a(0)\Big[\int_\Omega a(u(x))\nabla u\nabla\varphi\,dx - \int_{\partial\Omega} g\varphi\,d\sigma(x)\Big] + \int_\Omega u\varphi\,dx = 0, \quad \forall\varphi \in V. \tag{2.6}$$

Therefore, if $\int_{\partial\Omega} g\,d\sigma(x) = 0$ the study of the existence of solutions for (1.1) reduces to the study of the existence of solutions for the problem

$$\begin{gathered} -a(0)\operatorname{div}(a(u(x))\nabla u(x)) + u(x) = 0, \quad x \in \Omega,\\ a(u(x))\frac{\partial u}{\partial \nu}(x) = g(x), \quad x \in \partial\Omega. \end{gathered} \tag{2.7}$$

This case can be treated in a similar manner as the proof of Theorem 2.3 just by replacing $a(\mu)$ in (2.2), by $a(0)$ and analyzing the resulting problem in V instead of $H^1(\Omega)$. This completes the proof. □

Remark 2.4. In the particular case when

$$\int_{\partial\Omega} g\,d\sigma(x) = 1,$$

by Theorem 2.2 we deduce that we can prescribe the number of solutions of problem (1.1) just by prescribing the number of fixed points of function a, and thus, we may have a unique solution of the problem (1.1), a finite number of solutions or infinitely many solutions.

Next we point out a few examples of situations which can occur:

(1) If $a : [1,2] \to [1,2]$ with $a(t) = t$ for each $t \in [1,2]$ then problem (1.1) possesses infinitely many solutions corresponding to each point $t \in [1,2]$ which are all fixed points of function a.

(2) If $a : [1,2] \to [1,2]$ with $a(t) = 3 - t$ for each $t \in [1,2]$ then problem (1.1) possesses a unique solution corresponding to the unique fixed point of function a, namely $t = 3/2$.

(3) If $a : [1,2] \to [1,2]$ with

$$a(t) = \begin{cases} \frac{7}{3} - t, & \text{if } t \in [1, \frac{4}{3}], \\ 3t - 3, & \text{if } t \in [\frac{4}{3}, \frac{5}{3}], \\ \frac{16}{3} - 2t, & \text{if } t \in [\frac{5}{3}, 2], \end{cases}$$

then problem (1.1) possesses exactly three solutions, corresponding to the fixed points of function a from the set $\{\frac{7}{6}, \frac{3}{2}, \frac{16}{9}\}$.

(4) If $a : [1,2] \to [1,2]$ with

$$a(t) = \begin{cases} \frac{13}{6} - t, & \text{if } t \in [1, \frac{7}{6}], \\ 2t - \frac{4}{3}, & \text{if } t \in \left[\frac{7}{6}, \frac{8}{6}\right], \\ t, & \text{if } t \in [\frac{8}{6}, \frac{9}{6}], \\ 6 - 3t, & \text{if } t \in [\frac{9}{6}, \frac{10}{6}], \\ 1, & \text{if } t \in [\frac{10}{6}, \frac{11}{6}], \\ 6t - 10, & \text{if } t \in [\frac{11}{6}, 2], \end{cases}$$

then (1.1) possesses infinitely many solutions, corresponding on the one hand to the fixed points of function a from the interval $[\frac{8}{6}, \frac{9}{6}]$ plus two other solutions corresponding to the isolated fixed points of function a from the set $\{13/12, 2\}$.

3. Proof of Theorem 2.3

Fix $\mu \in \mathbb{R}$. The main tool in proving Theorem 2.3 will be Schauder's fixed point theorem (see [1, Theorem 3.21]).

Theorem 3.1 (Schauder's Fixed Point Theorem)**.** *Assume that K is a compact and convex subset of the Banach space B and $S : K \to K$ is a continuous map. Then S possesses a fixed point.*

We will give the proof of Theorem 2.3 only in the case when $\int_{\partial\Omega} g \, d\sigma \neq 0$. The case $\int_{\partial\Omega} g \, d\sigma = 0$ can be treated similarly with the difference that we have to take $\mu = 0$ this time and consider the weak formulation of the resulting problem on $\{u \in H^1(\Omega) : \int_\Omega u(x) \, dx = 0\}$ which is a closed subspace of $H^1(\Omega)$ (see the last part of the proof of Theorem 2.2 for more details). We start by establishing some auxiliary results which will be useful in obtaining the conclusion of Theorem 2.3.

Lemma 3.2. *For each $v \in L^2(\Omega)$, the problem*

$$\begin{gathered} -a(\mu) \operatorname{div}(a(v(x)) \nabla u(x)) + u(x) = 0, \quad x \in \Omega, \\ a(v(x)) \frac{\partial u}{\partial \nu}(x) = g(x), \quad x \in \partial\Omega \end{gathered} \tag{3.1}$$

has a weak solution $u \in H^1(\Omega)$, i.e. u satisfies

$$a(\mu)\Big[\int_\Omega a(v(x))\nabla u(x)\nabla\varphi(x)\,dx - \int_{\partial\Omega} g(x)\varphi(x)\,d\sigma(x)\Big] + \int_\Omega u(x)\varphi(x)\,dx = 0, \tag{3.2}$$

for all $\varphi \in H^1(\Omega)$.

Proof. Fix $v \in L^2(\Omega)$. By hypotheses (1.2) we obtain $a(v) \in L^\infty(\Omega)$. Consider the energy functional associated with (3.1), $J : H^1(\Omega) \to \mathbb{R}$ defined by

$$J(u) = a(\mu)\int_\Omega \frac{a(v)}{2}|\nabla u|^2\,dx + \frac{1}{2}\int_\Omega u^2\,dx - a(\mu)\int_{\partial\Omega} gu\,d\sigma(x).$$

Standard arguments imply that $J \in C^1(H^1(\Omega), \mathbb{R})$ with the derivative given by

$$\langle J'(u), \varphi\rangle = a(\mu)\int_\Omega a(v)\nabla u\nabla\varphi\,dx + \int_\Omega u\varphi\,dx - a(\mu)\int_{\partial\Omega} g\varphi\,d\sigma(x),$$

for all $u, \varphi \in H^1(\Omega)$. Thus, the weak solutions of (3.1) are exactly the critical points of J.

For each $u \in H^1(\Omega)$, using the fact that $H^1(\Omega)$ is continuously embedded in $L^2(\partial\Omega)$ (see, e.g. [3, Theorem 5.6.1]) and conditions (1.2) and (1.3) holds, we deduce that

$$\begin{aligned} J(u) &\geq \frac{a_1^2}{2}\int_\Omega |\nabla u|^2\,dx + \frac{1}{2}\int_\Omega u^2\,dx - a_2\|g\|_{L^2(\partial\Omega)}\|u\|_{L^2(\partial\Omega)} \\ &\geq \min\{\frac{a_1^2}{2}, \frac{1}{2}\}\|u\|^2_{H^1(\Omega)} - a_2C\|g\|_{L^2(\partial\Omega)}\|u\|_{H^1(\Omega)}\,, \end{aligned}$$

where C is a positive constant. The above estimates show that J is coercive. On the other hand, it is standard to check that J is weakly lower semi-continuous. Then, the Direct Method in the Calculus of Variations (see, e.g. [11, Theorem 1.2]) guarantees the existence of a global minimum point of J, $u \in H^1(\Omega)$ and consequently a weak solution of (3.1). The proof of Lemma 3.2 is complete. □

Next, for each $v \in L^2(\Omega)$ let $u = T(v) \in H^1(\Omega)$ be the weak solution of (3.1) given by Lemma 3.2. Thus, we can actually introduce a mapping

$$T : L^2(\Omega) \to H^1(\Omega)$$

associating to each $v \in L^2(\Omega)$ the weak solution of problem (3.1), $T(v) \in H^1(\Omega)$.

Lemma 3.3. *There exists a universal constant $\mathcal{C} > 0$, which does not depend on μ or v, such that*

$$\int_\Omega |\nabla T(v)|^2\,dx + \int_\Omega |T(v)|^2\,dx \leq \mathcal{C}, \quad \forall v \in L^2(\Omega). \tag{3.3}$$

Proof. Since $T(v)$ is a weak solution of (3.1), taking $\varphi = T(v)$ in (3.2) we find that

$$a(\mu)\int_\Omega a(v(x))|\nabla T(v)|^2\,dx + \int_\Omega |T(v)|^2\,dx = a(\mu)\int_{\partial\Omega} gT(v)\,d\sigma(x).$$

Using relation (1.2), Hölder's inequality and the fact that $H^1(\Omega)$ is continuously embedded in $L^2(\partial\Omega)$ we deduce

$$\begin{aligned} \min\Big\{\frac{a_1^2}{2}, \frac{1}{2}\Big\}\|T(v)\|^2_{H^1(\Omega)} &= \min\{\frac{a_1^2}{2}, \frac{1}{2}\}\Big(\int_\Omega |\nabla T(v)|^2\,dx + \int_\Omega |T(v)|^2\,dx\Big) \\ &\leq a_2\|g\|_{L^2(\partial\Omega)}\|T(v)\|_{L^2(\partial\Omega)} \end{aligned}$$

$$\leq a_2 D\|g\|_{L^2(\partial\Omega)}\|T(v)\|_{H^1(\Omega)},$$

where D is a positive constant. Leting

$$\mathcal{C} := \Big(\frac{Da_2\|g\|_{L^2(\partial\Omega)}}{\min\{\frac{a_1^2}{2},\frac{1}{2}\}}\Big)^2$$

we obtain inequality (3.3). The proof is complete. □

Lemma 3.4. *The mapping $T : L^2(\Omega) \to H^1(\Omega)$ is continuous.*

Proof. Let $\{v_n\} \subset L^2(\Omega)$ and $v \in L^2(\Omega)$ such that $\{v_n\}$ converges strongly to v in $L^2(\Omega)$. Set $u_n := T(v_n)$ for any positive integer n.

By Lemma 3.3 we infer that

$$\int_\Omega \big(|\nabla u_n|^2 + u_n^2\big)\,dx = \int_\Omega \big(|\nabla T(v_n)|^2 + |T(v_n)|^2\big)\,dx \leq \mathcal{C}, \quad \forall n;$$

that is, the sequence $\{u_n\}$ is bounded in $H^1(\Omega)$. It follows that there exists $u \in H^1(\Omega)$ such that, up to a subsequence still denoted by $\{u_n\}$, converges weakly to u in $H^1(\Omega)$ and by Rellich-Kondrachov theorem (see, e.g. [3, Theorem 5.5.2]) we deduce that $\{u_n\}$ converges strongly to u in $L^2(\Omega)$. On the other hand, we have u_n is a weak solution of problem (3.1) and thus by (3.2) we obtain

$$a(\mu)\int_\Omega a(v_n)\nabla u_n\nabla\varphi\,dx + \int_\Omega u_n\varphi\,dx = a(\mu)\int_{\partial\Omega} g\varphi\,d\sigma(x), \tag{3.4}$$

for all $\varphi \in H^1(\Omega)$ and all n.

Since $\{v_n\}$ converges strongly to v in $L^2(\Omega)$, it follows that $v_n(x) \to v(x)$ a.e. $x \in \Omega$, too. Combining that fact with the one that function a is continuous a.e. on $\mathbb{R}$, we find

$$a(v_n(x)) \to a(v(x)) \quad \text{for a.e. } x \in \Omega. \tag{3.5}$$

Moreover, since $\{u_n\}$ converges weakly to u in $H^1(\Omega)$ we deduce that

$$\{\nabla u_n\} \text{ converges weakly to } \nabla u \text{ in } (L^2(\Omega))^N. \tag{3.6}$$

Lebesgue's dominated convergence theorem (see, e.g. [4, Theorem 4.2]) and (3.5) imply that

$$\{a(v_n)\nabla\varphi\} \text{ converges strongly to } a(v)\nabla\varphi \text{ in } (L^2(\Omega))^N,\ \forall\varphi \in H^1(\Omega). \tag{3.7}$$

Thus, we deduce that

$$\int_\Omega a(v_n)\nabla u_n\nabla\varphi\,dx \to \int_\Omega a(v)\nabla u\nabla\varphi\,dx, \quad \forall\varphi \in H^1(\Omega).$$

In particular, for $\varphi = u$ we have

$$\int_\Omega a(v_n)\nabla u_n\nabla u\,dx \to \int_\Omega a(v)|\nabla u|^2\,dx. \tag{3.8}$$

Taking $\varphi = u_n - u$ in (3.4) and taking into account the above pieces of information we also find that

$$\int_\Omega a(v_n)\nabla u_n(\nabla u_n - \nabla u)\,dx = o(1) \tag{3.9}$$

and consequently,

$$\int_\Omega [a(v_n) - a(v)]\nabla u_n(\nabla u_n - \nabla u)\,dx + \int_\Omega a(v)\nabla u_n(\nabla u_n - \nabla u)\,dx = o(1). \tag{3.10}$$

By (3.5) and the fact that $\{|\nabla u_n|^2\}$ is a bounded sequence in $L^1(\Omega)$ we obtain by Hölder's inequality that

$$\Big|\int_\Omega [a(v_n)-a(v)]|\nabla u_n|^2\,dx\Big| \le \|a(v_n)-a(v)\|_{L^\infty(\Omega)}\|\nabla u_n\|^2_{L^2(\Omega)} \to 0 \tag{3.11}$$

as $n\to\infty$. Then (3.6), (3.8) and (3.11) yield

$$\int_\Omega [a(v_n)-a(v)]\nabla u_n(\nabla u_n-\nabla u)\,dx = o(1). \tag{3.12}$$

By (3.10) and (3.12) we find

$$\int_\Omega a(v)\ \nabla u_n(\nabla u_n-\nabla u)\,dx = o(1). \tag{3.13}$$

We deduce that

$$\begin{aligned}
&\int_\Omega a(v)|\nabla u_n-\nabla u|^2\,dx\\
&= \int_\Omega a(v)|\nabla u_n|^2\,dx - 2\int_\Omega a(v)\nabla u_n\nabla u\,dx + \int_\Omega a(v)|\nabla u|^2\,dx.
\end{aligned} \tag{3.14}$$

By (3.13) we infer that

$$\lim_{n\to\infty}\int_\Omega a(v)|\nabla u_n|^2\,dx = \lim_{n\to\infty}\int_\Omega a(v)\nabla u_n\nabla u\,dx = \int_\Omega a(v)|\nabla u|^2\,dx$$

and using (3.14) we finally obtain

$$\int_\Omega a(v)|\nabla(u_n-u)|^2\,dx = o(1)$$

which implies that

$$\int_\Omega |\nabla(u_n-u)|^2\,dx = o(1).$$

Moreover, taking into account that $\{u_n\}$ converges strongly to u in $L^2(\Omega)$ we conclude that $\{u_n\}$ converges strongly to u in $H^1(\Omega)$, that means application T is continuous. The proof is complete. □

Remark 3.5. Since $H^1(\Omega)$ is compactly embedded in $L^2(\Omega)$, that is the inclusion operator $i: H^1(\Omega)\to L^2(\Omega)$ is compact, it follows by Lemma 3.4 that the operator $S: L^2(\Omega)\to L^2(\Omega)$ defined by $S=i\circ T$ is compact.

Proof of Theorem 2.3. Let $\mathcal{C}$ be the positive constant given by Lemma 3.3. We have

$$\int_\Omega |\nabla S(v)|^2\,dx + \int_\Omega |S(v)|^2\,dx \le \mathcal{C},\ \ \forall\, v\in L^2(\Omega).$$

In particular,

$$\int_\Omega |S(v)|^2\,dx \le \mathcal{C},\quad \forall v\in L^2(\Omega).$$

In $L^2(\Omega)$, define the set

$$B_{\mathcal{C}}(0) := \big\{v\in L^2(\Omega): \int_\Omega |v(x)|^2\,dx \le \mathcal{C}\big\}.$$

Clearly, $B_{\mathcal{C}}(0)$ is a convex, closed subset of $L^2(\Omega)$ and $S(B_{\mathcal{C}}(0))\subset B_{\mathcal{C}}(0)$. By Remark 3.5 it follows that $S(B_{\mathcal{C}}(0))$ is relatively compact in $L^2(\Omega)$.

Finally, by Lemma 3.4 and Remark 3.5, we deduce that $S : S(B_{\mathcal{C}}(0)) \to S(B_{\mathcal{C}}(0))$ is a continuous map. Hence, we can apply the Schauder's fixed point theorem (Theorem 3.1) to obtain that S possesses a fixed point. This gives us a weak solution of problem (2.2) and thus the proof of Theorem 2.3 is finally complete. $\square$

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