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EXISTENCE AND CONCENTRATION OF POSITIVE BOUND STATES FOR SCHRÖDINGER-POISSON SYSTEMS WITH POTENTIAL FUNCTIONS

PATRÍCIA L. CUNHA

Abstract. In this article we study the existence and concentration behavior of bound states for a nonlinear Schrödinger-Poisson system with a parameter $\varepsilon > 0$. Under suitable conditions on the potential functions, we prove that for ε small the system has a positive solution that concentrates at a point which is a global minimum of the minimax function associated to the related autonomous problem.

1. Introduction

In this article we study the Schrödinger-Poisson system

$$\begin{gathered} -\varepsilon^2\Delta v + V(x)v + K(x)\phi(x)v = |v|^{q-2}v \quad \text{in } \mathbb{R}^3 \\ -\Delta\phi = K(x)v^2 \quad \text{in } \mathbb{R}^3 \end{gathered} \tag{1.1}$$

where $\varepsilon > 0$ is a parameter, $q \in (4, 6)$ and $V, K : \mathbb{R}^3 \to \mathbb{R}$ are, respectively, an external potential and a charge density. The unknowns of the system are the field u associated with the particles and the electric potential ϕ. We are interested in the existence and concentration behavior of solutions of (1.1) in the semiclassical limit $\varepsilon \to 0$.

The first equation of (1.1) is a nonlinear equation in which the potential ϕ satisfies a nonlinear Poisson equation. For this reason, (1.1) is called a Schrödinger-Poisson system, also known as Schrödinger-Maxwell system. For more information about physical aspects, we refer the reader to [6, 10] and references therein.

We observe that when $\phi \equiv 0$, (1.1) reduces to the well known Schrödinger equation

$$-\varepsilon^2\Delta u + V(x)u = f(x, u) \quad x \in \mathbb{R}^N. \tag{1.2}$$

In the previous years, the nonlinear stationary Schrödinger equation has been widely investigated, mainly in the semiclassical limit as $\varepsilon \to 0$ (see e.g. [21, 23, 24] and its references). Rabinowitz [21] studied problem (1.2) using mountain pass arguments to find least energy solutions, for $\varepsilon > 0$ sufficiently small. Then, Wang [23] proved that the solution in [21] concentrates around the global minimal of V when ε tends to 0.

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Wang and Zeng [24] considered the Schrödinger equation

$$-\varepsilon^2\Delta u + V(x)u = K(x)|u|^{p-1}u + Q(x)|u|^{q-1}u, \quad x \in \mathbb{R}^N \tag{1.3}$$

where $1 < q < p < (n+2)/(n-2)^+$. They proved the existence of least energy solutions and their concentration around a point in the semiclassical limit. The authors used the energy function $C(s)$ defined as the minimal energy of the functional associated with $\Delta u + V(s)u = K(s)|u|^{p-1}u + Q(s)|u|^{q-1}u$, where $s \in \mathbb{R}^N$ acts as a parameter instead of an independent variable. For each $\varepsilon > 0$ sufficiently small, they proved the existence of a solution u_ε for (1.3), whose global maximum approaches to a point y^* when ε tends to 0. Moreover, under suitable hypothesis on the potentials V and W, the function $\xi \mapsto C(\xi)$ assumes a minimum at y^*.

Motivated by these results, Alves and Soares [2] investigated the same phenomenon for the gradient system

$$\begin{gathered}
-\varepsilon^2\Delta u + V(x)u = Q_u(u,v) \quad \text{in } \mathbb{R}^N \\
-\varepsilon^2\Delta v + W(x)v = Q_v(u,v) \quad \text{in } \mathbb{R}^N \\
u(x), v(x) \to 0, \quad \text{as } |x| \to \infty \\
u, v > 0 \quad \mathbb{R}^N
\end{gathered} \tag{1.4}$$

In this system is natural to expect some competition between the potentials V and W, each one trying to attract the local maximum points of the solutions to its minimum points. In fact, in [2] the authors proved that functions u_ε and v_ε satisfies (1.4) and concentrate around the same point which is the minimum of the respective function $C(s)$.

Ianni and Vaira [17] studied the Schrödinger-Poisson system (1.1) proving that if V has a non-degenerated critical point x_0, then there exists a solution that concentrates around this point. Moreover, they also proved that if x_0 is degenerated for V and a local minimum for K, then there exist a solution concentrating around x_0. The proof was based in the Lyapunov-Schmidt reduction.

The double parameter perturbation was also considered for system (1.1) by [15, 16]. He and Zhou [16] studied the existence and behavior of a ground state solution which concentrates around the global minimum of the potential V. They considered $K \equiv 1$ and the presence of the nonlinear term $f(x,u)$.

Yang and Han [26] studied the Schrödinger-Poisson system

$$\begin{gathered}
-\Delta v + V(x)v + K(x)\phi(x)v = |v|^{q-2}v \quad \text{in } \mathbb{R}^3 \\
-\Delta\phi = K(x)v^2 \quad \text{in } \mathbb{R}^3
\end{gathered}. \tag{1.5}$$

Under suitable assumptions on V, K and f they proved existence and multiplicity results by using the mountain pass theorem and the fountain theorem. Later, Zhao, Liu and Zhao [27], using variational methods, proved the existence and concentration of solutions for the system

$$\begin{gathered}
-\Delta v + \lambda V(x)v + K(x)\phi(x)v = |v|^{q-2}v \quad \text{in } \mathbb{R}^3 \\
-\Delta\phi = K(x)v^2 \quad \text{in } \mathbb{R}^3
\end{gathered} \tag{1.6}$$

when $\lambda > 0$ is a parameter and $2 < p < 6$.

Several papers dealt with system (1.5) under variety assumptions on potentials V and K. Most part of the literature focuses on the study of the system with V or K constant or radially symmetric, mainly studying existence, nonexistence and multiplicity of solutions see e.g. [4, 9, 10, 11, 12, 18, 20, 22].

Using variational methods as in [2, 21, 24], we prove that there exists a solution u_ε for the Schrödinger-Poisson system (1.1) which concentrates around a point, without any additional assumption on the degenerability of such point related with the potentials V and K, as used in [17].

More precisely, denote C_∞ as the minimax value related to

$$\begin{gathered} -\Delta v + V_\infty v + K_\infty \phi v = |v|^{q-2}v \quad \text{in } \mathbb{R}^3 \\ -\Delta\phi = K_\infty v^2 \quad \text{in } \mathbb{R}^3 \end{gathered}$$

where the following conditions hold

(H0) There exists $\alpha > 0$ such that $V(x), K(x) \geq \alpha > 0$ for all $x \in \mathbb{R}^3$,

(H1) $V(x)$ and $K(x)$ are continuous functions and V_∞, K_∞ are defined by

$$\begin{gathered} V_\infty = \liminf_{|x|\to\infty} V(x) > \inf_{x\in\mathbb{R}^3} V(x) \\ K_\infty = \liminf_{|x|\to\infty} K(x) > \inf_{x\in\mathbb{R}^3} K(x). \end{gathered}$$

We prove that if

$$C_\infty > \inf_{\xi\in\mathbb{R}^3} C(\xi),$$

then (1.1) has a positive solution v_ε as ε tends to zero. After passing to a subsequence, v_ε concentrates at a global minimum point of $C(\xi)$ for $\xi \in \mathbb{R}^3$, where the energy function $C(\xi)$ is defined to be the minimax function associated with the problem

$$\begin{gathered} -\Delta u + V(\xi)u + K(\xi)\phi(\xi)u = |u|^{q-2}u \quad \text{in } \mathbb{R}^3 \\ -\Delta\phi = K(\xi)u^2 \quad \text{in } \mathbb{R}^3 \end{gathered} \tag{1.7}$$

Therefore, $C(\xi)$ plays a central role in our study. The main result for (1.1)) reads as follows.

Theorem 1.1. *Suppose* (H0)–(H1) *hold. If*

$$C_\infty > \inf_{\xi\in\mathbb{R}^3} C(\xi), \tag{1.8}$$

then there exists $\varepsilon^* > 0$ such that system (1.1)*) has a positive solution v_ε for $\varepsilon \in (0, \varepsilon^*)$. Moreover, v_ε concentrates at a local (hence global) maximum point $y^* \in \mathbb{R}^3$ such that*

$$C(y^*) = \min_{\xi\in\mathbb{R}^3} C(\xi).$$

Theorem 1.1 complements the study made in [12, 17, 26, 27] in the following sense: we deal with the perturbation problem (1.1) and study the concentration behavior of positive bound states.

To the best of our knowledge, the only previous article regarding the concentration of solutions for the perturbed Schrödinger-Poisson system with potentials V and K is [17], where the smoothness of such potentials is considered. We only need the boundedness of V and K. Moreover, we do not assume that the concentration point of solutions v_ε for the system (1.1) is a local minimum (or maximum) of such potentials, as in the previous paper. In our research we shall consider a different variational approach.

The outline of this paper is as follows: in Section 2 we set the variational framework. In Section 3 we study the autonomous system related to (1.1). In section 4 we establish an existence result for system (1.1) with $\varepsilon = 1$. In section 5, we prove Theorem 1.1.

2. Variational framework and preliminary results

Throughout this article we use the following notation:

• $H^1(\mathbb{R}^3)$ is the usual Sobolev space endowed with the standard scalar product and norm

$$(u,v)=\int_{\mathbb{R}^3}(\nabla u\nabla v+uv)\,dx,\quad \|u\|^2=\int_{\mathbb{R}^3}(|\nabla u|^2+u^2)\,dx.$$

• $\mathcal{D}^{1,2}=\mathcal{D}^{1,2}(\mathbb{R}^3)$ represents the completion of $C_0^\infty(\mathbb{R}^3)$ with respect to the norm

$$\|u\|^2_{\mathcal{D}^{1,2}}=\int_{\mathbb{R}^3}|\nabla u|^2\,dx.$$

• $L^p(\Omega)$, $1\le p\le\infty$, $\Omega\subset\mathbb{R}^3$, denotes a Lebesgue space; the norm in $L^p(\Omega)$ is denoted by $\|u\|_{L^p(\Omega)}$, where Ω is a proper subset of $\mathbb{R}^3$; $\|u\|_p$ is the norm in $L^p(\mathbb{R}^3)$.

We recall that by the Lax-Milgram theorem, for every $v\in H^1(\mathbb{R}^3)$, the Poisson equation $-\Delta\phi=v^2$ has a unique positive solution $\phi=\phi_v\in\mathcal{D}^{1,2}(\mathbb{R}^3)$ given by

$$\phi_v(x)=\int_{\mathbb{R}^3}\frac{v^2(y)}{|x-y|}\,dy. \tag{2.1}$$

The function $\phi:H^1(\mathbb{R}^3)\to\mathcal{D}^{1,2}(\mathbb{R}^3)$, $\phi[v]=\phi_v$ has the following properties (see for instance Cerami and Vaira [8]).

Lemma 2.1. *For any $v\in H^1(\mathbb{R}^3)$, we have*

(i) *ϕ is continuous and maps bounded sets into bounded sets;*
(ii) *$\phi_v\ge 0$;*
(iii) *there exists $C>0$ such that $\|\phi\|_{D^{1,2}}\le C\|v\|^2$ and*
$$\int_{\mathbb{R}^3}|\nabla v|^2\,dx=\int_{\mathbb{R}^3}\phi_v v^2\,dx\le C\|v\|^4;$$
(iv) *$\phi_{tv}=t^2\phi_v$, $\forall\,t>0$;*
(v) *if $v_n\rightharpoonup v$ in $H^1(\mathbb{R}^3)$, then $\phi_{v_n}\rightharpoonup\phi_v$ in $\mathcal{D}^{1,2}(\mathbb{R}^3)$.*

As in [4], for every $v\in H^1(\mathbb{R}^3)$, there exist a unique solution $\phi=\phi_{K,v}\in\mathcal{D}^{1,2}(\mathbb{R}^3)$ of $-\Delta\phi=K(x)v^2$ where

$$\phi_{K,v}(x)=\int_{\mathbb{R}^3}\frac{K(y)v^2(y)}{|x-y|}dy. \tag{2.2}$$

and it is easy to see that $\phi_{K,v}$ satisfies Lemma 2.1 if K satisfies conditions (H0)–(H1).

Substituting (2.2) into the first equation of (1.1), we obtain

$$-\varepsilon^2\Delta v+V(x)v+K(x)\phi_{K,v}(x)v=|v|^{q-2}v. \tag{2.3}$$

Making the changing of variables $x\mapsto\varepsilon x$ and setting $u(x)=v(\varepsilon x)$, (2.3) becomes

$$-\Delta u+V(\varepsilon x)u+K(\varepsilon x)\phi_{K,v}(\varepsilon x)u=|u|^{q-2}u. \tag{2.4}$$

A simple computation shows that

$$\phi_{K,v}(\varepsilon x)=\varepsilon^2\phi_{\varepsilon,u}(x),$$

where

$$\phi_{\varepsilon,u}(x)=\int_{\mathbb{R}^3}\frac{K(\varepsilon y)u^2(y)}{|x-y|}dy.$$

Substituting this into (2.4), Equation (1.1) can be rewritten in the equivalent equation

$$-\Delta u + V(\varepsilon x)u + \varepsilon^2 K(\varepsilon x)\phi_{\varepsilon,u}u = |u|^{q-2}u. \tag{2.5}$$

Note that if u_ε is a solution of (2.5), then $v_\varepsilon(x) = u_\varepsilon(x/\varepsilon)$ is a solution of (2.3). We denote by $H_\varepsilon = \{u \in H^1(\mathbb{R}^3) : \int_{\mathbb{R}^3} V(\varepsilon x)u^2 < \infty\}$ is a Sobolev space endowed with the norm

$$\|u\|_\varepsilon^2 = \int_{\mathbb{R}^3} (|\nabla u|^2 + V(\varepsilon x)u^2)\,dx.$$

At this step, we see that (5.1) is variational and its solutions are critical points of the functional

$$\mathcal{I}_\varepsilon(u) = \frac{1}{2}\int_{\mathbb{R}^3} (|\nabla u|^2 + V(\varepsilon x)u^2)\,dx + \frac{\varepsilon^2}{4}\int_{\mathbb{R}^3} K(\varepsilon x)\phi_{\varepsilon,u}(x)u^2\,dx - \frac{1}{q}\int_{\mathbb{R}^3} |u|^q\,dx.$$

3. Autonomous Case

In this section we study the autonomous system

$$\begin{gathered} -\Delta u + V(\xi)u + K(\xi)\phi(x)u = |u|^{q-2}u \quad \text{in } \mathbb{R}^3 \\ -\Delta\phi = K(\xi)u^2 \quad \text{in } \mathbb{R}^3 \end{gathered} \tag{3.1}$$

where $\xi \in \mathbb{R}^3$. To this system we associate the functional $I_\xi : H_\xi \mapsto \mathbb{R}$,

$$I_\xi(u) = \frac{1}{2}\int_{\mathbb{R}^3} (|\nabla u|^2 + V(\xi)u^2)\,dx + \frac{1}{4}\int_{\mathbb{R}^3} K(\xi)\phi_u(x)u^2\,dx - \frac{1}{q}\int_{\mathbb{R}^3} |u|^q\,dx. \tag{3.2}$$

Hereafter, the Sobolev space $H_\xi = H^1(\mathbb{R}^3)$ is endowed with the norm

$$\|u\|_\xi = \int_{\mathbb{R}^3} (|\nabla u|^2 + V(\xi)u^2)\,dx.$$

By standard arguments, the functional I_ξ verifies the Mountain-Pass Geometry, more exactly it satisfies the following lemma.

Lemma 3.1. *The functional I_ξ satisfies*

(i) *There exist positive constants β, ρ such that $I_\xi(u) \geq \beta$ for $\|u\|_\xi = \rho$,*
(ii) *There exists $u_1 \in H^1(\mathbb{R}^3)$ with $\|u_1\|_\xi > \rho$ such that $I_\xi(u_1) < 0$.*

Applying a variant of the Mountain Pass Theorem (see [25]), we obtain a sequence $(u_n) \subset H^1(\mathbb{R}^3)$ such that

$$I_\xi(u_n) \to C(\xi) \quad \text{and} \quad I'_\xi(u_n) \to 0,$$

where

$$C(\xi) = \inf_{\gamma\in\Gamma}\max_{0\leq t\leq 1} I_\xi(\gamma(t)), \quad C(\xi) \geq \alpha, \tag{3.3}$$

$$\Gamma = \{\gamma \in \mathcal{C}([0,1], H^1(\mathbb{R}^3)) | \gamma(0) = 0, \gamma(1) = u_1\}. \tag{3.4}$$

We observe that $C(\xi)$ can be also characterized as

$$C(\xi) = \inf_{u\neq 0}\max_{t>0} I_\xi(tu).$$

Proposition 3.2. *Let $\xi \in \mathbb{R}^3$. Then system* (3.1) *has a positive solution $u \in H^1(\mathbb{R}^3)$ such that $I'_\xi(u) = 0$ and $I_\xi(u) = C(\xi)$, for any $q \in (4,6)$.*

The proof of the above propostion is an easy adaptation of Azzollini and Pomponio [5, Theorem 1.1] and we omit it.

Lemma 3.3. *The function $\xi \mapsto C(\xi)$ is continuous.*

Proof. The proof consists in proving that there exist sequences (ζ_n) and (λ_n) in $\mathbb{R}^3$ such that $C(\zeta_n), C(\lambda_n) \to C(\xi)$ as $n \to 0$, where

- $\zeta_n \to \xi$ and $C(\zeta_n) \geq C(\xi)$ for all n,
- $\lambda_n \to \xi$ and $C(\lambda_n) \geq C(\xi)$ for all n,

as we know by Alves and Soares [2]. □

4. System (1.1) with $\varepsilon = 1$

Setting $\varepsilon = 1$, in this section we consider the system

$$\begin{gathered} -\Delta u + V(x)u + K(x)\phi(x)u = |u|^{q-2}u \quad \text{in } \mathbb{R}^3 \\ -\Delta\phi = K(x)u^2 \quad \text{in } \mathbb{R}^3 \end{gathered} \tag{4.1}$$

whose solutions are critical points of the corresponding functional

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^3} (|\nabla u|^2 + V(x)u^2)\,dx + \frac{1}{4}\int_{\mathbb{R}^3} K(x)\phi_u(x)u^2\,dx - \frac{1}{p}\int_{\mathbb{R}^3} |u|^q\,dx$$

which is well defined for $u \in H_1$, where

$$H_1 = \{u \in H^1(\mathbb{R}^3) : \int_{\mathbb{R}^3} V(x)u^2\,dx < \infty\}$$

with the same norm notation of the Sobolev space $H^1(\mathbb{R}^3)$.

Similar to the autonomous case, the functional I satisfies the mountain pass geometry, then there exists a sequence $(u_n) \subset H_1$ such that

$$I(u_n) \to c \quad \text{and} \quad I'(u_n) \to 0\,, \tag{4.2}$$

where

$$c = \inf_{\gamma\in\Gamma}\max_{0\leq t\leq 1} I(\gamma(t)),$$
$$\Gamma = \{\gamma \in \mathcal{C}([0,1], H_1(\mathbb{R}^3)) | \gamma(0) = 0, I(\gamma(1)) < 0\}.$$

Remark 4.1. The function $(\mu,\nu) \mapsto c_{\mu,\nu}$ is continuous, where $c_{\mu,\nu}$ is the minimax level of

$$I_{\mu,\nu}(u) = \frac{1}{2}\int_{\mathbb{R}^3} (|\nabla u|^2 + \mu u^2)\,dx + \frac{1}{4}\int_{\mathbb{R}^3} \nu\phi_u(x)u^2\,dx - \frac{1}{q}\int_{\mathbb{R}^3} |u|^q\,dx. \tag{4.3}$$

Remark 4.2. We denote by C_∞ the minimax value related to the functional

$$I_\infty(u) = \frac{1}{2}\int_{\mathbb{R}^3} (|\nabla u|^2 + V_\infty u^2)\,dx + \frac{1}{4}\int_{\mathbb{R}^3} K_\infty\phi_u u^2\,dx - \frac{1}{q}\int_{\mathbb{R}^3} |u|^q\,dx\,,$$

where V_∞ and K_∞, given by condition (H_1), belong to $(0,\infty)$. Otherwise, define $C_\infty = \infty$. $I_\infty(u)$ is well defined for $u \in E_\infty$, where E_∞ is a Sobolev space endowed with the norm

$$\|u\|_\infty = \int_{\mathbb{R}^3} (|\nabla u|^2 + V_\infty u^2)\,dx$$

equivalent to the usual Sobolev norm on $H^1(\mathbb{R}^3)$.

An important tool in our analysis is the following theorem.

Theorem 4.3. *If $c < C_\infty$, then c is a nontrivial critical value for I.*

Proof. From (4.2), (u_n) is bounded in H_1. As a consequence, passing to a subsequence if necessary, $u_n \rightharpoonup u$ in H_1. From Proposition 2.1 (v), $\phi_{u_n} \rightharpoonup \phi_u$ in $\mathcal{D}^{1,2}(\mathbb{R}^3)$, as $n \to \infty$. Then, (u, ϕ_u) is a weak solution of (4.1). Similar to the proof of Proposition 3.1, $I(u) = c$. It remains to show that $u \neq 0$.

From Alves, Souto and Soares [3], if there exist constants η, R such that

$$\liminf_{n\to+\infty} \int_{B_R(0)} u_n^2\,dx \geq \eta > 0,$$

then $u \neq 0$.

By contradiction, consider $u \equiv 0$. Hence, there exists a subsequence of (u_n), still denoted by (u_n), such that

$$\lim_{n\to+\infty} \int_{B_R(0)} u_n^2\,dx = 0.$$

Let μ and ν be such that

$$\inf_{x\in\mathbb{R}^3} V(x) < \mu < \liminf_{|x|\to\infty} V(x) = V_\infty \inf_{x\in\mathbb{R}^3} K(x) < \nu < \liminf_{|x|\to\infty} K(x) = K_\infty$$

and take $R > 0$ such that

$$V(x) > \mu, \quad \forall\, x \in \mathbb{R}^3\backslash B_R(0) K(x) > \nu, \quad \forall\, x \in \mathbb{R}^3\backslash B_R(0).$$

For each $n \in \mathbb{N}$, there exist $t_n > 0$, $t_n \to 1$ such that $I(t_n u_n) = \max_{t\geq 0} I(tu_n)$. The convergence of (t_n) follows from (4.2). In fact, since $I'(u_n)u_n = o_n(1)$ and $I'(t_n u_n)t_n u_n = o_n(1)$, we have

$$\|u_n\|^2 + \int_{\mathbb{R}^3} K(x)\phi_{u_n} u_n^2\,dx = \int_{\mathbb{R}^3} |u_n|^q\,dx + o_n(1)$$

we have

$$t_n^2\|u_n\|^2 + t_n^4 \int_{\mathbb{R}^3} K(x)\phi_{u_n} u_n^2\,dx = t_n^q \int_{\mathbb{R}^3} |u_n|^q\,dx + o_n(1).$$

Then

$$(1 - t_n^2)\|u_n\|^2 = (t_n^{q-2} - t_n^2) \int_{\mathbb{R}^3} |u_n|^q\,dx + o_n(1)$$

Observe that t_n neither converge to 0 nor to ∞, otherwise we would have $\|u_n\| \to \infty$ as $n \to \infty$, which is impossible since $c > 0$. See e.g. [1].

Suppose $t_n \to t_0$. Letting $n \to +\infty$,

$$0 = (t_0^2 - 1)\ell_1 + t_0^2(t_0^{q-4} - 1)\ell_2$$

where $\ell_1, \ell_2 > 0$. Hence, $t_0 = 1$. Consequently, we have

$$\begin{aligned} &I(u_n) - I(t_n u_n)\\ &= \frac{1-t_n^2}{2}\|u_n\|^2 + \frac{1}{4}(1 - t_n^4) \int_{\mathbb{R}^3} K(x)\phi_{u_n} u_n^2\,dx + \frac{t_n^q - 1}{q} \int_{\mathbb{R}^3} |u_n|^q\,dx = o_n(1) \end{aligned}$$

which implies, for every $t \geq 0$,

$$
\begin{aligned}
I(u_n) &\geq I(tu_n) + o_n(1) \\
&= \frac{t^2}{2}\int_{\mathbb{R}^3} |\nabla u_n|^2 + V(x)u_n^2\,dx + \frac{t^4}{4}\int_{\mathbb{R}^3} K(x)\phi_{u_n}u_n^2\,dx \\
&\quad - \frac{t^q}{q}\int_{\mathbb{R}^3} |u_n|^q\,dx + I_{\mu,\nu}(tu_n) - I_{\mu,\nu}(tu_n) + o_n(1) \\
&\geq \frac{t^2}{2}\int_{B_R(0)} (V(x)-\mu)u_n^2\,dx + \frac{t^4}{4}\int_{B_R(0)} (K(x)-\nu)\phi_{u_n}u_n^2\,dx \\
&\quad + I_{\mu,\nu}(tu_n) + o_n(1),
\end{aligned}
\tag{4.4}
$$

where $I_{\mu,\nu}(u)$ is given by (4.3).

Consider τ_n such that $I_{\mu,\nu}(\tau_n u_n) = \max_{t\geq 0} I_{\mu,\nu}(tu_n)$. As in the above arguments, $\tau_n \to 1$. Letting $t = \tau_n$ in (4.4), we have

$$
\begin{aligned}
I(u_n) \geq \frac{\tau_n^2}{2}\int_{B_R(0)} (V(x)-\mu)u_n^2\,dx + \frac{\tau_n^4}{4}\int_{B_R(0)} (K(x)-\nu)\phi_{u_n}u_n^2\,dx \\
+ c_{\mu,\nu} + o_n(1).
\end{aligned}
$$

Taking the limit $n \to +\infty$, we have $c \geq c_{\mu,\nu}$. Next, taking $\mu \to V_\infty$ and $\nu \to K_\infty$, we obtain $c \geq C_\infty$, proving Theorem 4.3. $\square$

5. Proof of Theorem 1.1

This section is devoted to study the existence, regularity and the asymptotic behavior of solutions for the system (1.1)), which is equivalent to

$$
-\Delta u + V(\varepsilon x)u + \varepsilon^2 K(\varepsilon x)\phi_{\varepsilon,u}u = |u|^{q-2}u. \tag{5.1}
$$

where

$$
\mathcal{I}_\varepsilon(u) = \frac{1}{2}\int_{\mathbb{R}^3} (|\nabla u|^2 + V(\varepsilon x)u^2)\,dx + \frac{\varepsilon^2}{4}\int_{\mathbb{R}^3} K(\varepsilon x)\phi_{\varepsilon,u}(x)u^2\,dx - \frac{1}{q}\int_{\mathbb{R}^3} |u|^q\,dx.
$$

is the Euler-Lagrange functional related to (5.1).

The proof of Theorem 1.1 is divided into three subsections as follows:

5.1. **Existence of a solution.**

Theorem 5.1. *Suppose* (H0)–(H1) *hold and consider*

$$
C_\infty > \inf_{\xi\in\mathbb{R}^3} C(\xi)\,. \tag{5.2}
$$

Then, there exists $\varepsilon^* > 0$ such that system (5.1) *has a positive solution for every $0 < \varepsilon < \varepsilon^*$.*

Proof. By hypothesis (5.2), there exists $b \in \mathbb{R}^3$ and $\delta > 0$ such that

$$
C(b) + \delta < C_\infty. \tag{5.3}
$$

Define $u_\varepsilon(x) = u(x - \frac{b}{\varepsilon})$, where, from Proposition 3.2, u is a solution of the autonomous Schrödinger-Poisson system

$$
\begin{gathered}
-\Delta u + V(b)u + K(b)\phi(x)u = |u|^{q-2}u \quad \text{in } \mathbb{R}^3 \\
-\Delta\phi = K(b)u^2 \quad \text{in } \mathbb{R}^3
\end{gathered}
\tag{5.4}
$$

with $I_b(u) = C(b)$.

Let t_ε be such that $\mathcal{I}_\varepsilon(t_\varepsilon u_\varepsilon) = \max_{t\geq 0}\mathcal{I}_\varepsilon(tu_\varepsilon)$. Similar to the proof of Theorem 4.3, we have $\lim_{\varepsilon\to 0} t_\varepsilon = 1$.

Since

$$c_\varepsilon = \inf_{\gamma\in\Gamma}\max_{0\leq t\leq 1}\mathcal{I}_\varepsilon(\gamma(t)) = \inf_{\substack{u\in H^1\\ u\neq 0}}\max_{t\geq 0}\mathcal{I}_\varepsilon(tu) \leq \max_{t\geq 0}\mathcal{I}_\varepsilon(tu_\varepsilon) = \mathcal{I}_\varepsilon(t_\varepsilon u_\varepsilon),$$

we have

$$\limsup_{\varepsilon\to 0} c_\varepsilon \leq \limsup_{\varepsilon\to 0}\mathcal{I}_\varepsilon(t_\varepsilon u_\varepsilon) = I_b(u) = C(b) < C(b)+\delta,$$

which, from (5.3), implies

$$\limsup_{\varepsilon\to 0} c_\varepsilon < C_\infty.$$

Therefore, there exists $\varepsilon^* > 0$ such that $c_\varepsilon < C_\infty$ for every $0<\varepsilon<\varepsilon^*$. In view of Theorem 4.3, system (5.1) has a positive solution for every $0<\varepsilon<\varepsilon^*$. $\square$

5.2. **Regularity of the solution.** The first result is a suitable version of Brezis and Kato [7] and the second one is a particular version of from Gilbarg and Trudinger [14, Theorem 8.17].

Proposition 5.2. *Consider $u\in H^1(\mathbb{R}^3)$ satisfying*

$$-\Delta u + b(x)u = f(x,u) \quad \text{in } \mathbb{R}^3,$$

where $b:\mathbb{R}^3\to\mathbb{R}$ is a $L^\infty_{\rm loc}(\mathbb{R}^3)$ function and $f:\mathbb{R}^3\to\mathbb{R}$ is a Caratheodory function such that

$$0\leq f(x,s)\leq C_f(s^r+s), \quad \forall s>0,\ x\in\mathbb{R}^3.$$

Then, $u\in L^t(\mathbb{R}^3)$ for every $t\geq 2$. Moreover, there exists a positive constant $C=C(t,C_f)$ such that

$$\|u\|_{L^t(\mathbb{R}^3)} \leq C\|u\|_{H^1(\mathbb{R}^3)}.$$

Proposition 5.3. *Consider $t>3$ and $g\in L^{1/2}(\Omega)$, where Ω is an open subset of $\mathbb{R}^3$. Then, if $u\in H^1(\Omega)$ is a subsolution of*

$$\Delta u = g \quad \text{in } \Omega,$$

we have that for any $y\in\mathbb{R}^3$ and $B_{2R}(y)\subset\Omega$, $R>0$ and

$$\sup_{B_R(y)} u \leq C\Big(\|u^+\|_{L^2(B_{2R}(y))} + \|g\|_{L^{1/2}(B_{2R}(y))}\Big)$$

where $C=C(t,R)$.

In view of Propositions 5.2 and 5.3, the positive solutions of (1.1) are in $C^2(\mathbb{R}^3)\cap L^\infty(\mathbb{R}^3)$ for all $\varepsilon>0$. Similar arguments was employed by He and Zou [16].

5.3. **Concentration of solutions.**

Lemma 5.4. *Suppose* (H0)–(H1) *hold. Then, there exists $\beta_0>0$ such that*

$$c_\varepsilon \geq \beta_0,$$

for every $\varepsilon>0$. Moreover,

$$\limsup_{\varepsilon\to 0} c_\varepsilon \leq \inf_{\xi\in\mathbb{R}^3} C(\xi).$$

Proof. Let $w_\varepsilon \in H_\varepsilon$ be such that $c_\varepsilon = \mathcal{I}_\varepsilon(w_\varepsilon)$. Then, from condition (H_0)

$$c_\varepsilon = \mathcal{I}_\varepsilon(w_\varepsilon) \geq \inf_{\substack{u\in H^1\\ u\neq 0}} \sup_{t\geq 0} J(tu) = \beta_0, \quad \forall \varepsilon > 0,$$

where

$$J(u) = \frac{1}{2}\int_{\mathbb{R}^3}(|\nabla u|^2 + \alpha u^2)\,dx + \frac{1}{4}\int_{\mathbb{R}^3}\alpha\phi_u u^2\,dx - \frac{1}{q}\int_{\mathbb{R}^3}|u|^q\,dx.$$

Let $\xi \in \mathbb{R}^3$ and consider $w \in H^1(\mathbb{R}^3)$ a least energy solution for system (1.7), that is, $I_\xi(w) = C(\xi)$ and $I'_\xi(w) = 0$. Let $w_\varepsilon(x) = w(x - \frac{\xi}{\varepsilon})$ and take $t_\varepsilon > 0$ such that

$$c_\varepsilon \leq \mathcal{I}_\varepsilon(t_\varepsilon w_\varepsilon) = \max_{t\geq 0}\mathcal{I}_\varepsilon(tw_\varepsilon).$$

Similar to the proof of Theorem 4.3, $t_\varepsilon \to 1$ as $\varepsilon \to 0$, then

$$c_\varepsilon \leq \mathcal{I}_\varepsilon(t_\varepsilon w_\varepsilon) \to I_\xi(w) = C(\xi), \quad \text{as } \varepsilon \to 0$$

which implies that $\limsup_{\varepsilon\to 0} c_\varepsilon \leq C(\xi)$ for all $\xi \in \mathbb{R}^3$. Therefore,

$$\limsup_{\varepsilon\to 0} c_\varepsilon \leq \inf_{\xi\in\mathbb{R}^3} C(\xi).$$

□

Lemma 5.5. *There exist a family $(y_\varepsilon) \subset \mathbb{R}^3$ and constants $R, \beta > 0$ such that*

$$\liminf_{\varepsilon\to 0}\int_{B_R(y_\varepsilon)} u_\varepsilon^2\,dx \geq \beta, \quad \textit{for each } \varepsilon > 0.$$

Proof. By contradiction, suppose that there exists a sequence $\varepsilon_n \to 0$ such that

$$\lim_{n\to\infty}\sup_{y\in\mathbb{R}^3}\int_{B_R(y)} u_n^2\,dx = 0, \quad \text{for all } R > 0,$$

where, for the sake of simplicity, we denote $u_n(x) = u_{\varepsilon_n}(x)$. Hereafter, denote $\phi_{\varepsilon_n,u_n}(x) = \phi_{u_n}(x)$. From [19, Lemma I.1], we have

$$\int_{\mathbb{R}^3}|u_n|^q\,dx \to 0, \quad \text{as } n \to \infty.$$

But, since

$$\int_{\mathbb{R}^3}(|\nabla u_n|^2 + V(\varepsilon_n x)u_n^2)\,dx + \int_{\mathbb{R}^3}\varepsilon_n^2 K(\varepsilon_n x)\phi_{u_n}u_n^2\,dx = \int_{\mathbb{R}^3}|u_n|^q\,dx\,,$$

we have

$$\int_{\mathbb{R}^3}(|\nabla u_n|^2 + V(\varepsilon_n x)u_n^2)\,dx \to 0, \quad \text{as } n \to \infty.$$

Therefore,

$$\lim_{n\to\infty} c_{\varepsilon_n} = \lim_{n\to\infty} I_{\varepsilon_n}(u_n) = 0$$

which is an absurd, since for some $\beta_0 > 0$, $c_\varepsilon \geq \beta_0$, from Lemma 5.4. □

Lemma 5.6. *The family $(\varepsilon y_\varepsilon)$ is bounded. Moreover, if y^* is the limit of the sequence $(\varepsilon_n y_{\varepsilon_n})$ in the family $(\varepsilon y_\varepsilon)$, then we have*

$$C(y^*) = \inf_{\xi\in\mathbb{R}^3} C(\xi).$$

Proof. Consider $u_n(x) = u_{\varepsilon_n}(x + y_{\varepsilon_n})$. Suppose by contradiction that $(\varepsilon_n y_{\varepsilon_n})$ approaches infinity. It follows from Lemma 5.5 that there exists constants $R, \beta > 0$ such that

$$\int_{B_R(0)} u_n^2(x)\,dx \geq \beta > 0, \quad \text{for all } n \in \mathbb{N}. \tag{5.5}$$

Since $u_n(x)$ satisfies

$$-\Delta u_n + V(\varepsilon_n x + \varepsilon_n y_{\varepsilon_n})u_n + \varepsilon_n^2 K(\varepsilon_n x + \varepsilon_n y_{\varepsilon_n})\phi_{\varepsilon_n, u_n} u_n = |u_n|^{q-2}u_n, \tag{5.6}$$

it follows that $u_n(x)$ is bounded in H_ε. Hence, passing to a subsequence if necessary, $u_n \to \hat{u} \geq 0$ weakly in H_ε, strongly in $L^p_{\rm loc}(\mathbb{R}^3)$ for $p \in (2, 6)$ and a.e. in $\mathbb{R}^3$. From (5.5), $\hat{u} \neq 0$.

Using $\hat{u}$ as a test function in (5.6) and taking the limit, we obtain

$$\int_{\mathbb{R}^3} (|\nabla \hat{u}|^2 + \mu \hat{u}^2)\,dx \leq \int_{\mathbb{R}^3} (|\nabla \hat{u}|^2 + \mu \hat{u}^2)\,dx + \int_{\mathbb{R}^3} \nu \phi_{\hat{u}} \hat{u}^2\,dx \leq \int_{\mathbb{R}^3} |\hat{u}|^q\,dx \tag{5.7}$$

where, μ and ν are positive constantes such that

$$\mu < \liminf_{|x|\to\infty} V(x) \quad \text{and} \quad \nu < \liminf_{|x|\to\infty} K(x).$$

Consider the functional $I_{\mu,\nu} : H^1(\mathbb{R}^3) \to \mathbb{R}$ given by

$$I_{\mu,\nu}(u) = \frac{1}{2}\int_{\mathbb{R}^3} (|\nabla u|^2 + \mu u^2)\,dx + \frac{1}{4}\int_{\mathbb{R}^3} \nu \phi_u(x) u^2\,dx - \frac{1}{q}\int_{\mathbb{R}^3} |u|^q\,dx.$$

Let $\sigma > 0$ be such that $I_{\mu,\nu}(\sigma\hat{u}) = \max_{t>0} I_{\mu,\nu}(t\hat{u})$. We claim that

$$\sigma^2 \int_{\mathbb{R}^3} (|\nabla \hat{u}|^2 + \mu \hat{u}^2)\,dx + \sigma^4 \int_{\mathbb{R}^3} \nu \phi_{\hat{u}} \hat{u}^2\,dx = \sigma^q \int_{\mathbb{R}^3} |\hat{u}|^q\,dx. \tag{5.8}$$

In fact, from (5.7)

$$\begin{aligned}
I_{\mu,\nu}(\sigma\hat{u}) &= \frac{\sigma^2}{2}\int_{\mathbb{R}^3} (|\nabla \hat{u}|^2 + \mu \hat{u}^2)\,dx + \frac{\sigma^4}{4}\int_{\mathbb{R}^3} \nu \phi_{\hat{u}} \hat{u}^2\,dx - \frac{\sigma^q}{q}\int_{\mathbb{R}^3} |\hat{u}|^q\,dx \\
&\leq \frac{\sigma^2}{2}\int_{\mathbb{R}^3} |\hat{u}|^q\,dx + \frac{\sigma^4}{4}\int_{\mathbb{R}^3} \nu \phi_{\hat{u}} \hat{u}^2\,dx - \frac{\sigma^q}{q}\int_{\mathbb{R}^3} |\hat{u}|^q\,dx
\end{aligned}$$

it follows that $\sigma \leq 1$, and since $\frac{d}{dt} I_{\mu,\nu}(t\hat{u})\Big|_{t=\sigma} = 0$, we obtain

$$\frac{d}{dt} I_{\mu,\nu}(t\hat{u})\Big|_{t=\sigma} = \sigma \int_{\mathbb{R}^3} (|\nabla \hat{u}|^2 + \mu \hat{u}^2)\,dx + \sigma^3 \int_{\mathbb{R}^3} \nu \phi_{\hat{u}} \hat{u}^2\,dx - \sigma^{q-1}\int_{\mathbb{R}^3} |\hat{u}|^q\,dx = 0$$

proving (5.8).

From Lemma 5.4, equation (5.8) and the fact that $\sigma \leq 1$, we have

$$\begin{aligned}
c_{\mu,\nu} &= \inf_{u\neq 0} \max_{t>0} I_{\mu,\nu}(tu) = \inf_{u\neq 0} I_{\mu,\nu}(\sigma u) \leq I_{\mu,\nu}(\sigma\hat{u}) \\
&= \frac{\sigma^2}{2}\int_{\mathbb{R}^3} (|\nabla \hat{u}|^2 + \mu \hat{u}^2)\,dx + \frac{\sigma^4}{4}\int_{\mathbb{R}^3} \nu \phi_{\hat{u}} \hat{u}^2\,dx - \frac{\sigma^q}{q}\int_{\mathbb{R}^3} |\hat{u}|^q\,dx \\
&= \frac{\sigma^2}{4}\int_{\mathbb{R}^3} (|\nabla \hat{u}|^2 + \mu \hat{u}^2)\,dx + \frac{\sigma^q(q-4)}{4q}\int_{\mathbb{R}^3} |\hat{u}|^q\,dx \\
&\leq \frac{1}{4}\int_{\mathbb{R}^3} (|\nabla \hat{u}|^2 + \mu \hat{u}^2)\,dx + \frac{q-4}{4q}\int_{\mathbb{R}^3} |\hat{u}|^q\,dx \\
&\leq \liminf_{n\to\infty} \Big(\mathcal{I}_{\varepsilon_n}(u_n) - \frac{1}{4}\mathcal{I}'_{\varepsilon_n}(u_n)u_n\Big)
\end{aligned}$$

$$= \liminf_{n\to\infty} c_{\varepsilon_n} \le \limsup_{n\to\infty} c_{\varepsilon_n} \le \inf_{\xi\in\mathbb{R}^3} C(\xi)$$

hence, $c_{\mu,\nu} \le \inf_{\xi\in\mathbb{R}^3} C(\xi)$.

If we consider

$$\mu \to \liminf_{|x|\to\infty} V(x) = V_\infty \quad \text{and} \quad \nu \to \liminf_{|x|\to\infty} K(x) = K_\infty,$$

then by the continuity of the function $(\mu,\nu) \mapsto c_{\mu\nu}$ we obtain $C_\infty \le \inf_{\xi\in\mathbb{R}^3} C(\xi)$, which contradicts condition (C^∞). Therefore, $(\varepsilon y_\varepsilon)$ is bounded and there exists a subsequence of $(\varepsilon y_\varepsilon)$ such that $\varepsilon_n y_{\varepsilon_n} \to y^*$.

Now we proceed to prove that $C(y^*) = \inf_{\xi\in\mathbb{R}^3} C(\xi)$. Recalling that $u_n(x) = u_{\varepsilon_n}(x + y_{\varepsilon_n})$ and from the arguments above, $\hat{u}$ satisfies the equation

$$-\Delta u + V(y^*)u + K(y^*)\phi_u u = |u|^{q-2}u \tag{5.9}$$

The Euler-Lagrange functional associated to this equation is $I_{y^*} : H_{y^*}(\mathbb{R}^3)$, defined as in (3.2) with $\xi = y^*$.

Using $\hat{u}$ as a test function in (5.9) and taking the limit, we obtain

$$\int_{\mathbb{R}^3} (|\nabla \hat{u}|^2 + V(y^*)\hat{u}^2)\,dx \le \int_{\mathbb{R}^3} |\hat{u}|^q\,dx.$$

Then

$$I_{y^*}(\sigma\hat{u}) = \max_{t>0} I_{y^*}(t\hat{u}).$$

Finally, from Lemma 5.4 and since $0 < \sigma \le 1$ we have

$$\begin{aligned}
&\inf_{\xi\in\mathbb{R}^3} C(\xi)\\
&\le C(y^*) \le I_{y^*}(\sigma\hat{u})\\
&= \frac{\sigma^2}{4}\int_{\mathbb{R}^3} (|\nabla\hat{u}|^2 + V(y^*)\hat{u}^2)\,dx + \frac{\sigma^q(q-4)}{4q}\int_{\mathbb{R}^3} |\hat{u}|^q\,dx\\
&\le \frac{1}{4}\int_{\mathbb{R}^3} (|\nabla\hat{u}|^2 + V(y^*)\hat{u}^2)\,dx + \frac{q-4}{4q}\int_{\mathbb{R}^3} |\hat{u}|^q\,dx\\
&\le \liminf_{n\to\infty}\Big[\frac{1}{4}\int_{\mathbb{R}^3}\Big(|\nabla u_n|^2 + V(\varepsilon_n x + \varepsilon_n y_{\varepsilon_n})u_n^2\Big)\,dx + \frac{q-4}{4q}\int_{\mathbb{R}^3} |u_n|^q\,dx\Big]\\
&\le \liminf_{n\to\infty}\Big(\mathcal{I}_{\varepsilon_n}(u_n) - \frac{1}{4}\mathcal{I}'_{\varepsilon_n}(u_n)u_n\Big)\\
&= \liminf_{n\to\infty} c_{\varepsilon_n} \le \inf_{\xi\in\mathbb{R}^3} C(\xi)
\end{aligned}$$

which implies that $C(y^*) = \inf_{\xi\in\mathbb{R}^3} C(\xi)$. □

As a consequence of the previous lemma, there exists a subsequence of $(\varepsilon_n y_{\varepsilon_n})$ such that $\varepsilon_n y_{\varepsilon_n} \to y^*$.

Let $u_{\varepsilon_n}(x + y_{\varepsilon_n}) = u_n(x)$ and consider $\tilde{u} \in H^1$ such that $u_n \rightharpoonup \tilde{u}$.

Lemma 5.7. *$u_n \to \tilde{u}$ in $H^1(\mathbb{R}^3)$, as $n \to \infty$. Moreover, there exists $\varepsilon^* > 0$ such that $\lim_{|x|\to\infty} u_\varepsilon(x) = 0$ uniformly on $\varepsilon \in (0, \varepsilon^*)$.*

Proof. By Lemmas 5.4 and 5.6, we have

$$\begin{aligned}
&\inf_{\xi\in\mathbb{R}^3} C(\xi)\\
&= C(y^*) \le I_{y^*}(\tilde{u}) - \frac{1}{4}I'_{y^*}(\tilde{u})\tilde{u}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{4}\int_{\mathbb{R}^3}(|\nabla \tilde{u}|^2 + V(y^*)\tilde{u}^2)\,dx + \Big(\frac{q-4}{4q}\Big)\int_{\mathbb{R}^3}|\tilde{u}|^q\,dx \\
&\le \liminf_{n\to\infty}\Big\{\frac{1}{4}\int_{\mathbb{R}^3}(|\nabla u_n|^2 + V(\varepsilon_n x + \varepsilon_n y_{\varepsilon_n})u_n^2)\,dx + \Big(\frac{q-4}{4q}\Big)\int_{\mathbb{R}^3}|u_n|^q\,dx\Big\} \\
&\le \limsup_{n\to\infty}\Big\{\frac{1}{4}\int_{\mathbb{R}^3}(|\nabla u_n|^2 + V(\varepsilon_n x + \varepsilon_n y_{\varepsilon_n})u_n^2)\,dx + \Big(\frac{q-4}{4q}\Big)\int_{\mathbb{R}^3}|u_n|^q\,dx\Big\} \\
&= \limsup_{n\to\infty}\Big\{\mathcal{I}_{\varepsilon_n}(u_{\varepsilon_n}) - \frac{1}{4}\mathcal{I}'_{\varepsilon_n}(u_{\varepsilon_n})u_{\varepsilon_n}\Big\} \\
&= \limsup_{n\to\infty} c_{\varepsilon_n} \le \inf_{\xi\in\mathbb{R}^3} C(\xi)\,.
\end{aligned}
$$

Then

$$
\lim_{n\to\infty}\int_{\mathbb{R}^3}(|\nabla u_n|^2 + V(\varepsilon_n x + \varepsilon_n y_{\varepsilon_n})u_n^2)\,dx = \int_{\mathbb{R}^3}(|\nabla \tilde{u}|^2 + V(y^*)\tilde{u}^2)\,dx.
$$

Now observe that

$$
\begin{aligned}
c_{\varepsilon_n} &= \mathcal{I}_{\varepsilon_n}(u_{\varepsilon_n}) - \frac{1}{4}\mathcal{I}'_{\varepsilon_n}(u_{\varepsilon_n})u_{\varepsilon_n} \\
&= \frac{1}{4}\int_{\mathbb{R}^3}(|\nabla u_{\varepsilon_n}|^2 + V(\varepsilon_n x)u_{\varepsilon_n}^2)\,dx + \Big(\frac{q-4}{4q}\Big)\int_{\mathbb{R}^3}|u_{\varepsilon_n}|^q\,dx \\
&= \frac{1}{4}\int_{\mathbb{R}^3}(|\nabla u_n|^2 + V(\varepsilon_n x + \varepsilon_n y_{\varepsilon_n})u_n^2)\,dx + \Big(\frac{q-4}{4q}\Big)\int_{\mathbb{R}^3}|u_n|^q\,dx \\
&:= \alpha_n;
\end{aligned}
$$

hence,

$$
\limsup_{n\to\infty}\alpha_n = \limsup_{n\to\infty} c_{\varepsilon_n} \le C(y^*).
$$

On the other hand, using Fatou's Lemma,

$$
\begin{aligned}
\liminf_{n\to\infty}\alpha_n &\ge \frac{1}{4}\int_{\mathbb{R}^3}(|\nabla \tilde{u}|^2 + V(y^*)\tilde{u}^2)\,dx + \Big(\frac{q-4}{4q}\Big)\int_{\mathbb{R}^3}|\tilde{u}|^q\,dx \\
&= I_{y^*}(\tilde{u}) - \frac{1}{4}I'_{y^*}(\tilde{u})\tilde{u} \\
&\ge C(y^*);
\end{aligned}
$$

then $\lim_{n\to\infty}\alpha_n = C(y^*)$.

Therefore, since $\tilde{u}$ is the weak limit of (u_n) in $H^1(\mathbb{R}^3)$, we conclude that $u_n \to \tilde{u}$ strongly in $H^1(\mathbb{R}^3)$. In particular, we have

$$
\lim_{R\to\infty}\int_{|x|\ge R} u_n^{2^*}\,dx = 0 \quad \text{uniformly on } n. \tag{5.10}
$$

Applying Proposition 5.2 with $b(x) = V(\varepsilon_n x + \varepsilon_n y_{\varepsilon_n}) + \varepsilon_n^2 K(\varepsilon_n x + \varepsilon_n y_{\varepsilon_n})\phi_{u_n}$, we obtain $u_n \in L^t(\mathbb{R}^3)$, $t \ge 2$ and

$$
\|u_n\|_t \le C\|u_n\|,
$$

where C does not depend on n.

Now consider

$$
\begin{aligned}
-\Delta u_n &\le -\Delta u_n + V(\varepsilon_n x + \varepsilon_n y_{\varepsilon_n})u_n + \varepsilon_n^2 K(\varepsilon_n x + \varepsilon_n y_{\varepsilon_n})\phi_{u_n}u_n \\
&= |u_n|^{q-2}u_n := g_n(x).
\end{aligned}
$$

For some $t > 3$, $\|g_n\|_{\frac{t}{2}} \le C$, for all n. Using Proposition 5.3, we have

$$\sup_{B_R(y)} u_n \le C\Big(\|u_n\|_{L^2(B_{2R}(y))} + \|g_n\|_{L^{1/2}(B_{2R}(y))}\Big)$$

for every $y \in \mathbb{R}^3$, which implies that $\|u_n\|_{L^\infty(\mathbb{R}^3)}$ is uniformly bounded. Then, from (5.10),

$$\lim_{|x|\to\infty} u_n(x) = 0 \quad \text{uniformly on } n \in \mathbb{N}.$$

Consequently, there exists $\varepsilon^* > 0$ such that

$$\lim_{|x|\to\infty} u_\varepsilon(x) = 0 \quad \text{uniformly on } \varepsilon \in (0, \varepsilon^*).$$

□

To complete the proof of Theorem 1.1, it remains to show that the solutions of (1.1) have at most one local (hence global) maximum point y^* such that $C(y^*) = \min_{\xi\in\mathbb{R}^3} C(\xi)$.

From the previous Lemma, we can focus our attention only in a fixed ball $B_R(0) \subset \mathbb{R}^3$. If $w \in L^\infty(\mathbb{R}^3)$ is the limit in $C^2_{\rm loc}(\mathbb{R}^3)$ of

$$w_n(x) = u_n(x + y_n)$$

then, from Gidas, Ni and Nirenberg [13], w is radially symmetric and has a unique local maximum at zero which is a non-degenerate global maximum. Therefore, there exists $n_0 \in \mathbb{N}$ such that w_n does not have two critical points in $B_R(0)$ for all $n \ge n_0$. Consider $p_\varepsilon \in \mathbb{R}^3$ this local (hence global) maximum of w_ε.

Recall that if u_ε is a solution of (S_ε), then

$$v_\varepsilon(x) = u_\varepsilon(\frac{x}{\varepsilon})$$

is a solution of (1.1). Since p_ε is the unique maximum of w_ε, then $\hat{y_\varepsilon} = p_\varepsilon + y_\varepsilon$ is the unique maximum of u_ε. Hence, $\tilde{y}_\varepsilon = \varepsilon p_\varepsilon + \varepsilon y_\varepsilon$ is the unique maximum of v_ε. Once $p_\varepsilon \in B_R(0)$, that is, it is bounded, and $\varepsilon y_\varepsilon \to y^*$, we have

$$\tilde{y}_\varepsilon \to y^*.$$

where $C(y^*) = \inf_{\xi\in\mathbb{R}^3} C(\xi)$. Consequently, the concentration of functions v_ε approaches y^*.

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Patrícia L. Cunha
Departamento de Informática e Métodos Quantitativos, Fundação Getulio Vargas, São Paulo, Brazil
E-mail address: patcunha80@gmail.com