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ASYMPTOTIC BEHAVIOR OF SOLUTIONS TO A DEGENERATE QUASILINEAR PARABOLIC EQUATION WITH A GRADIENT TERM

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Abstract. This article concerns the asymptotic behavior of solutions to the Cauchy problem of a degenerate quasilinear parabolic equations with a gradient term. A blow-up theorem of Fujita type is established and the critical Fujita exponent is formulated by the spacial dimension and the behavior of the coefficient of the gradient term at ∞.

1. Introduction

In this article, we study the asymptotic behavior of solutions to the Cauchy problem

$$\frac{\partial u}{\partial t} = \Delta u^m + b(|x|)x \cdot \nabla u^m + u^p, \quad x \in \mathbb{R}^n,\ t > 0, \tag{1.1}$$

$$u(x,0) = u_0(x), \quad x \in \mathbb{R}^n, \tag{1.2}$$

where $p > m > 1$, $b \in C^{0,1}([0,+\infty))$ and $0 \le u_0 \in L^\infty(\mathbb{R}^n)$. The equation (1.1) is a typical quasilinear parabolic equaton which is called the Newtonian filtration equation. It is noted that (1.1) is degenerate at the points where $u = 0$. In the semilinear case $m = 1$, (1.1) is the heat equation.

The studies on asymptotic behavior of solutions to diffusion equations with nonlinear reaction began in 1966 by Fujita [6]. There it was proved that for (1.1)-(1.2) with $m = 1$ and $b \equiv 0$, there does not exist a nontrivial nonnegative global solution if $1 < p < p_c = 1 + 2/n$, whereas if $p > p_c$, there exist both nontrivial nonnegative global and blow-up solutions. This result shows that the exponent p of the nonlinear reaction affects the properties of solutions directly. We call p_c the critical Fujita exponent and such a result a blow-up theorem of Fujita type.

The elegant work of Fujita revealed a new phenomenon of nonlinear evolution equations. There have been a number of extensions of Fujita's results in several directions since then, including similar results for numerous of quasilinear parabolic equations and systems in various of geometries (whole spaces, cones and exterior domains) with nonlinear reactions or nonhomogeneous boundary conditions, and even degenerate equations in domains with non-compact boundary [1, 2]. We refer

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to the survey papers [4, 11] and the references therein, and more recent works [5, 12, 15, 17, 18, 19, 21, 22, 23, 24]. Among those extensions, it is Galaktionov [7, 8] who first investigated the blow-up theorem of Fujita type for (1.1)-(1.2) with $b \equiv 0$ and obtained that $p_c = m + 2/n$. As to nonlinear evolution equations with gradient terms, there are some studies for the semilinear case. In 1990, Meier [13] studied the critical Fujita exponent for the Cauchy problem of

$$\frac{\partial u}{\partial t} = \Delta u + \vec{b}(x) \cdot \nabla u + u^p, \quad x \in \mathbb{R}^n,\ t > 0, \tag{1.3}$$

where $\vec{b} \in L^\infty(\mathbb{R}^n; \mathbb{R}^n)$. It was proved that

$$p_c = 1 + \frac{1}{\lambda^*},$$

where λ^* is the maximal decay rate for solutions to

$$\frac{\partial w}{\partial t} = \Delta w + \vec{b}(x) \cdot \nabla w, \quad x \in \mathbb{R}^n,\ t > 0, \tag{1.4}$$

i.e.

$$\begin{aligned} \lambda^* = \sup \big\{ &\lambda \in \mathbb{R} :\ \text{there exists a nontrivial solution } w \text{ of } (1.4) \\ &\text{such that } \limsup_{t \to +\infty} t^\lambda \|w(\cdot, t)\|_{L^\infty(\mathbb{R})} < +\infty \big\}. \end{aligned}$$

If $\vec{b}$ is constant, it is clear that $\lambda^* = n/2$ and $p_c = 1 + 2/n$. However, for nonconstant $\vec{b} \in L^\infty(\mathbb{R}^n; \mathbb{R}^n)$, λ^* and p_c are unknown generally. In 1993, Aguirre and Escobedo [3] considered the Cauchy problem of

$$\frac{\partial u}{\partial t} = \Delta u + \vec{b}_0 \cdot \nabla u^q + u^p, \quad x \in \mathbb{R}^n,\ t > 0, \quad (0 \neq \vec{b}_0 \in \mathbb{R}^n,\ q > 1)$$

and proved that

$$p_c = \min \big\{ 1 + \frac{2}{n}, 1 + \frac{2q}{n+1} \big\}.$$

In [24], the semilinear problem (1.1)-(1.2) with $m = 1$ was studied and it was shown that if b satisfies

$$\lim_{s \to +\infty} s^2 b(s) = \kappa, \quad (-\infty \le \kappa \le +\infty), \tag{1.5}$$

$$\inf\{s^2 b(s) : s > 0\} > -n \text{ in the case } -n < \kappa \le +\infty, \tag{1.6}$$

then the critical Fujita exponent is

$$p_c = \begin{cases} 1, & \kappa = +\infty, \\ 1 + 2/(n + \kappa), & -n < \kappa < +\infty, \\ +\infty, & -\infty \le \kappa \le -n. \end{cases}$$

As to the quasilinear parabolic equations with gradient terms, Suzuki [16] in 1998 considered the Cauchy problem of

$$\frac{\partial u}{\partial t} = \Delta u^m + \vec{b}_0 \cdot \nabla u^q + u^p, \quad x \in \mathbb{R}^n,\ t > 0, \quad (m \ge 1,\ 0 \neq \vec{b}_0 \in \mathbb{R}^n,\ p, q > 1)$$

and proved that if $q > m - 1$ and $\max\{m, q\} \le p < \min\{m + 2/n, m + 2(q - m + 1)/(n + 1)\}$, then there does not exist any nontrivial nonnegative global solutions. In [23], the case

$$b(s) = \frac{\kappa}{s^2}, \quad s > 0, \quad (-\infty < \kappa < +\infty)$$

was studied. Since such a function is singular at 0 when $\kappa \neq 0$, the authors considered the Neumann exterior problem

$$
\begin{gathered}
\frac{\partial u}{\partial t} = \Delta u^m + \frac{\kappa}{|x|^2} x \cdot \nabla u^m + u^p, \quad x \in \mathbb{R}^n \setminus B_1,\ t > 0, \\
\frac{\partial u^m}{\partial \nu}(x,t) = 0, \quad x \in \partial B_1,\ t > 0, \\
u(x,0) = u_0(x), \quad x \in \mathbb{R}^n \setminus \overline{B}_1
\end{gathered}
$$

and showed that its critical Fujita exponent is

$$
p_c = \begin{cases} m + 2/(n+\kappa), & -n < \kappa < +\infty, \\ +\infty, & -\infty < \kappa \le -n, \end{cases}
$$

where B_1 is the unit ball in $\mathbb{R}^n$ and ν is the unit inner normal vector to ∂B_1. Also they considered a special case for the Dirichlet exterior problem.

In this article, we study the asymptotic behavior of solutions to the Cauchy problem (1.1)-(1.2), where b satisfies (1.5) and (1.6). It is proved that the critical Fujita exponent to (1.1)-(1.2) can be formulated as

$$
p_c = \begin{cases} m, & \kappa = +\infty, \\ m + 2/(n+\kappa), & -n < \kappa < +\infty, \\ +\infty, & -\infty \le \kappa \le -n. \end{cases} \tag{1.7}
$$

That is to say, if $m < p < p_c$, there does not exist any nontrivial nonnegative global solution to (1.1)-(1.2), whereas if $p > p_c$, there exist both nontrivial nonnegative global and blow-up solutions to (1.1)-(1.2). It is shown from (1.7) that the behavior of the coefficient of the gradient term at ∞, together with the spacial dimension, determines precisely the critical Fujita exponent to (1.1)-(1.2). The technique used in this paper is mainly inspired by [14, 17, 23, 24]. To prove the blow-up of solutions, we determine the interactions among the diffusion, the gradient and the reaction by a precise energy integral estimate instead of pointwise comparisons. The key is to choose a suitable weight for the energy integral. For the existence of global nontrivial solutions, we construct a global nontrivial supersolution. Noting that (1.1) does not possess a self-similar construct, we have to seek a complicated supersolution and do some precise calculations. By the way, (1.6) is used only for constructing a global nontrivial supersolution and it seems necessary when one constructs such a supersolution.

This article is organized as follows. We give some preliminaries in §2, such as the well-posedness of (1.1)-(1.2) and some auxiliary lemmas. The blow-up theorems of Fujita type for (1.1)-(1.2) are obtained in §3.

2. Preliminaries

Equation (1.1) is degenerate at the points where $u = 0$. So, weak solutions are considered at those points in this paper.

Definition 2.1. Let $0 < T \le +\infty$. A nonnegative function u is called a super (sub) solution to the problem (1.1), (1.2) in $(0,T)$, if

$$
u \in C([0,T), L^m_{\rm loc}(\mathbb{R}^n)) \cap L^\infty_{\rm loc}(0,T; L^\infty(\mathbb{R}^n))
$$

and the integral inequality

$$\begin{aligned} &\int_0^T \int_{\mathbb{R}^n} u(x,t)\frac{\partial \varphi}{\partial t}(x,t)\,dx\,dt + \int_0^T \int_{\mathbb{R}^n} u^m(x,t)\Delta\varphi(x,t)\,dx\,dt \\ &- \int_0^T \int_{\mathbb{R}^n} u^m(x,t)\operatorname{div}(b(|x|)\varphi(x,t)x)\,dx\,dt \\ &+ \int_0^T \int_{\mathbb{R}^n} u^p(x,t)\varphi(x,t)\,dx\,dt + \int_{\mathbb{R}^n} u_0(x)\varphi(x,0)dx \le (\ge) 0 \end{aligned}$$

is satisfied for each $0 \le \varphi \in C^{1,1}(\mathbb{R}^n \times [0,T))$ vanishing when t near T or $|x|$ being sufficiently large. A nonnegative function u is called a solution to the problem (1.1), (1.2) in $(0,T)$, if it is both a supersolution and a subsolution.

Definition 2.2. A solution u to the problem (1.1), (1.2) is said to blow up in a finite time $0 < T < +\infty$, if

$$\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \to +\infty \quad \text{as } t \to T^-.$$

Otherwise, u is said to be global.

For $0 \le u_0 \in L^\infty(\mathbb{R}^n)$, by using the theory on the Newtonian filtration equations (see, e.g., [9, 10, 20]), one can establish the existence, uniqueness and the comparison principle to the solutions of (1.1)-(1.2) locally in time. Moreover, it can be proved that

Lemma 2.3. *Assume that u is a solution to* (1.1)-(1.2) *in* $(0,T)$ *with* $0 < T \le +\infty$. *Then, for each* $\psi \in C_0^{1,1}(\mathbb{R}^n)$,

$$\begin{aligned} \frac{d}{dt}\int_{\mathbb{R}^n} u(x,t)\psi(x)dx = \int_{\mathbb{R}^n} u^m(x,t)\Delta\psi(x)\,dx\,dt - \int_{\mathbb{R}^n} u^m(x,t)\operatorname{div}(b(|x|)\psi(x)x)dx \\ + \int_{\mathbb{R}^n} u^p(x,t)\psi(x)dx \end{aligned}$$

in the distribution sense.

To investigate the blow-up property of solutions to (1.1)-(1.2), we need the following auxiliary lemma.

Lemma 2.4. *Assume that* $b \in C^{0,1}([0,+\infty))$ *satisfies* (1.5) *with* $-\infty \le \kappa < +\infty$. *Let u be a solution to the problem* (1.1), (1.2). *Then there exist three numbers* $R_0 > 0$, $\delta > 1$ *and* $M_0 > 0$ *depending only on n and b, such that for each* $R > R_0$,

$$\begin{aligned} \frac{d}{dt}\int_{\mathbb{R}^n} u(x,t)\psi_R(|x|)dx \ge -M_0R^{-2}\int_{B_{\delta R}\setminus B_R} u^m(x,t)\psi_R(|x|)dx \\ + \int_{\mathbb{R}^n} u^p(x,t)\psi_R(|x|)dx, \end{aligned} \tag{2.1}$$

for $t > 0$, *where*

$$\psi_R(r) = \begin{cases} h(r), & 0 \le r \le R, \\ \frac{1}{2}h(r)\Big(1 + \cos\frac{(r-R)\pi}{(\delta-1)R}\Big), & R < r < \delta R, \\ 0, & r \ge \delta R \end{cases}$$

with

$$h(r) = \exp\Big\{\int_0^r sb(s)ds\Big\}, \quad r \ge 0,$$

while B_r denotes the open ball in $\mathbb{R}^n$ with radius r and centered at the origin.

Proof. It is clear that $\psi_R \in C^{1,1}([0,+\infty))$ with $\psi_R'(0) = 0$. Choosing $\psi(x) = \psi_R(|x|)$ in Lemma 2.3, one gets that

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^n} u(x,t)\psi_R(|x|)dx \\
&= \int_{\mathbb{R}^n} u^m(x,t)\Delta\psi_R(|x|)dx - \int_{\mathbb{R}^n} u^m(x,t)\operatorname{div}(b(|x|)\psi_R(|x|)x)dx \\
&\quad + \int_{\mathbb{R}^n} u^p(x,t)\psi_R(|x|)dx \\
&= \int_{B_{\delta R}} u^m(x,t)\big(\Delta\psi_R(|x|) - \operatorname{div}(b(|x|)\psi_R(|x|)x)\big)dx \\
&\quad + \int_{\mathbb{R}^n} u^p(x,t)\psi_R(|x|)dx.
\end{aligned} \tag{2.2}$$

As shown in [24], there exist three numbers $R_0 > 0$, $\delta > 1$ and $M_0 > 0$ depending only on n and b, such that for each $R > R_0$,

$$\Delta\psi_R(|x|) - \operatorname{div}(b(|x|)\psi_R(|x|)x) \geq -M_0R^{-2}\psi_R(|x|), \quad |x| > 0. \tag{2.3}$$

Substituting (2.3) into (2.2) leads to (2.1). □

Remark 2.5. Lemma 2.4 still holds if (1.5) is replaced by

$$\limsup_{s\to+\infty} s^2b(s) = \kappa.$$

Remark 2.6. The proof of Lemma 2.4 is invalid if $\kappa = +\infty$. In this case, (2.1) holds for each fixed $R > 0$, but $\delta > 1$ and M_0 depend also on R.

Next, we study self-similar supersolutions of (1.1) of the form

$$u(x,t) = (t+T)^{-\alpha}v((t+T)^{-\beta}|x|), \quad x \in \mathbb{R}^n,\ t \geq 0, \tag{2.4}$$

where

$$\alpha = \frac{1}{p-1}, \quad \beta = \frac{p-m}{2(p-1)},$$

$T \geq 1$ will be determined. If $v \in C^{0,1}([0,+\infty))$ with $v^m \in C^{1,1}([0,+\infty))$ solves

$$\begin{aligned}
&(v^m)''(r) + \frac{n-1}{r}(v^m)'(r) + (t+T)^{2\beta}b((t+T)^{\beta}r)r(v^m)'(r) \\
&+ \beta rv'(r) + \alpha v(r) + v^p(r) \leq 0, \quad r > 0
\end{aligned} \tag{2.5}$$

for each $t > 0$, then u given by (2.4) is a supersolution to (1.1).

Lemma 2.7. *Assume that $b \in C^{0,1}([0,+\infty))$ satisfies* (1.5) *and* (1.6) *with $-n < \kappa \leq +\infty$, and $p > p_c$. Choose $-n < \kappa_1 < \kappa_2 < \kappa$ such that*

$$\inf\{s^2b(s) : s > 0\} > \kappa_1, \quad \kappa_2 > \frac{1}{p-m} - n. \tag{2.6}$$

Let $T = \eta^{-(m+2)/\beta}$ and

$$v(r) = (\eta - A(r))_+^{1/(m-1)}, \quad r \geq 0, \tag{2.7}$$

where $s_+ = \max\{0, s\}$, $0 < \eta < 1$ *will be determined, while* $A \in C^{1,1}([0, +\infty))$ *satisfies* $A(0) = 0$ *and*

$$A'(r) = \begin{cases} A_1 r, & 0 \le r \le \eta^{m+1}, \\ A_2 r + (A_1 - A_2)\frac{\eta^{(m+1)(n+\kappa_2)}}{r^{n+\kappa_2-1}}, & \eta^{m+1} < r < \eta^m, \\ A_2 r + (A_1 - A_2)\eta^{n+\kappa_2} r, & r \ge \eta^m \end{cases}$$

with

$$A_1 = \max\Big\{\frac{2(m-1)}{m(n+\kappa_1)(p+p_c-2)}, A_2\Big\},$$
$$A_2 = \frac{m-1}{2m(n+\kappa_2)(p-1)} + \frac{(m-1)(p-m)}{4m(p-1)}.$$

Then, there exists sufficiently small $0 < \eta < 1$ *such that* u *given by* (2.4) *and* (2.7) *is a supersolution to* (1.1).

Proof. Choose $\eta_0 \in (0, 1)$ such that for each $0 < \eta < \eta_0$,

$$A(r) < r, \quad 0 < r < \eta_0. \tag{2.8}$$

By the first formula in (2.6), one has

$$(t+T)^{2\beta} b((t+T)^\beta r) r \ge \frac{\kappa_1}{r}, \quad r > 0,\ t > 0. \tag{2.9}$$

Therefore, for each $0 < \eta < \eta_0$, it follows from (2.8) and (2.9) that

$$\begin{aligned}
&(v^m)''(r) + \frac{n-1}{r}(v^m)'(r) + (t+T)^{2\beta} b((t+T)^\beta r) r (v^m)'(r) \\
&+ \beta r v'(r) + \alpha v(r) + v^p(r) \\
&\le \frac{m}{(m-1)^2}(A'(r))^2 v^{2-m}(r) + \Big(\alpha - \frac{m}{m-1}A''(r) - \frac{m(n+\kappa_1-1)}{m-1}\frac{A'(r)}{r}\Big) v(r) \\
&\quad + v^p(r) \\
&= \frac{A_1^2 m}{(m-1)^2} r^2 v^{2-m}(r) + \Big(\alpha - \frac{A_1 m(n+\kappa_1)}{m-1}\Big) v(r) + v^p(r) \\
&\le \Big(\frac{A_1^2 m}{(m-1)^2}\eta^{2m-1}(1-\eta^{m-1})^{-1} + \Big(\alpha - \frac{A_1 m(n+\kappa_1)}{m-1}\Big) + \eta^{(p-1)/(m-1)}\Big) v(r),
\end{aligned}$$

for $0 < r < \eta^{m+1}$ and $t > 0$. The choice of A_1 implies

$$\alpha < \frac{A_1 m(n+\kappa_1)}{m-1}.$$

Thus, there exists $0 < \eta_1 < \eta_0$ such that for each $0 < \eta < \eta_1$,

$$\begin{gathered}
(v^m)''(r) + \frac{n-1}{r}(v^m)'(r) + (t+T)^{2\beta} b((t+T)^\beta r) r (v^m)'(r) \\
+ \beta r v'(r) + \alpha v(r) + v^p(r) \le 0, \\
0 < r < \eta^{m+1}, \quad t > 0.
\end{gathered} \tag{2.10}$$

By (1.5) and $\kappa_2 < \kappa$, there exists $0 < \eta_2 < \eta_0$ such that

$$s^2 b(s) \ge \kappa_2, \quad s > \frac{1}{\eta_2},$$

which, together with the choice of T, implies that for each $0 < \eta < \eta_2$,

$$(t+T)^{2\beta} b((t+T)^{\beta} r) r \geq \frac{\kappa_2}{r}, \quad r > \eta^{m+1},\ t > 0. \tag{2.11}$$

For each $0 < \eta < \eta_2$, it follows from (2.11) and (2.8) that

$$\begin{aligned}
&(v^m)''(r) + \frac{n-1}{r}(v^m)'(r) + (t+T)^{2\beta} b((t+T)^{\beta} r) r (v^m)'(r) \\
&+ \beta r v'(r) + \alpha v(r) + v^p(r) \\
&\leq \frac{m}{(m-1)^2}(A'(r))^2 v^{2-m}(r) + \Big(\alpha - \frac{m}{m-1}A''(r) - \frac{m(n+\kappa_2-1)}{m-1}\frac{A'(r)}{r}\Big)v(r) \\
&\quad + v^p(r) \\
&= \frac{m}{(m-1)^2}\Big(A_2 + (A_1 - A_2)\frac{\eta^{(m+1)(n+\kappa_2)}}{r^{n+\kappa_2}}\Big)^2 r^2 v^{2-m}(r) \\
&\quad + \Big(\alpha - \frac{A_2 m(n+\kappa_2)}{m-1}\Big)v(r) + v^p(r) \\
&\leq \Big(\frac{mA_1^2}{(m-1)^2}\eta^{2m-1}(1-\eta^{m-1})^{-1} + \Big(\alpha - \frac{A_2 m(n+\kappa_2)}{m-1}\Big) + \eta^{(p-1)/(m-1)}\Big)v(r),
\end{aligned}$$

for $\eta^{m+1} < r < \eta^m$ and $t > 0$. The choice of A_2 implies

$$\alpha < \frac{A_2 m(n+\kappa_2)}{m-1}.$$

Thus, there exists $0 < \eta_3 < \eta_2$ such that for each $0 < \eta < \eta_3$,

$$\begin{aligned}
&(v^m)''(r) + \frac{n-1}{r}(v^m)'(r) + (t+T)^{2\beta} b((t+T)^{\beta} r) r (v^m)'(r) \\
&+ \beta r v'(r) + \alpha v(r) + v^p(r) \leq 0, \quad \eta^{m+1} < r < \eta^m,\ t > 0.
\end{aligned} \tag{2.12}$$

For each $0 < \eta < \eta_0$, (2.8) yields $A^{-1}(\eta) > \eta$. Thus, for each $0 < \eta < \eta_2$, it follows from (2.11) that

$$\begin{aligned}
&(v^m)''(r) + \frac{n-1}{r}(v^m)'(r) + (t+T)^{2\beta} b((t+T)^{\beta} r) r (v^m)'(r) \\
&+ \beta r v'(r) + \alpha v(r) + v^p(r) \\
&\leq \frac{1}{m-1} r A'(r)\Big(\frac{m}{m-1}\frac{A'(r)}{r} - \beta\Big)v^{2-m}(r) \\
&\quad + \Big(\alpha - \frac{m}{m-1}A''(r) - \frac{m(n+\kappa_2-1)}{m-1}\frac{A'(r)}{r}\Big)v(r) + v^p(r) \\
&= \frac{A_2}{m-1} r^2 \Big(\frac{mA_2}{m-1} - \beta\Big)v^{2-m}(r) + \Big(\alpha - \frac{A_2 m(n+\kappa_2)}{m-1}\Big)v(r) + v^p(r),
\end{aligned}$$

for $\eta^m < r < A^{-1}(\eta)$ and $t > 0$. The choice of A_2 implies

$$\frac{mA_2}{m-1} < \beta, \quad \alpha < \frac{A_2 m(n+\kappa_2)}{m-1}.$$

Thus, there exists $0 < \eta_4 < \eta_2$ such that for each $0 < \eta < \eta_4$,

$$\begin{aligned}
&(v^m)''(r) + \frac{n-1}{r}(v^m)'(r) + (t+T)^{2\beta} b((t+T)^{\beta} r) r (v^m)'(r) \\
&+ \beta r v'(r) + \alpha v(r) + v^p(r) \leq 0, \quad \eta^m < r < A^{-1}(\eta),\ t > 0.
\end{aligned} \tag{2.13}$$

Summing up, from (2.10), (2.12) and (2.13), for $0 < \eta < \min\{\eta_1, \eta_3, \eta_4\}$, the function u given by (2.4) and (2.7) is a supersolution of (1.1). $\square$

Remark 2.8. Lemma 2.7 still holds if (1.5) is replaced by

$$\liminf_{s\to+\infty} s^2 b(s) = \kappa.$$

Remark 2.9. In Lemma 2.7, (1.6) is necessary to get a supersolution to (1.1) of the form (2.4) and (2.7). But, it is not clear at this moment whether (1.1) admits a global supersolution of other form if (1.6) is invalid.

3. Blow-up theorem of Fujita type

In this section, we establish the blow-up theorem of Fujita type for the problem (1.1), (1.2) by using Lemmas 2.4 and 2.7. First consider the case $m < p < p_c$ with $-\infty \le \kappa < +\infty$.

Theorem 3.1. *Assume that $b \in C^{0,1}([0, +\infty))$ satisfies (1.5) with $-\infty \le \kappa < +\infty$. Let $m < p < p_c$. Then for each nontrivial $0 \le u_0 \in L^\infty(\mathbb{R}^n)$, the solution to the problem (1.1), (1.2) must blow up in a finite time.*

Proof. Let ψ_R, h, R_0, δ and M_0 be given by Lemma 2.4. Owing to $-\infty \le \kappa < +\infty$ and $1 < p < p_c$,

$$\kappa < \frac{2}{p-m} - n.$$

Fix $\tilde\kappa$ to satisfy

$$\kappa < \tilde\kappa < \frac{2}{p-m} - n. \tag{3.1}$$

By (1.5) and (3.1), there exists $R_1 > 0$ such that

$$s^2 b(s) < \tilde\kappa, \quad s > R_1.$$

For each $R > R_1$, one obtain

$$\begin{aligned}
\int_{\mathbb{R}^n} \psi_R(|x|)dx &\le n\omega_n \int_0^{\delta R} r^{n-1} h(r)dr \\
&\le \omega_n(\delta R)^n \exp\Big\{\int_0^{\delta R} sb(s)ds\Big\} \\
&\le \omega_n(\delta R)^n \exp\Big\{\int_0^{R_1} sb(s)ds\Big\} \exp\Big\{\tilde\kappa \int_{R_1}^{\delta R} \frac{1}{s} ds\Big\} \\
&= M_1 R^{n+\tilde\kappa},
\end{aligned} \tag{3.2}$$

where ω_n is the volume of the unit ball in $\mathbb{R}^n$, while $M_1 > 0$ depends only on n, b, R_1, δ and $\tilde\kappa$. Let u be the solution to the problem (1.1), (1.2). Denote

$$w_R(t) = \int_{\mathbb{R}^n} u(x,t)\psi_R(|x|)dx, \quad t \ge 0.$$

For each $R > \max\{R_0, R_1\}$, it follows from Lemma 2.4 that

$$\frac{d}{dt} w_R(t) \ge -M_0 R^{-2} \int_{\mathbb{R}^n} u^m(x,t)\psi_R(|x|)dx + \int_{\mathbb{R}^n} u^p(x,t)\psi_R(|x|)dx, \tag{3.3}$$

for $t>0$. The Hölder inequality and (3.2) yield

$$\begin{aligned}
&\int_{\mathbb{R}^n} u^m(x,t)\psi_R(|x|)dx\\
&\le \Big(\int_{\mathbb{R}^n}\psi_R(|x|)dx\Big)^{(p-m)/p}\Big(\int_{\mathbb{R}^n}u^p(x,t)\psi_R(|x|)dx\Big)^{m/p}\\
&\le M_1^{(p-m)/p}R^{(p-m)(n+\tilde\kappa)/p}\Big(\int_{\mathbb{R}^n}u^p(x,t)\psi_R(|x|)dx\Big)^{m/p},\quad t>0.
\end{aligned}\tag{3.4}$$

Substitute (3.4) into (3.3) to obtain

$$\begin{aligned}
\frac{d}{dt}w_R(t)\ge{}&\Big(\int_{\mathbb{R}^n}u^p(x,t)\psi_R(|x|)dx\Big)^{m/p}\Big(-M_0M_1^{(p-m)/p}R^{-2+(p-m)(n+\tilde\kappa)/p}\\
&+\Big(\int_{\mathbb{R}^n}u^p(x,t)\psi_R(|x|)dx\Big)^{(p-m)/p}\Big),\quad t>0.
\end{aligned}\tag{3.5}$$

It follows from the Hölder inequality and (3.2) that

$$\begin{aligned}
\int_{\mathbb{R}^n}u(x,t)\psi_R(|x|)dx&\le\Big(\int_{\mathbb{R}^n}\psi_R(|x|)dx\Big)^{(p-1)/p}\Big(\int_{\mathbb{R}^n}u^p(x,t)\psi_R(|x|)dx\Big)^{1/p}\\
&\le M_1^{(p-1)/p}R^{(p-1)(n+\tilde\kappa)/p}\Big(\int_{\mathbb{R}^n}u^p(x,t)\psi_R(|x|)dx\Big)^{1/p},
\end{aligned}$$

for $t>0$, which implies

$$\int_{\mathbb{R}^n}u^p(x,t)\psi_R(|x|)dx\ge M_1^{-(p-1)}R^{-(p-1)(n+\tilde\kappa)}w_R^p(t),\quad t>0.\tag{3.6}$$

Substituting (3.6) into (3.5), one gets that for each $R>\max\{R_0,R_1\}$,

$$\begin{aligned}
\frac{d}{dt}w_R(t)\ge{}&M_1^{-m(p-1)/p}R^{-m(p-1)(n+\tilde\kappa)/p}w_R^m(t)\\
&\times\Big(-M_0M_1^{(p-m)/p}R^{-2+(p-m)(n+\tilde\kappa)/p}\\
&+M_1^{-(p-1)(p-m)/p}R^{-(p-1)(p-m)(n+\tilde\kappa)/p}w_R^{(p-m)}(t)\Big),\quad t>0.
\end{aligned}\tag{3.7}$$

Note that (3.1) implies $2>(p-m)(n+\tilde\kappa)$, while $w_R(0)$ is nondecreasing with respect to $R\in(0,+\infty)$ and

$$\sup\{w_R(0):R>0\}>0.$$

Therefore, there exists $R_2>0$ such that for each $R>R_2$,

$$\begin{aligned}
&M_0M_1^{(p-m)/p}R^{-2+(p-m)(n+\tilde\kappa)/p}\\
&\le\frac12M_1^{-(p-1)(p-m)/p}R^{-(p-1)(p-m)(n+\tilde\kappa)/p}w_R^{(p-m)}(0).
\end{aligned}\tag{3.8}$$

Fix $R>\max\{R_0,R_1,R_2\}$. Then, (3.7) and (3.8) yield

$$\frac{d}{dt}w_R(t)\ge\frac12M_1^{-p(p-1)/p}R^{-p(p-1)(n+\tilde\kappa)/p}w_R^p(t),\quad t>0.$$

Since $p>m>1$, there exists $T>0$ such that

$$w_R(t)=\int_{\mathbb{R}^n}u(x,t)\psi_R(|x|)dx\to+\infty\quad\text{as }t\to T^-.$$

Noting that $\operatorname{supp}\psi_R(|x|)$ is bounded, one gets

$$\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^n)}\to+\infty\quad\text{as }t\to T^-.$$

That is to say, u blows up in a finite time. $\square$

Turn to the case $p > p_c$ with $-n < \kappa \le +\infty$.

Theorem 3.2. *Assume that $b \in C^{0,1}([0, +\infty))$ satisfies (1.5) and (1.6) with $-n < \kappa \le +\infty$. Let $p > p_c$. Then there exist both nontrivial nonnegative global and blow-up solutions of problem (1.1)-(1.2).*

Proof. The comparison principle and Lemma 2.7 yield that problem (1.1)-(1.2) admits a global nontrivial solution. Let us show there also exists a blow-up solutions to (1.1)-(1.2). Fix $R > R_0$. Assume that u is a solution to (1.1)-(1.2). It follows from Lemma 2.4 (the case $-n < \kappa < +\infty$), Remark 2.6 (the case $\kappa = +\infty$) that there exist a nontrivial function $0 \le \psi \in C_0^{1,1}(\mathbb{R}^n)$, with $\|\psi\|_{L^1(\mathbb{R}^n)} \le 1$, and a constant $M > 0$, both depending only on n, R and b, such that

$$\frac{d}{dt}w(t) \ge -M\int_{\mathbb{R}^n} u^m(x,t)\psi(x)dx + \int_{\mathbb{R}^n} u^p(x,t)\psi(x)dx, \quad t > 0, \tag{3.9}$$

where

$$w(t) = \int_{\mathbb{R}^n} u(x,t)\psi(x)dx, \quad t \ge 0.$$

The Hölder inequality yields

$$\int_{\mathbb{R}^n} u^m(x,t)\psi(x)dx \le \Big(\int_{\mathbb{R}^n} \psi(x)dx\Big)^{(p-m)/p}\Big(\int_{\mathbb{R}^n} u^p(x,t)\psi(x)dx\Big)^{m/p}, \tag{3.10}$$

for $t > 0$. Substitute (3.10) into (3.9) to get

$$\frac{d}{dt}w(t) \ge \Big(\int_{\mathbb{R}^n} u^p(x,t)\psi(x)dx\Big)^{m/p}\Big(-M + \Big(\int_{\mathbb{R}^n} u^p(x,t)\psi(x)dx\Big)^{(p-m)/p}\Big), \tag{3.11}$$

for $t > 0$. It follows from the Hölder inequality that

$$\int_{\mathbb{R}^n} u(x,t)\psi(x)dx \le \Big(\int_{\mathbb{R}^n} \psi(x)dx\Big)^{(p-1)/p}\Big(\int_{\mathbb{R}^n} u^p(x,t)\psi(x)dx\Big)^{1/p}, \quad t > 0,$$

which implies

$$\int_{\mathbb{R}^n} u^p(x,t)\psi(x)dx \ge \Big(\int_{\mathbb{R}^n} \psi(x)dx\Big)^{-(p-1)} w^p(t), \quad t > 0. \tag{3.12}$$

Substituting (3.12) into (3.11), one gets that

$$\begin{aligned} \frac{d}{dt}w(t) \ge \Big(\int_{\mathbb{R}^n} \psi(x)dx\Big)^{-m(p-1)/p} w^m(t)\Big(-M \\ + \Big(\int_{\mathbb{R}^n} \psi(x)dx\Big)^{-(p-m)(p-1)/p} w^{p-m}(t)\Big), \quad t > 0. \end{aligned} \tag{3.13}$$

If u_0 is so large that

$$w(0) = \int_{\mathbb{R}^n} u_0(x)\psi(x)dx \ge (2M)^{1/(p-m)}\Big(\int_{\mathbb{R}^n} \psi(x)dx\Big)^{(p-1)/p}.$$

Then, (3.13) leads to

$$\frac{d}{dt}w(t) \ge \frac{1}{2}\Big(\int_{\mathbb{R}^n} \psi(x)dx\Big)^{-p(p-1)/p} w^p(t), \quad t > 0.$$

By the same argument as in the end of the proof of Theorem 3.1, u must blow up in a finite time. $\square$

Remark 3.3. In Theorem 3.1, b need not to satisfy (1.6) even if $-n < \kappa < +\infty$. However, (1.6) is needed in the proof of Lemma 2.7 and thus in the proof of Theorem 3.2.

According to Remarks 2.5 and 2.8, one gets the following statement.

Remark 3.4. *Theorem 3.1 still holds if* (1.5) *is replaced by*

$$\limsup_{s\to+\infty} s^2 b(s) = \kappa,$$

while Theorem 3.2 still holds if (1.5) *is replaced by*

$$\liminf_{s\to+\infty} s^2 b(s) = \kappa.$$

From Theorems 3.1 and 3.2 we have the following statement.

Remark 3.5. For problem (1.1)-(1.2), $p_c = m$ if $\lim_{s\to+\infty} s^2 b(s) = +\infty$, while $p_c = +\infty$ if $\limsup_{s\to+\infty} s^2 b(s) \le -n$. In particular, $p_c = m$ for the Cauchy problems of

$$\frac{\partial u}{\partial t} = \Delta u^m + x \cdot \nabla u^m + u^p, \quad x \in \mathbb{R}^n,\ t > 0$$

and

$$\frac{\partial u}{\partial t} = \Delta u^m + \frac{x}{|x|+1} \cdot \nabla u^m + u^p, \quad x \in \mathbb{R}^n,\ t > 0,$$

while $p_c = +\infty$ for the Cauchy problems of

$$\frac{\partial u}{\partial t} = \Delta u^m - x \cdot \nabla u^m + u^p, \quad x \in \mathbb{R}^n,\ t > 0$$

and

$$\frac{\partial u}{\partial t} = \Delta u^m - \frac{x}{|x|+1} \cdot \nabla u^m + u^p, \quad x \in \mathbb{R}^n,\ t > 0.$$

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