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POSITIVE RADIALLY SYMMETRIC SOLUTION FOR A SYSTEM OF QUASILINEAR BIHARMONIC EQUATIONS IN THE PLANE

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Abstract. We study the boundary value system for the two-dimensional quasilinear biharmonic equations

$$\begin{gathered}\Delta(|\Delta u_i|^{p-2}\Delta u_i) = \lambda_i w_i(x) f_i(u_1,\dots,u_m), \quad x \in B_1,\\ u_i = \Delta u_i = 0, \quad x \in \partial B_1, \quad i = 1,\dots,m,\end{gathered}$$

where $B_1 = \{x \in \mathbb{R}^2 : |x| < 1\}$. Under some suitable conditions on w_i and f_i, we discuss the existence, uniqueness, and dependence of positive radially symmetric solutions on the parameters $\lambda_1,\dots,\lambda_m$. Moreover, two sequences are constructed so that they converge uniformly to the unique solution of the problem. An application to a special problem is also presented.

1. Introduction

In this article, we are concerned with the existence, uniqueness, and dependence of positive radially symmetric solutions of the following boundary value system for the two-dimensional quasilinear biharmonic equations

$$\begin{gathered}\Delta(|\Delta u_i|^{p-2}\Delta u_i) = \lambda_i w_i(x) f_i(u_1,\dots,u_m), \quad x \in B_1,\\ u_i = \Delta u_i = 0, \quad x \in \partial B_1, \quad i = 1,\dots,m,\end{gathered} \tag{1.1}$$

where $B_1 = \{x \in \mathbb{R}^2 : |x| < 1\}$ with $x = (x_1, x_2)$ and $|x| = \sqrt{x_1^2 + x_2^2}$, $\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$, $p > 1$ is a constant, $m \ge 1$ is an integer, λ_i is a positive parameter, $w_i(x)$ is radially symmetric, namely, $w_i(x) = w(|x|)$, f_i is continuous and positive on $(0,\infty)^m$, and $f_i(y_1,\dots,y_m)$ may be singular at $y_i = 0$.

Assume that $(u_1(t),\dots,u_m(t)) = (u_1(|x|),\dots,u_m(|x|))$ with $t = |x|$ is a radially symmetric solution of (1.1). Then, direct calculations show that for $i = 1,\dots,m$,

$$\begin{gathered}\mathcal{L}(|\mathcal{L}u|^{p-2}\mathcal{L}u) = \lambda_i w_i(t) f_i(u_1,\dots,u_m), \quad t \in (0,1),\\ u_i'(0) = u_i(1) = (|\mathcal{L}u|^{p-2}\mathcal{L}u)'|_{t=0} = (|\mathcal{L}u|^{p-2}\mathcal{L}u)|_{t=1} = 0,\end{gathered} \tag{1.2}$$

where $\mathcal{L}$ denotes the polar form of the two-dimensional Laplacian operator Δ, i.e.,

$$\mathcal{L} = \frac{1}{t}\frac{d}{dt}\Big(t\frac{d}{dt}\Big).$$

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Thus, to study positive radially symmetric solutions of (1.1), it suffices to study positive solutions of problem (1.2).

Recently, Guo, Yin, and Ke [9] studied the scalar case of (1.1), i.e.,

$$\begin{gathered} \Delta(|\Delta u|^{p-2}\Delta u) = \lambda w(x) f(u), \quad x \in B_1, \\ u = \Delta u = 0, \quad x \in \partial B_1. \end{gathered} \tag{1.3}$$

Using fixed point index theory, they found some sufficient conditions under which there exists $\lambda^* > 0$ such that (1.3) has two positive radially symmetric solutions for each $0 < \lambda < \lambda^*$, has one positive radially symmetric solution for $\lambda = \lambda^*$, and does not have a positive radially symmetric solution for any $\lambda > \lambda^*$. In [9], it was assumed, among others, that f is continuous on $[0,\infty)$ and nondecreasing on $[0,\infty)$. Such assumptions are not needed in this paper.

The background information on problem (1.3) and its related applications can be found, for example, in [2, 3, 16]. In recent years, fourth order nonlinear differential equations have become increasingly popular due to their possible applications in the fields of image and signal processing, nuclear physics, and engineering. We refer the reader to [3, 4, 5, 8, 6, 10, 11, 13] for a small sample of the work. Among these works, papers [3, 5, 8] studied the existence of solutions of biharmonic equations with singular nonlinearities and only paper [5] considered the uniqueness of positive solutions. Specifically, paper [5] established criteria for the existence and uniqueness of positive solutions of the problem

$$\begin{gathered} \Delta^2 u = u^{-\beta}, \quad x \in \Omega, \\ u = \partial u/\partial \nu = 0, \quad x \in \partial\Omega, \end{gathered}$$

where $0 < \beta < 1$, $\Omega \subset \mathbb{R}^n$ with $n \geq 2$ is a smooth bounded domain and ν is the exterior unit normal vector at $\partial\Omega$.

To the best of our knowledge, little work has been done in the literature on the uniqueness and dependence of positive solutions of biharmonic equations. In this paper, we not only investigate the existence and uniqueness of positive solutions of problem (1.2), but also discuss the dependence of positive solutions on the parameters $\lambda_1, \dots, \lambda_m$. Moreover, in our theorem, two sequences are constructed in such a way that they converge uniformly to the unique positive solution of the problem. As a simple application of our theory, we also present some uniqueness and dependence results for the following special case of problem (1.2)

$$\begin{gathered} \mathcal{L}(|\mathcal{L}u|^{p-2}\mathcal{L}u) = \lambda_i w_i(t)\Big(\sum_{k=1}^m a_{ik} u_k^{b_{ik}} + \sum_{k=1}^m c_{ik} u_k^{-d_{ik}}\Big), \quad t \in (0,1), \\ u_i'(0) = u_i(1) = (|\mathcal{L}u|^{p-2}\mathcal{L}u)'|_{t=0} = (|\mathcal{L}u|^{p-2}\mathcal{L}u)|_{t=1} = 0, \quad i = 1, \dots, m, \end{gathered} \tag{1.4}$$

where a_{ik}, b_{ik}, $d_{ik} \geq 0$, and $c_{ik} > 0$, i, $k = 1, \dots, m$. In our proof, part of the analysis relies on some results from mixed monotone operator theory. This technique was introduced by Guo and Lakshmikantham [7] in 1987. Since then, many authors have investigated such operators and related applications to a variety of problems; see, for example, [12, 14, 15, 17] and the references therein.

In this article, we need the following assumptions:

- (H1) for $i = 1, \dots, m$ and $y \in (0,\infty)^m$, $f_i(y)$ can be written as $f_i(y) = g_i(y) + h_i(y)$, where $g_i : [0,\infty)^m \to [0,\infty)$ is continuous and nondecreasing in each of its arguments, and $h_i : (0,\infty)^m \to (0,\infty)$ is continuous and nonincreasing in each of its arguments;

(H2) for $i = 1, \dots, m$, $w_i : (0,1) \to [0,\infty)$ is continuous and
$$0 < -\int_0^1 s(\ln s) w_i(s) h_i(\rho(1-s))ds < \infty \quad \text{for any constant } \rho > 0;$$
(H3) for $i = 1, \dots, m$, there exists $\alpha \in (0,1)$ such that
$$g_i(\kappa y) \geq \kappa^{(p-1)\alpha} g_i(y), \tag{1.5}$$
$$h_i(\kappa^{-1} y) \geq \kappa^{(p-1)\alpha} h_i(y) \tag{1.6}$$
for $\kappa \in (0,1)$ and $y \in (0,\infty)^m$.

The rest of this article is organized as follows. Section 2 contains some preliminary lemmas, Section 3 contains the main results of this paper and their proofs.

2. Preliminary lemmas

Lemma 2.1 below provides the equivalent integral form of problem (1.2).

Lemma 2.1. *The function $u(t) = (u_1(t), \dots, u_m(t))$ is a solution of (1.2) if and only if*
$$u_i(t) = \lambda_i^{\frac{1}{p-1}} \int_0^1 k(t,\tau)\phi_q\Big(\int_0^1 k(\tau,s) w_i(s) f_i(u(s))ds\Big)d\tau,$$
where $\phi_q(s) = |s|^{q-2}s$ with $1/p + 1/q = 1$ and
$$k(t,s) = \begin{cases} -s\ln t, & 0 \leq s \leq \tau \leq 1, \\ -s\ln s, & 0 \leq t \leq s \leq 1. \end{cases} \tag{2.1}$$

A similar version of Lemma 2.1 with $m = 1$ has been proved in [9, Pages 1322–1323]. The present version can be proved in the same way.

The following lemma summarizes some properties of the function $k(t,s)$.

Lemma 2.2. *The function $k(t,s)$ defined by (2.1) satisfies*

(a) $k(t,s) > 0$ *for* $t, s \in (0,1)$;
(b) $k(t,s) \leq k(s,s)$ *for* $t, s \in [0,1]$;
(c) $0 \leq k(t,s) \leq 1/e$ *for* $t, s \in [0,1]$;
(d) $s(1-t) \leq k(t,s) \leq \frac{-s\ln s}{1-s}(1-t)$ *for* $0 \leq s \leq t \leq 1$.

Proof. Parts (a)–(c) were shown in [9, Remark 2]. We now show part (d). For any fixed $0 \leq s \leq 1$, let $l(t) = -s\ln t$, $s \leq t \leq 1$. Assume first that $s > 0$. It is easy to check that the tangent line of $y = l(t)$ at $t = 1$ is given by $y = s(1-t)$, and the secant line of $y = l(t)$ connecting the points $(1, l(1))$ and $(s, l(s))$ is $y = \frac{-s\ln s}{1-s}(1-t)$. Note that $l''(t) = s/t^2 > 0$ for $t \in [s,1]$. Then, $l(t)$ is concave upward. Hence, we have
$$s(1-t) \leq k(t,s) \leq \frac{-s\ln s}{1-s}(1-t) \quad \text{for } 0 < s \leq t \leq 1.$$
When $s = 0$, the above inequalities obviously hold. This completes the proof. $\square$

In the sequel, let $(X, \|\cdot\|)$ be a real Banach space that is partially ordered by a normal cone $P \subset X$, i.e., $u \leq v$ if and only if $v - u \in P$. If $u \leq v$ and $u \neq v$, then we write $u < v$ or $v > u$. By θ we denote the zero element of X.

Recall that a nonempty closed convex subset $P \subset X$ is called a *cone* if it satisfies:

(i) $u \in P$ and $\lambda > 0$ implies $\lambda u \in P$;
(ii) $u \in P$ and $-u \in P$ implies $u = \theta$.

Moreover, a cone P is said to be *normal* if there exists a constant $C > 0$ such that, for all $u, v \in X$, $\theta \le u \le v$ implies $\|u\| \le C\|v\|$.

Given $\omega \in P \setminus \{\theta\}$, let

$$P_\omega = \big\{u \in X : \text{ there exist } d > c > 0 \text{ such that } c\omega \le u \le d\omega\big\}. \tag{2.2}$$

It is easy to see that $P_\omega \subset P$.

To prove our theorem, we need some results from monotone operator theory. The following definition and lemma are well known. For instance, Definition 2.3 can be found in [7, 12, 14, 15, 17], and Lemma 2.4 is a special case of [17, Corollary 4.1]; see also [12, Theorem 2.1] and [15, Theorem 2.6].

Definition 2.3. An operator $T : P \times P \to X$ is called *mixed monotone* if $T(u, v)$ is nondecreasing in u and nonincreasing in v. Moreover, an element $u \in P$ is said to be a fixed point of T if $T(u, u) = u$.

Lemma 2.4. *Let P be a normal cone in a real Banach space X, $\omega \in P \setminus \{\theta\}$, and $T : P_\omega \times P_\omega \to P_\omega$ be a mixed monotone operator. Assume that there exists $\alpha \in (0, 1)$ such that*

$$T(\kappa u, \kappa^{-1}v) \ge \kappa^\alpha T(u, v) \quad \text{for } u, v \in P_\omega \text{ and } \kappa \in (0, 1).$$

Then T has a unique fixed point u in P_ω. Moreover, if constructing successively the sequences $\{u_n\}$ and $\{v_n\}$

$$u_n = T(u_{n-1}, v_{n-1}), \quad v_n = T(v_{n-1}, u_{n-1}), \quad n = 1, 2, \dots,$$

for any initial values $u_0, v_0 \in P_\omega$, we have $\|u_n - u_\lambda\| \to 0$ and $\|v_n - u_\lambda\| \to 0$ as $n \to \infty$.

3. Main results

In this section, we let $0 = (0, \dots, 0)$ and $\infty = (\infty, \dots, \infty)$. For any vectors $y = (y_1, \dots, y_m)$ and $z = (z_1, \dots, z_m)$, the following notations will be used:

- $y \to z$ if every component of y approaches the corresponding one of z;
- $y \to z^+$ (z^-) if every component of y approaches the corresponding one of z from the right (left);
- $y \to \infty$ if every component of y approaches ∞;
- $y > z$ ($y < z$) if every component of y is strictly larger (smaller) than the corresponding one of z.

In the remainder of the paper, we let X be the Banach space $(C[0, 1])^m$ equipped with the norm

$$\|u\| = \max\big\{\max_{t\in[0,1]} |u_i(t)| : i = 1, \dots, m\big\}, \quad u = (u_1, \dots, u_m) \in X,$$

and define a cone $P \subset X$ by

$$P = \big\{u = (u_1, \dots, u_m) \in X) : u_i(t) \ge 0 \text{ for } t \in [0, 1] \text{ and } i = 1, \dots, m\big\}.$$

Then, P is a normal cone in X. Choose $\omega(t) = 1-t \in P \setminus \{\theta\}$ and let P_ω be defined by (2.2) with the above P, i.e.,

$$\begin{aligned} P_\omega = \big\{u = (u_1, \dots, u_m) \in X &: \text{there exist } d > c > 0 \text{ such that} \\ &c\omega(t) \le u_i(t) \le d\omega(t) \text{ for } t \in [0, 1] \text{ and } i = 1, \dots, m\big\}. \end{aligned}$$

Now, we state the main results in this paper.

Theorem 3.1. *Assume that* (H1)–(H3) *hold. Then:*

(i) *for any* $\lambda = (\lambda_1, \dots, \lambda_m) > 0$, *problem* (1.2) *has a unique positive solution* $u_\lambda(t) = (u_{\lambda,1}(t), \dots, u_{\lambda,m}(t))$ *in* X;

(ii) *for any* $u_0, v_0 \in P_\omega$, *consider the sequences* $\{u_n\} = \{u_{n,1}, \dots, u_{n,m}\}$ *and* $\{v_n\} = \{v_{n,1}, \dots, v_{n,m}\}$ *defined by*

$$u_{n,i}(t) = \lambda_i^{\frac{1}{p-1}} \int_0^1 k(t,\tau)\phi_q\Big(\int_0^1 k(\tau,s)w_i(s)\big(g_i(u_{n-1}(s)) + h_i(v_{n-1}(s))ds\Big)d\tau,$$

$$v_{n,i}(t) = \lambda_i^{\frac{1}{p-1}} \int_0^1 k(t,\tau)\phi_q\Big(\int_0^1 k(\tau,s)w_i(s)\big(g_i(v_{n-1}(s)) + h_i(u_{n-1}(s))ds\Big)d\tau$$

for $i = 1, \dots, m$ *and* $n = 1, 2, \dots$. *Then*

$$\|u_n - u_\lambda\| \to 0 \quad \text{and} \quad \|v_n - u_\lambda\| \to 0 \quad \text{as } n \to \infty;$$

(iii) *if, in addition,* $0 < \alpha < 1/2$, *then the unique solution* $u_\lambda(t)$ *satisfies the following properties:*

(a) $u_\lambda(t)$ *is strictly increasing in* λ, *i.e.,* $\mu > \nu > 0 \Rightarrow u_\mu(t) > u_\nu(t)$ *on* $[0, 1)$;

(b) $\lim_{\nu\to 0^+} \|u_\nu\| = 0$ *and* $\lim_{\mu\to\infty} \|u_\mu\| = \infty$;

(c) $u_\lambda(t)$ *is continuous in* λ, *i.e.,* $\mu \to \nu > 0 \Rightarrow \|u_\mu - u_\nu\| \to 0$.

The following corollary is a direct consequence of Theorem 3.1.

Corollary 3.2. *Assume that the following conditions hold:*

(A1) $0 < \zeta < p - 1$, *where* $\zeta = \max\{b_{ik},\ d_{ik} : i, k = 1, \dots, m\}$;

(A2) $0 < -\int_0^1 s(\ln s)(1 - s)^{-d_{ik}} w_i(s)ds < \infty$ *for* $i = 1, \dots, m$.

Then:

(i) *for any* $\lambda = (\lambda_1, \dots, \lambda_m) > 0$, *problem* (1.4) *has a unique positive solution* $u_\lambda(t) = (u_{\lambda,1}(t), \dots, u_{\lambda,m}(t))$ *in* X;

(ii) *with* $g_i(y_1, \dots, y_m) = \sum_{k=1}^m a_{ik}y_k^{b_{ik}}$, *and* $h_i(y_1, \dots, y_m) = \sum_{k=1}^m c_{ik}y_k^{-d_{ik}}$, *part (ii) of Theorem 3.1 holds for problem* (1.4);

(iii) *if, in addition,* $0 < \zeta < 1/2$, *then the unique solution* $u_\lambda(t)$ *satisfies the three properties stated in part (iii) of Theorem 3.1.*

Remark 3.3. In Theorem 3.1 (ii) and Corollary 3.2 (ii), if we choose $u_0 = v_0$ in P_ω, then it is easy to see that $u_{n,i}(t) = v_{n,i}(t)$ on $[0, 1]$ for $i = 1, \dots, m$ and $n = 1, 2, \dots$. Hence, $u_n(t) = v_n(t)$ on $[0, 1]$ for $n = 1, 2, \dots$. Thus, to use iterations to approximate the unique solution of problem (1.2), in the nth step, we only need to solve the equations

$$u_{n,i}(t) = \lambda_i^{\frac{1}{p-1}} \int_0^1 k(t,\tau)\phi_q\Big(\int_0^1 k(\tau,s)w_i(s)\big(g_i(u_{n-1}(s)) + h_i(u_{n-1}(s))ds\Big)d\tau$$

for $i = 1, \dots, m$.

Proof of Theorem 3.1. Let $\lambda = (\lambda_1, \dots, \lambda_m) > 0$ be fixed. For $u = (u_1, \dots, u_m)$, $v = (v_1, \dots, v_m) \in P_\omega$, define an operator $T_\lambda : P_\omega \times P_\omega \to X$ by

$$T_\lambda(u, v)(t) = \big(T_{\lambda,1}(u, v)(t), \dots, T_{\lambda,m}(u, v)(t)\big),$$

where

$$T_{\lambda,i}(u, v)(t) = \lambda_i^{\frac{1}{p-1}} \int_0^1 k(t,\tau)\phi_q\Big(\int_0^1 k(\tau,s)w_i(s)[g_i(u(s)) + h_i(v(s))]ds\Big)d\tau. \tag{3.1}$$

In view of (H1), (H2), and Lemma 2.2 (b), T_λ is well defined. Moreover, from the monotonicity of g_i and h_i assumed in (H2), it is easy to verify that $T_{\lambda,i}$ is mixed monotone, and so is T_λ.

We now show that $T_\lambda(P_\omega \times P_\omega) \subset P_\omega$. For $i = 1, \dots, m$ and $u = (u_1, \dots, u_m), v = (v_1, \dots, v_m) \in P_\omega$, let

$$z_i(u,v)(\tau) = \phi_q\Big(\int_0^1 k(\tau,s) w_i(s)[g_i(u(s)) + h_i(v(s))]ds\Big).$$

Then, from (2.1), Lemma 2.2 (d), (3.1), and the fact that $-\ln\tau \geq 1-\tau$ for $\tau \in [t,1]$, we have

$$\begin{aligned}
&T_{\lambda,i}(u,v)(t)\\
&= \lambda_i^{\frac{1}{p-1}} \int_0^t k(t,\tau) z_i(u,v)(\tau)d\tau + \lambda_i^{\frac{1}{p-1}} \int_t^1 k(t,\tau) z_i(u,v)(\tau)d\tau\\
&\geq (1-t)\lambda_i^{\frac{1}{p-1}} \int_0^t \tau z_i(u,v)(\tau)d\tau + \lambda_i^{\frac{1}{p-1}} \int_t^1 \tau(-\ln\tau) z_i(u,v)(\tau)d\tau\\
&\geq (1-t)\lambda_i^{\frac{1}{p-1}} \int_0^t \tau z_i(u,v)(\tau)d\tau + \lambda_i^{\frac{1}{p-1}} \int_t^1 \tau(1-\tau) z_i(u,v)(\tau)d\tau\\
&\geq (1-t)\lambda_i^{\frac{1}{p-1}} \Big(\int_0^t \tau(1-\tau) z_i(u,v)(\tau)d\tau + \int_t^1 \tau(1-\tau) z_i(u,v)(\tau)d\tau\Big)\\
&= (1-t)\lambda_i^{\frac{1}{p-1}} \int_0^1 \tau(1-\tau) z_i(u,v)(\tau)d\tau\\
&\geq c_1(1-t) = c_1\omega(t),
\end{aligned} \tag{3.2}$$

where

$$c_1 = \min_{1\leq i\leq m}\big\{\lambda_i^{\frac{1}{p-1}} \int_0^1 \tau(1-\tau) z_i(u,v)(\tau)d\tau\big\}.$$

Similarly, from (2.1), Lemma 2.2 (d), (3.1), and the fact that $(1-t)/(1-\tau) \geq 1$ for $\tau \in [t,1]$, it follows that

$$\begin{aligned}
&T_{\lambda,i}(u,v)(t)\\
&= \lambda_i^{\frac{1}{p-1}} \int_0^t k(t,\tau) z_i(u,v)(\tau)d\tau + \lambda_i^{\frac{1}{p-1}} \int_t^1 k(t,\tau) z_i(u,v)(\tau)d\tau\\
&\leq (1-t)\lambda_i^{\frac{1}{p-1}} \int_0^t \frac{-\tau\ln\tau}{1-\tau} z_i(u,v)(\tau)d\tau + \lambda_i^{\frac{1}{p-1}} \int_t^1 -\tau(\ln\tau) z_i(u,v)(\tau)d\tau\\
&\leq (1-t)\lambda_i^{\frac{1}{p-1}} \Big(\int_0^t \frac{-\tau\ln\tau}{1-\tau} z_i(u,v)(\tau)d\tau + \int_t^1 \frac{-\tau\ln\tau}{1-\tau} z_i(u,v)(\tau)d\tau\Big)\\
&= (1-t)\lambda_i^{\frac{1}{p-1}} \int_0^1 \frac{-\tau\ln\tau}{1-\tau} z_i(u,v)(\tau)d\tau\\
&\leq d_1(1-t) = d_1\omega(t),
\end{aligned} \tag{3.3}$$

where

$$d_1 = \max_{1\leq i\leq m}\big\{\lambda_i^{\frac{1}{p-1}} \int_0^1 \frac{-\tau\ln\tau}{1-\tau} z_i(u,v)(\tau)d\tau\big\}.$$

From (3.2) and (3.3), we see that $T_\lambda(P_\omega \times P_\omega) \subset P_\omega$.

Next, for $i = 1, \dots, m$ and $u = (u_1, \dots, u_m), v = (v_1, \dots, v_m) \in P_\omega$ and $\kappa \in (0, 1)$, from (H3) and (3.1), we have

$$\begin{aligned}
&T_{\lambda,i}(\kappa u, \kappa^{-1} v)(t) \\
&= \lambda_i^{\frac{1}{p-1}} \int_0^1 k(t,\tau)\phi_q\Big(\int_0^1 k(\tau,s)w_i(s)[g_i(\kappa u(s)) + h_i(\kappa^{-1}v(s))]ds\Big)d\tau \\
&\geq \kappa^\alpha \lambda_i^{\frac{1}{p-1}} \int_0^1 k(t,\tau)\phi_q\Big(\int_0^1 k(\tau,s)w_i(s)[g_i(u(s)) + h_i(v(s))]ds\Big)d\tau \\
&= \kappa^\alpha T_{\lambda,i}(u,v)(t).
\end{aligned}$$

Thus,

$$T_\lambda(\kappa u, \kappa^{-1} v)(t) \geq \kappa^\alpha T_\lambda(u,v)(t). \tag{3.4}$$

We have shown that all the conditions of Lemma 2.4 hold, so there exists a unique $u_\lambda = (u_{\lambda,1}, \dots, u_{\lambda,m}) \in P_\omega$ such that $T_\lambda(u_\lambda, v_\lambda) = u_\lambda$. Hence, in view of the fact that $f_i(y) = g_i(y) + h_i(y)$ (see (H1)), we have

$$\begin{aligned}
u_{\lambda,i}(t) &= \lambda_i^{\frac{1}{p-1}} \int_0^1 k(t,\tau)\phi_q\Big(\int_0^1 k(\tau,s)w_i(s)[g_i(u_\lambda(s)) + h_i(u_\lambda(s))]ds\Big)d\tau \\
&= \lambda_i^{\frac{1}{p-1}} \int_0^1 k(t,\tau)\phi_q\Big(\int_0^1 k(\tau,s)w_i(s)f_i(u_\lambda(s))ds\Big)d\tau.
\end{aligned}$$

By Lemma 2.1, we see that (1.2) has a unique solution $u_\lambda = (u_{\lambda,1}, \dots, u_{\lambda,m})$ in P_ω, which is obviously positive on $[0, 1)$. From the "moreover" part of Lemma 2.4, part (ii) of Theorem 3.1 holds.

It is clear that, to show that (1.2) has a unique positive solution in X, it suffices to prove the following claim.

Claim: If, for any $\lambda = (\lambda_1, \dots, \lambda_m) > 0$, $\hat{u}_\lambda(t) = (\hat{u}_{\lambda,1}, \dots, \hat{u}_{\lambda,m})$ is a positive solution of problem (1.2), then $\hat{u}_\lambda \in P_\omega$.

In fact, if $\hat{u}_\lambda(t) = (\hat{u}_{\lambda,1}, \dots, \hat{u}_{\lambda,m})$ is a positive solution of problem (1.2), then by Lemma 2.1,

$$\hat{u}_{\lambda,i}(t) = \lambda_i^{\frac{1}{p-1}} \int_0^1 k(t,\tau)\phi_q\Big(\int_0^1 k(\tau,s)w_i(s)[g_i(\hat{u}_\lambda(s)) + h_i(\hat{u}_\lambda(s))]ds\Big)d\tau.$$

Hence, by similar arguments as in showing (3.2) and (3.3), we see that there exist $d_2 > c_2 > 0$ such that

$$c_2\omega(t) \leq \hat{u}_{\lambda,i}(t) \leq d_2\omega(t) \quad \text{for } t \in [0,1] \text{ and } i = 1, \dots, m.$$

This shows that $\hat{u}_\lambda \in P_\omega$, i.e., the claim is true. Now, by the claim, problem (1.2) has a unique positive solution in X. Thus, part (i) of Theorem 3.1 holds.

In the rest of the proof, we show part (iii) of Theorem 3.1. For $i = 1, \dots, m$, define an operator $A_i : P_\omega \times P_\omega \to X$ by

$$A_i(u,v)(t) = \int_0^1 k(t,\tau)\phi_q\Big(\int_0^1 k(\tau,s)w_i(s)[g_i(u(s)) + h_i(v(s))]ds\Big)d\tau. \tag{3.5}$$

Clearly, A_i is mixed monotone, and as in showing (3.4), we have

$$A_i(\kappa u, \kappa^{-1} v)(t) \geq \kappa^\alpha A_i(u,v)(t) \quad \text{for } \kappa \in (0,1). \tag{3.6}$$

Moreover, in view of (3.1), we have

$$T_{\lambda,i}(u,v)(t) = \lambda_i^{\frac{1}{p-1}} A_i(u,v)(t), \quad i = 1, \dots, m. \tag{3.7}$$

We first show property (a). Assume that $\mu = (\mu_1, \dots, \mu_m) > \nu = (\nu_1, \dots, \nu_m) > 0$. Let $u_\mu = (u_{\mu,1}, \dots, u_{\mu,m})$ and $u_\nu = (u_{\nu,1} \dots, u_{\nu,m})$ be the unique positive solutions of (1.2) corresponding to $(\lambda_1, \dots, \lambda_m) = (\mu_1, \dots, \mu_m)$ and $(\lambda_1, \dots, \lambda_m) = (\nu_1, \dots, \nu_m)$, respectively. Then, from (3.7), it follows that

$$\begin{aligned} u_{\mu,i}(t) = T_{\mu,i}(u_\mu, u_\mu)(t) = \mu_i^{\frac{1}{p-1}} A_i(u_\mu, v_\mu)(t) \\ u_{\nu,i}(t) = T_{\nu,i}(u_\nu, u_\nu)(t) = \nu_i^{\frac{1}{p-1}} A_i(u_\nu, v_\nu)(t). \end{aligned} \tag{3.8}$$

Define the set

$$S(\mu, \nu) = \big\{\gamma > 0 : u_{\mu,i}(t) \geq \gamma u_{\nu,i}(t) \text{ and } u_{\nu,i}(t) \geq \gamma u_{\mu,i}(t) \text{ on } [0,1],\ i = 1, \dots, m\big\}.$$

We show that $S(\mu, \nu) \neq \emptyset$. In fact, by the above claim, $u_\mu, u_\nu \in P_\omega$. Then, for $t \in [0,1]$ and $i = 1, \dots, m$, there exist $d_\mu > c_\mu > 0$ and $d_\nu > c_\nu > 0$ such that

$$c_\mu \omega(t) \leq u_{\mu,i}(t) \leq d_\mu \omega(t) \quad \text{and} \quad c_\nu \omega(t) \leq u_{\nu,i}(t) \leq d_\nu \omega(t).$$

Thus, we have

$$u_{\mu,i} \geq c_\mu \omega(t) \geq \frac{c_\mu}{d_\nu} u_{\nu,i} \quad \text{and} \quad u_{\nu,i} \geq c_\nu \omega(t) \geq \frac{c_\nu}{d_\mu} u_{\mu,i}.$$

Hence, any γ satisfying $0 < \gamma < \min\{c_\mu/d_\nu, c_\nu/d_\mu\}$ is in $S(\mu, \nu)$. This shows that $S(\mu, \nu) \neq \emptyset$. Let $\overline{\gamma} = \sup S(\mu, \nu)$. Then, $0 < \overline{\gamma} < 1$ and

$$u_{\mu,i}(t) \geq \overline{\gamma} u_{\nu,i}(t) \text{ and } u_{\nu,i}(t) \geq \overline{\gamma} u_{\mu,i}(t) \quad \text{for } t \in [0,1] \text{ and } i = 1, \dots, m. \tag{3.9}$$

In fact, (3.9) is obviously true. If $\overline{\gamma} > 1$, then (3.9) implies that $u_{\mu,i}(t) > u_{\nu,i}(t) > u_{\mu,i}(t)$ on $t \in [0,1)$. This is a contradiction. If $\overline{\gamma} = 1$, then, by (3.9), $u_{\mu,i}(t) = u_{\nu,i}(t)$ for $t \in [0,1]$ and $i = 1, \dots, m$. So, $u_\mu(t) = u_\nu(t)$ on $[0,1]$. Hence, from (3.8),

$$u_{\mu,i}(t) = \mu_i^{\frac{1}{p-1}} A_i(u_\mu, u_\mu)(t) > \nu_i^{\frac{1}{p-1}} A_i(u_\nu, u_\nu)(t) = u_{\nu,i}.$$

Again, this is a contradiction. Thus, $0 < \overline{\gamma} < 1$.

Since A_i is mixed monotone, from (3.6), (3.8), and (3.9), we have

$$\begin{aligned} u_{\mu,i}(t) = \mu_i^{\frac{1}{p-1}} A_i(u_\mu, u_\mu)(t) &\geq \mu_i^{\frac{1}{p-1}} A_i(\overline{\gamma} u_\nu, \overline{\gamma}^{-1} u_\nu)(t) \\ &\geq (\overline{\gamma})^\alpha \mu_i^{\frac{1}{p-1}} \nu_i^{-\frac{1}{p-1}} \nu_i^{\frac{1}{p-1}} A_i(u_\nu, u_\nu)(t) \\ &= (\overline{\gamma})^\alpha \mu_i^{\frac{1}{p-1}} \nu_i^{-\frac{1}{p-1}} u_{\nu,i}(t) \end{aligned}$$

and

$$\begin{aligned} u_{\nu,i}(t) = \nu_i^{\frac{1}{p-1}} A_i(u_\nu, u_\nu)(t) &\geq \nu_i^{\frac{1}{p-1}} A_i(\overline{\gamma} u_\mu, \overline{\gamma}^{-1} u_\mu)(t) \\ &\geq (\overline{\gamma})^\alpha \nu_i^{\frac{1}{p-1}} \mu_i^{-\frac{1}{p-1}} \mu_i^{\frac{1}{p-1}} A_i(u_\mu, u_\mu)(t) \\ &= (\overline{\gamma})^\alpha \nu_i^{\frac{1}{p-1}} \mu_i^{-\frac{1}{p-1}} u_{\mu,i}(t), \end{aligned}$$

i.e.,

$$u_{\mu,i}(t) \geq (\overline{\gamma})^\alpha \mu_i^{\frac{1}{p-1}} \nu_i^{-\frac{1}{p-1}} u_{\nu,i}(t) \quad \text{and} \quad u_{\nu,i}(t) \geq (\overline{\gamma})^\alpha \nu_i^{\frac{1}{p-1}} \mu_i^{-\frac{1}{p-1}} u_{\mu,i}(t) \tag{3.10}$$

for $t \in [0,1]$ and $i = 1, \dots, m$. In view of the fact $(\overline{\gamma})^\alpha \mu_i^{\frac{1}{p-1}} \nu_i^{-\frac{1}{p-1}} \geq (\overline{\gamma})^\alpha \nu_i^{\frac{1}{p-1}} \mu_i^{-\frac{1}{p-1}}$, we have

$$u_{\mu,i}(t) \geq (\overline{\gamma})^\alpha \nu_i^{\frac{1}{p-1}} \mu_i^{-\frac{1}{p-1}} u_{\nu,i}(t).$$

Then, by the definition of $\overline{\gamma}$, we have $(\overline{\gamma})^\alpha \nu_i^{\frac{1}{p-1}} \mu_i^{-\frac{1}{p-1}} \leq \overline{\gamma}$. Hence,

$$\overline{\gamma} \geq \big(\frac{\nu_i}{\mu_i}\big)^{\frac{1}{(p-1)(1-\alpha)}}.$$

Thus, from (3.10), we obtain that

$$u_{\mu,i}(t) \geq \big(\frac{\nu_i}{\mu_i}\big)^{\frac{\alpha}{(p-1)(1-\alpha)}} \mu_i^{\frac{1}{p-1}} \nu_i^{-\frac{1}{p-1}} u_{\nu,i}(t) = \big(\frac{\mu_i}{\nu_i}\big)^{\frac{1-2\alpha}{(p-1)(1-\alpha)}} u_{\nu,i}(t) \tag{3.11}$$

and

$$u_{\nu,i}(t) \geq \big(\frac{\nu_i}{\mu_i}\big)^{\frac{\alpha}{(p-1)(1-\alpha)}} \nu_i^{\frac{1}{p-1}} \mu_i^{-\frac{1}{p-1}} u_{\mu,i}(t) = \big(\frac{\nu_i}{\mu_i}\big)^{\frac{1}{(p-1)(1-\alpha)}} u_{\mu,i}(t) \tag{3.12}$$

for $t \in [0,1]$ and $i = 1,\dots,m$. Since $\alpha \in (0,1/2)$, we have $\big(\frac{\mu_i}{\nu_i}\big)^{\frac{1-2\alpha}{(p-1)(1-\alpha)}} > 1$, and so from (3.11), $u_{\mu,i}(t) > u_{\nu,i}(t)$ for $t \in [0,1)$ and $i = 1,\dots,m$. This proves property (a) of part (iii).

Next, we prove property (b). Assume $\mu = (\mu_1,\dots,\mu_m) > \nu = (\nu_1,\dots,\nu_m) > 0$. From (3.11), we have

$$u_{\nu,i}(t) \leq \big(\frac{\nu_i}{\mu_i}\big)^{\frac{1-2\alpha}{(p-1)(1-\alpha)}} u_{\mu,i}(t) \quad \text{for } t \in [0,1] \text{ and } i = 1,\dots,m,$$

which in turn implies that

$$\|u_\nu\| \leq \max\Big\{\big(\frac{\nu_i}{\mu_i}\big)^{\frac{1-2\alpha}{(p-1)(1-\alpha)}},\ i = 1,\dots,m\Big\}\|u_\mu\|.$$

Thus, $\|u_\nu\| \to 0$ as $\nu \to 0^+$. Similarly, (3.11) also implies that

$$\|u_\mu\| \geq \min\Big\{\big(\frac{\mu_i}{\nu_i}\big)^{\frac{1-2\alpha}{(p-1)(1-\alpha)}},\ i = 1,\dots,m\Big\}\|u_\nu\|.$$

Hence, $\|u_\mu\| \to \infty$ as $\mu \to \infty$.

Finally, we prove property (c). When $\mu = (\mu_1,\dots,\mu_m) > \nu = (\nu_1,\dots,\nu_m) > 0$, from (3.12), we have

$$u_{\mu,i}(t) \leq \big(\frac{\mu_i}{\nu_i}\big)^{\frac{1}{(p-1)(1-\alpha)}} u_{\nu,i}(t) \quad \text{for } t \in [0,1] \text{ and } i = 1,\dots,m. \tag{3.13}$$

Then,

$$\|u_\mu - u_\nu\| \leq \max\Big\{\Big(\big(\frac{\mu_i}{\nu_i}\big)^{\frac{1}{(p-1)(1-\alpha)}} - 1\Big),\ i = 1,\dots,m\Big\}\|u_\nu\|.$$

As a result, $\|u_\mu - u_\nu\| \to 0$ as $\mu \to \nu^+$.

When $0 < \mu = (\mu_1,\dots,\mu_m) < \nu = (\nu_1,\dots,\nu_m)$, from (3.13) with μ and ν switched, we have

$$u_{\nu,i}(t) \geq \big(\frac{\nu_i}{\mu_i}\big)^{\frac{1}{(p-1)(1-\alpha)}} u_{\mu,i}(t) \quad \text{for } t \in [0,1] \text{ and } i = 1,\dots,m.$$

This, together with $u_{\mu,i}(t) \leq u_{\nu,i}(t)$ on $[0,1]$, implies that

$$\|u_\mu - u_\nu\| \leq \max\Big\{\Big(1 - \big(\frac{\nu_i}{\mu_i}\big)^{\frac{1}{(p-1)(1-\alpha)}}\Big),\ i = 1,\dots,m\Big\}\|u_\mu\|.$$

Then, $\|u_\mu - u_\nu\| \to 0$ as $\mu \to \nu^-$. Hence, property (c) holds. This completes the proof of the theorem. $\square$

Finally, we present a proof for Corollary 3.2.

Proof of Corollary 3.2. With $f_i(y_1,\dots,y_m) = g_i(y_1,\dots,y_m) + h_i(y_1,\dots,y_m)$, where

$$g_i(y_1,\dots,y_m) = \sum_{k=1}^{m} a_{ik} y_k^{b_{ik}} \quad \text{and} \quad h_i(y_1,\dots,y_m) = \sum_{k=1}^{m} c_{ik} y_k^{-d_{ik}},$$

it is clear that problem (1.3) is of the form of problem (1.2) and (H1) holds. Let ζ be defined in (A1). Then, (A1) and (A2) imply that (H2) and (H3) hold with $\alpha = \zeta$. The conclusion now readily follows from Theorem 3.1. $\square$

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