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PERIODIC ORBITS FOR SEASONAL SIRS MODELS WITH NON-LINEAR INCIDENCE RATES

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Abstract. In this work, we prove the existence of periodic solutions for a seasonally-dependent SIRS model using Leray-Schauder degree theory. We obtain criteria for the uniqueness and asymptotic stability of the periodic solution of the system. We also present suitable examples of a seasonal epidemiological disease.

1. Introduction

We consider a Susceptible-Infectious-Recovered-Susceptible (SIRS) compartmental epidemiological model with a periodically forced transmission rate. For simplicity, we assume that the population size is constant, $N = 1$, and that the population is divided into three disjoint classes which change with time. Let $S(t)$, $R(t)$ and $I(t)$ be the fractions of the populations that are susceptible, recovered and infectious, respectively. Then the differential equations of a SIRS model with seasonally-dependent incidence rates are

$$\begin{aligned} S' &= \eta R + \mu(1-S) - f(t,S,I),\\ I' &= f(t,S,I) - (\gamma+\mu)I, \\ R' &= \gamma I - (\mu+\eta)R. \end{aligned} \tag{1.1}$$

The parameters of this model are positive constants. The natural death rate and birth rate are assumed to be equal, denoted by μ. It is assumed that the infected population recovers at a rate of γ and joins the recovered class. Also, the recovered class can lose immunity and rejoin the susceptible class (at a rate of η). The interaction between susceptible and infected population will produce new infected individuals. These contagious processes are characterized by the incidence function $f(t,S,I)$. In particular, the use of a periodic incidence function accounts for the variability of diseases according to climate seasons, school calendars, etc.

In the study of epidemiological models the analysis of periodic solutions is seen as an important goal as this periodicity reveals the recurrence of an epidemic in a population. Hence, determining existence of such solutions under different parameter configurations and incidence functions is crucial.

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Different types of incidence functions have been used to analyze SIR and SIRS models in the literature. Katriel [3] proved the existence of periodic orbits of an epidemiological SIR model with periodic linear incidence rates, i.e., $f(t,S,I) = \beta(t)SI$, by using Leray-Schauder degree theory. Jódar and coworkers [2] studied the existence of periodic solutions in a class of epidemic SIRS models with the same incidence rate, $\beta(t)SI$, by using Mawhin's continuation theorem. In this work we consider models with nonlinear incidence rates of type

$$f(t,S,I) = \beta(t)I(1+\alpha I^n)S^k, \quad n,k \in \mathbb{N},\ \alpha \geq 0. \tag{1.2}$$

where $\beta(t)$ is a continuous T-periodic function, such that:

$$-\infty < \beta^l := \min_{t\in\mathbb{R}} \beta(t) \leq \beta(t) \leq \beta^u := \max_{t\in\mathbb{R}} \beta(t) < \infty.$$

Note that if $\alpha = 0$ and $k = 1$ we recover the linear incidence function.

Several incidence rates of type (1.2) (with β constant) have been proposed by authors, for example: Liu and coworkers studied the SEIRS and SIRS models with the incidence rate $\beta S^r I^k$ in [4, 5]. Van den Driessche and Watmough [6] studied an incidence rate of the form $\beta I(1+\alpha I^n)S$. For low I, bilinear term dominates, but if $\alpha > 1$, then for large I the higher order term dominates.

In this article, we consider a class of infectious disease models of SIRS type with periodic nonlinear incidence rate and give conditions which ensure the existence of periodic solutions. We show the global asymptotical stability of such solutions by constructing a Lyapunov function. We also present two examples related to the transmission of respiratory syncytial virus.

2. Results

The infectious disease models which we consider involve nonlinear incidence rates of type (1.2), given by

$$\begin{aligned} S' &= \eta R + \mu(1-S) - \beta(t)I(1+\alpha I^n)S^k, \\ I' &= \beta(t)I(1+\alpha I^n)S^k - (\gamma+\mu)I, \\ R' &= \gamma I - (\mu+\eta)R. \end{aligned} \tag{2.1}$$

Given that $S' + I' + R' = 0$, we obtain that $S + R + I = 1$ for all time t for initial conditions that satisfy $S(0) + I(0) + R(0) = 1$. Also, it is easy to see that if the per capita contact rate β is constant, the system (2.1) has an infection-free equilibrium state $(S_0, I_0) = (1, 0)$.

The basic reproductive number $\mathcal{R}_0$ of model (1.1), i.e. the average number of secondary cases produced by a single infective introduced into an entirely susceptible population is given (see [7]) by

$$\mathcal{R}_0 := \frac{1}{\gamma+\mu}\frac{\partial f}{\partial I}(S_0, I_0).$$

When β is constant, a direct calculation for the system (2.1) yields $\mathcal{R}_0 = \frac{\beta}{\gamma+\mu}$. Motivated by this, we define $\mathcal{R}_0$ for system (2.1) as follows

$$\mathcal{R}_0 := \frac{\overline{\beta}}{\gamma+\mu},$$

where $\overline{\beta} := \frac{1}{T}\int_0^T \beta(t)dt$.

To analyze the existence of periodic solutions, we reduce model (2.1) to a two-dimensional system as follows. We replace $R = 1 - S - I$ in the first equation of system (2.1), so we can eliminate R from this equation. This observation gives the simpler system

$$\begin{aligned} S' &= \eta(1-S-I) + \mu(1-S) - \beta(t)I(1+\alpha I^n)S^k, \\ I' &= \beta(t)I(1+\alpha I^n)S^k - (\gamma+\mu)I. \end{aligned} \tag{2.2}$$

Note that the existence of periodic orbits for (2.2) implies the existence of periodic solutions for system (2.1). The proof of the existence of periodic orbits for systems of type (2.2) will be done in two steps. First, we consider the case $\alpha = 0$, and prove the existence of solutions on this system. Then, we construct an homotopy to a convenient system and carry on one of the solutions through the homotopy. In our methods, we are following the approach used in [3], but we also develop proper adjustments for the non-linear incidence case.

Considering the case $\alpha = 0$, we obtain the system

$$\begin{aligned} S' &= -\eta I + (\mu+\eta)(1-S) - \beta(t)IS^k, \\ I' &= \beta(t)IS^k - (\gamma+\mu)I. \end{aligned} \tag{2.3}$$

We write

$$\beta(t) = \overline{\beta} + \beta_0(t), \quad \text{where } \int_0^T \beta_0(t)dt = 0.$$

For $\lambda \in [0,1]$ we define the homotopy

$$\begin{aligned} S' &= -\eta I + (\mu+\eta)(1-S) - \beta_\lambda IS^k, \\ I' &= \beta_\lambda IS^k - (\gamma+\mu)I, \end{aligned} \tag{2.4}$$

where $\beta_\lambda := \overline{\beta} + \lambda\beta_0(t)$. We note that when $\lambda = 0$, we recover system (2.3). We now state our first result.

Theorem 2.1. *If $\mathcal{R}_0 > 1$, then there is at least one T-periodic orbit of* (2.3) *whose components are positive.*

To prove this Theorem, we use the Leray-Schauder degree theory [1]. To do so, we need to reformulate the homotopy defined by system (2.4) as a functional problem defined on an adequate Banach space where periodic solutions correspond to the zeroes of convenient operators. Then, we need to find an open bounded subset on the Banach space such that the family of operators does not support zeroes on the boundary of such open set. After that, we can proceed to determine the Leray-Schauder degree and, if applicable, establish the existence of solutions.

We start the proof by introducing the Banach spaces

$$\mathcal{C}^l := \big\{(S,I) : S, I \in C^l(\mathbb{R},\mathbb{R}), S(t+T) = S(t), I(t+T) = I(t)\big\}, \quad l = 0,1.$$

Motivated by (2.4) we define the operators $L : \mathcal{C}^1 \to \mathcal{C}^0$ and $N_\lambda : \mathcal{C}^0 \to \mathcal{C}^0$ by

$$L(S,I) := (S' + (\mu+\eta)S, I' + (\gamma+\mu)I), \; N_\lambda(S,I) := (-\eta I + (\mu+\eta) - \beta_\lambda IS^k, \beta_\lambda IS^k).$$

This way, system (2.4) becomes $L(S,I) = N_\lambda(S,I)$ and since L is invertible the above equation can be rewritten as

$$F_\lambda(S,I) := (S,I) - L^{-1} \circ N_\lambda(S,I) = 0. \tag{2.5}$$

Since $\mathcal{C}^1$ is compactly embedded in $\mathcal{C}^0$, we can think of L^{-1} to go from $\mathcal{C}^0$ to $\mathcal{C}^0$. Therefore $L^{-1} \circ N_\lambda : \mathcal{C}^0 \to \mathcal{C}^0$ is a compact operator. In a similar fashion, we consider $F_\lambda : \mathcal{C}^0 \to \mathcal{C}^0$. Thus, equation (2.5) is a functional reformulation of problem (2.4).

We consider the open sets $D := \{(S, I) \in \mathcal{C}^0 : S > 0, I > 0, S + I < 1\}$ and $U := \{(S, I) \in D : \min_{[0,T]} S^k(t) < \delta\}$.

When $\lambda = 0$, system (2.4) has exactly two periodic orbits in $\mathcal{C}^1$ being these: $S_0 = 1$, $I_0 = 0$, and

$$S_1 = \Big(\frac{\gamma+\mu}{\overline{\beta}}\Big)^{1/k}, \quad I_1 = \frac{\mu+\eta}{\gamma+\mu+\eta}\Big(1 - \Big(\frac{\mu+\gamma}{\overline{\beta}}\Big)^{1/k}\Big).$$

which in fact are critical points.

Lemma 2.2. *The critical point (S_0, I_0) is the only solution of (2.4) on ∂D.*

For a proof of the above lemma see [3, Lemma 1].

Lemma 2.3. *If $\mathcal{R}_0 > 1$ and $1/\mathcal{R}_0 < \delta < 1$, then for any $\lambda \in [0, 1]$ there are no solutions (S, I) of (2.4) on ∂U.*

Proof. Assume that $(S, I) \in \partial U$; by Lemma 2.2, $(S, I) \notin \partial D$, so $(S, I) \in D$ and $S^k(t) \geq \delta$ for all t. Integrating the second equation in (2.4) and using the previous inequality we obtain

$$\gamma + \mu = \frac{1}{T}\int_0^T \beta_\lambda(t) S^k(t) dt \geq \delta\overline{\beta}. \tag{2.6}$$

Now from our hypothesis

$$\frac{1}{T}\int_0^T \beta_\lambda(t) S^k(t) dt \geq \delta\overline{\beta} > \gamma + \mu \tag{2.7}$$

which is a contradiction. □

Now, to establish the existence of periodic solutions of (2.4) we need to determine $\deg(F_0, U)$. For this we have the following lemma.

Lemma 2.4. *For the above open set U we have that* $\deg(F_0, U) \neq 0$.

Proof. Since (S_1, I_1) is the unique solution of $F_0(S, I) = 0$ in U, we need to prove that $DF_0(S_1, I_1)$ is invertible. We have that F_0 is a compact perturbation of the identity, by the Fredholm alternative it is enough to prove that the $\ker(DF_0(S_1, I_1)) = \{0\}$.

Consider $(V, W) \in \mathcal{C}^0$ so that $(V, W) \in \ker(DF_0(S_1, I_1))$, by the definition of F_0, we obtain that $L(V, W) = DN_0(S_1, I_1)(V, W)$ and thus $N_0(S_1, I_1) = (-\eta I + (\mu + \eta) - \bar{\beta} S_1^k I_1, \bar{\beta} S_1^k I_1)$. Then, we obtain

$$DN_0(S_1, I_1)(V, W) = \big(-\eta W - \bar{\beta}(k S_1^{k-1} I_1 V + S_1^k W), \bar{\beta}(k S_1^{k-1} I_1 V + S_1^k W)\big).$$

Using the definition of $L(V, W)$ and from the above equation we obtain

$$(V' + (\mu+\eta)V, W' + (\gamma+\mu)W) = \big(-\eta W - \bar{\beta}(k S_1^{k-1} I_1 V + S_1^k W), \bar{\beta}(k S_1^{k-1} I_1 V + S_1^k W)\big)$$

If we substitute S_1, I_1 into the previous equation, and isolate V' and W', we obtain

$$\begin{aligned}(V', W') &= \Big(-\eta W - \bar{\beta}(k S_1^{k-1} I_1 V + S_1^k W) - (\mu+\eta)V, \bar{\beta}(k S_1^{k-1} I_1 V \\ &\quad + S_1^k W) - (\gamma+\mu)W\Big) \\ &= \Big(-(\mu+\eta)\big(\frac{\gamma+\mu}{\gamma+\mu+\eta} k(\sqrt[k]{\mathcal{R}_0} - 1) + 1\big)V\end{aligned}$$

$$- (\gamma+\mu+\eta)W, \frac{(\gamma+\mu)(\mu+\eta)}{\gamma+\mu+\eta}k(\sqrt[k]{\mathcal{R}_0}-1)V\Big).$$

Setting

$$\phi = \frac{\gamma+\mu}{\gamma+\mu+\eta}k(\sqrt[k]{\mathcal{R}_0}-1),$$

we obtain

$$\begin{pmatrix} V \\ W \end{pmatrix}' = \begin{pmatrix} -(\mu+\eta)(\phi+1) & -(\gamma+\mu+\eta) \\ (\mu+\eta)\phi & 0 \end{pmatrix} \begin{pmatrix} V \\ W \end{pmatrix}. \tag{2.8}$$

The characteristic polynomial of the above matrix is

$$p(\lambda) = \lambda^2 + (\mu+\eta)(\phi+1)\lambda + (\mu+\eta)(\gamma+\mu+\eta)\phi,$$

which is a Hurwitz polynomial. Therefore the linear system (2.8) has no periodic orbits different to the trivial solution. □

Since the Leray-Schauder degree is invariant under homotopy and given the result obtained in Lemma 2.3, we obtain that Theorem 2.1 follows from Lemma 2.4. We now establish the existence of periodic solutions of system (2.4).

Theorem 2.5. *If $\mathcal{R}_0 > 1$, then there is at least one T-periodic orbit of* (2.4) *whose components are positive.*

For $\tau \in [0,1]$ we consider the family

$$\begin{aligned} S' &= -\eta I + (\mu+\eta)(1-S) - \beta(t)I(1+\tau\alpha I^n)S^k, \\ I' &= \beta(t)I(1+\tau\alpha I^n)S^k - (\gamma+\mu)I. \end{aligned} \tag{2.9}$$

We introduce the operator $K_\tau : \mathcal{C}^0 \to \mathcal{C}^0$ by

$$K_\tau(S,I) = (\mu - \beta(t)I(1+\tau\alpha I^n)S^k, \beta(t)I(1+\tau\alpha I^n)S^k).$$

System (2.9) becomes $L(S,I) = K_\tau(S,I)$; which can be rewritten as

$$G_\tau(S,I) := (S,I) - L^{-1}\circ K_\tau(S,I) = 0. \tag{2.10}$$

Note that $\deg(G_0,U) = \deg(F_1,U) \neq 0$, so to use the Leray-Schauder degree it remains to prove that there are no solutions of (2.10) on the boundary of U. For this we present the following lemma.

Lemma 2.6. *If $1/\mathcal{R}_0 < \delta < 1$, then for any $\tau \in [0,1]$ there are no solutions (S,I) of* (2.9) *on ∂U.*

Proof. Assume that $(S,I) \in \partial U$; again the critical point (S_0,I_0) is the only solution of (2.9) on ∂D, therefore $(S,I) \notin \partial D$, so $(S,I) \in D$ and $S^k(t) \geq \delta$ for all t. Integrating the second equation in (2.9), and by the above inequality when $\delta \in (1/\mathcal{R}_0, 1)$ we obtain

$$\gamma+\mu = \frac{1}{T}\int_0^T \beta(t)(1+\tau\alpha I^n)S^k(t)dt \geq \delta\overline{\beta} > \gamma+\mu. \tag{2.11}$$

which is a contradiction. Therefore there are no solutions (S,I) of (2.9) on ∂U. □

Using the invariance of the Leray-Schauder degree under homotopy, Lemma 2.6 and since $\deg(G_1,U) \neq 0$, we obtain that system (2.4) admits a periodic solution, which proves Theorem 2.5.

We now establish the uniqueness and global stability of the periodic solutions. To do so, we construct a Lyapunov function. We also present examples of SIRS models with non-linear incidence rate modeling the transmission of Respiratory Syncytial Virus (RSV) for the countries of Gambia and Singapore and exhibit the existence and stability of periodic solutions. Before we start the stability analysis we establish the following definition.

Definition 2.7. A bounded positive solution $(S^*(t), T^*(t))^T$ of (2.2) is globally asymptotically stable if, for any other solution $(S(t), T(t))^T$ of (2.2) with positive initial conditions we have

$$\lim_{t\to\infty} (|S(t) - S^*(t)| + |I(t) - I^*(t)|) = 0 \tag{2.12}$$

If property (2.12) holds for any two solutions with positive initial conditions, it is said that (2.2) is globally asymptotically stable. It can be proven that if (2.2) has a bounded positive solution that is globally asymptotically stable, then (2.2) is globally asymptotically stable with the converse also holding true.

Lemma 2.8. *If $\beta^u > 0$, $k \geq 1$, $\eta > 0$ and $\mu > 0$ there exist positive constants t^*, m_S, and M_I such that $m_S < S(t)$ and $I(t) < M_I$ for all $t > t^*$. Moreover, if $m_S^k > \mu$, then there exist constants m_I and M_S such that $M_S > S(t)$ and $I(t) > m_I$, for a sufficiently large t.*

We can bound the first equation of system (2.2) as:

$$\begin{aligned} S' &> \eta(1-S) - nI + \mu(1-S) - \beta(t)I(1+\alpha)S^2 \\ &> (\eta+\mu)(1-S)S - \beta(t)I(1+\alpha)S^2 \\ &> S(\mu+\eta - (\mu+\eta+\beta_u(1+\alpha))S) \\ &> S(\mu+\eta - (\mu+\eta+\beta^u(1+\alpha))S) \end{aligned}$$

hence

$$S(t) > \frac{\mu+\eta}{2(\mu+\eta+\beta_u(1+\alpha))} := m_S,$$

for $t > t_1$, for some t_1.

We can also bound the equation of the infected individuals as:

$$I(t) < 1 - S(t) < 1 - m_S := M_I,$$

for a sufficiently large t.

We obtain an upper bound of the second equation of (2.2) in the following way:

$$\begin{aligned} I' &> \beta_l I1(1+\alpha I^n)S^k - \mu I \\ &> \beta_l I(1-I)m_S^k - \mu I \\ &> I((m_S^k - \mu) - m_S^k I) \end{aligned}$$

Assuming $m_S^k > \mu$ we obtain

$$I(t) > \frac{\beta^l m_S^k - \mu}{2\beta^l m_S^k} := m_I,$$

for a sufficiently large $t > t_3$.

We also obtain an upper bound for the susceptible population:

$$S(t) < 1 - I = 1 - m_I = 1 - \Big(\frac{\beta^l m_S^k - \mu}{\beta^l 2 m_S^k}\Big) := M_S,$$

for a sufficiently large t. Finally, we set $t^* = \max\{t_1, t_2, t_3\}$.

Theorem 2.9. *With the hypothesis of Theorem 2.5 and Lemma 2.8, if* $\eta + \mu > 2\beta^u(M_I(1+2\alpha n M_S))$ *and* $\gamma+\mu+\eta > 2\beta^u(M_S(1+2\alpha n M_I)$*, then* (2.2) *has a unique globally asymptotically stable positive* T_0*-periodic solution, for a given positive initial condition.*

Proof. Let $(S^*(t), I^*(t))^T$ be a positive T_0-periodic solution of system (2.2). Let $(S(t), I(t))^T$ be a positive solution of system (2.2) with positive initial conditions. Define the Lyapunov function

$$V(t) = \big(|S(t) - S^*(t)| + |I(t) - I^*(t)|\big).$$

We compute the upper right derivative of $V(t)$ along the solutions of (2.2) and we obtain

$$\begin{aligned}
D^+V(t) &= sgn\,(S(t) - S^*(t))\,\big(S'(t) - {S^*}'(t)\big) + sgn\,(I(t) - I^*(t))\,\big(I'(t) - {I^*}'(t)\big)\\
&= \operatorname{sgn}(S(t) - S^*(t))\big[-(\eta+\mu)(S(t) - S^*(t)) - \gamma(I(t) - I^*(t))\\
&\quad + \beta(t)(I(t)S^k(t)(1+\alpha I^n(t)) - I^*(t){S^*}^k(t)(1+\alpha {I^*}^n(t)))\big]\\
&\quad + \operatorname{sgn}(I(t) - I^*(t))\big[-(\gamma+\mu)(I(t) - I^*(t))\\
&\quad + \beta(t)(I(t)S^k(t)(1+\alpha I^n(t)) - I^*(t){S^*}^k(t)(1+\alpha {I^*}^n(t))\big]\\
&\le -(\eta+\mu)|S(t) - S^*(t)| - (\gamma+\mu)|I(t) - I^*(t)|\\
&\quad - \eta\operatorname{sgn}(S(t) - S^*(t))(I(t) - I^*(t))\\
&\quad - \operatorname{sgn}(S(t) - S^*(t))\Big[\beta(t)(I(t)S^k(t)(1+\alpha I^n(t))\\
&\quad - I^*(t){S^*}^k(t)(1+\alpha {I^*}^n(t)))\Big] + \operatorname{sgn}(I(t) - I^*(t))\\
&\quad \times \Big[\beta(t)(I(t)S^k(t)(1+\alpha I^n(t)) - I^*(t){S^*}^k(t)(1+\alpha {I^*}^n(t)))\Big]
\end{aligned}$$

We can bound the following terms:

$$\begin{aligned}
&\operatorname{sgn}(S(t) - S^*(t))\beta(t)\Big[\Big(-I(t)S^k(t)(1+\alpha I^n(t))\\
&+ I^*(t)S^{k^*}(t)(1+\alpha {I^*}^{n+1}(t){S^*}^k(t))\Big)\Big]\\
&= \operatorname{sgn}(S(t) - S^*(t))\beta(t)\Big[I^*(t)S^*(t) - I(t)S(t) + \alpha({I^*}^{n+1}(t)({S^*}^k(t) - S^k(t))\\
&\quad + S^k(t)({I^*}^{n+1}(t) - I^{n+1}(t))\Big]\\
&= \operatorname{sgn}(S(t) - S^*(t))\beta(t)\big[{S^*}^k(t)(I^*(t) - I(t))\\
&\quad + I(t)(S^*(t) - S(t))(S^{k-1}(t) + \dots + S^{k-1}(t))\\
&\quad + \alpha({I^*}^{n+1}(t)(S^*(t) - S(t))({S^*}^{k-1}(t) + \dots + S^{k-1}(t))\\
&\quad + S^k(t)(I^*(t) - I(t))({I^*}^n(t) + \dots + I^n(t))\big]\\
&\le \beta_u\big[|I(t) - I^*(t)|M_S(1+2\alpha n M_I) + |S(t) - S^*(t)|M_I(1+2\alpha n M_S)\big]
\end{aligned}$$

Similarly, we obtain:

$$\begin{aligned}
&\operatorname{sgn}(I(t) - I^*(t))\beta(t)\Big[\Big(I(t)S^k(t)(1+\alpha I^n(t))\\
&- I^*(t)S^{k^*}(t)(1+\alpha {I^*}^{n+1}(t){S^*}^k(t))\Big)\Big]\\
&\le \beta_u\big[|I(t) - I^*(t)|M_S(1+2\alpha n M_I) + |S(t) - S^*(t)|M_I(1+2\alpha n M_S)\big]
\end{aligned}$$

Therefore,

$$\begin{aligned} D^+V(t) &\le -(\eta+\mu-2\beta^u(M_I(1+2\alpha n M_S)))|S(t)-S^*(t)| \\ &\quad -(\gamma+\mu+\eta-2\beta^u(M_S(1+2\alpha n M_I)))|I(t)-I^*(t)| \\ &\le -\phi\big[|S(t)-S^*(t)|+|I(t)-I^*(t)|\big] \end{aligned}$$

Hence, $D^+V(t) \le -\phi\big[|S(t)-S^*(t)|+|I(t)-I^*(t)|\big]$, where

$$0<\phi = min\{\eta+\mu-2\beta^u(M_I(1+2\alpha n M_S)), \gamma+\mu+\eta-2\beta^u(M_S(1+2\alpha n M_I)\}.$$

Integrating the previous equation on the interval $[0,t]$ for $t \ge 0$ we obtain:

$$\begin{gathered} V(t)-V(0) = -\phi\int_0^t |S(t)-S^*(t)|+|I(t)-I^*(t)|dt, \\ V(t)+\phi\int_0^t |S(t)-S^*(t)|dt + \phi\int_0^t |I(t)-I^*(t)|dt \\ = |S(0)-S^*(0)|+|I(0)-I^*(0)| < \infty \end{gathered}$$

Thus, $V(t)$ is bounded on $[0,t]$ and $\int_0^\infty |S(t)-S^*(t)|+|I(t)-I^*(t)|dt < \infty$.

From this we obtain that $|S(t)-S^*(t)|$ and $|I(t)-I^*(t)|$ are in $L^1[0,\infty)$. Since $(S(t),I(t))^T$ and $(S^*(t),I^*(t))^T$ as well as their derivatives are bounded on $\big[T,\infty)$, hence $|S(t)-S^*(t)|$ and $|I(t)-I^*(t)|$ are uniformly continuous on $[T,\infty)$. Then, we can use Barbalat's lemma to determine that the solution is globally asymptotically stable; that is,

$$\lim_{t\to\infty} |S(t)-S^*(t)| = 0, \quad \lim_{t\to\infty} |I(t)-I^*(t)| = 0.$$

The proof is complete. □

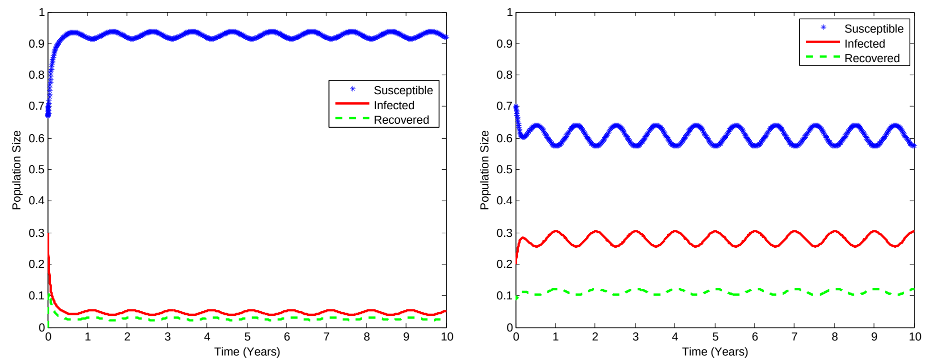
We now present two examples to exhibit the existence and stability of T-periodic solutions. To do so, we consider the system (2.3) with parameters fitted from data to model the transmission of respiratory syncytial virus (RSV) for the countries of Gambia and Singapore using a SIRS model. We consider parameter choices based on the estimation developed in [8]. RSV virus is known to be the most common cause of acute lower respiratory tract infection in children. As so, it can be modeled as having a seasonal pattern of increased infection due, in a determinant way, to the increased contact among children during school terms. To model the transmission rate of this infection the term $\beta(t)$ can be approximated by:

$$\beta(t) = \beta_0(1+\epsilon\cos(2\pi(t-\phi))),$$

where β_0 is the baseline transmission rate, ϵ is the relative seasonal forcing and ϕ accounts for the time when the transmission rate is maximal. This transmission rate assumes that the period of transmission is one year. In [8] the authors obtained parameter estimates for the corresponding transmission rate for the countries of Gambia and Singapore.

Example 2.10. Consider the SIRS model for the transmission of RSV in the country of Gambia determined by

$$\begin{aligned} S' &= 52R + 0.041(1-S) - (35(1+0.04\cos(2\pi t+0.15)))I(1+0.06I^7)S^2, \\ I' &= (35(1+0.04\cos(2\pi t+0.15))I(1+0.04I^7)S^2 - 30.041I, \\ R' &= 30I - 52.041R. \end{aligned} \tag{2.13}$$



(A) (B)

FIGURE 1. Existence of 1-periodic globally asymptotically stable solutions. (A) SIRS model for the transmission of RSV virus for the country of Gambia. The initial conditions are $S_0 = 0.7$, $I_0 = 0.3$ and $R_0 = 0.0$. (B) SIRS model for the transmission of RSV virus in the country of Singapore. The initial conditions are $S_0 = 0.7$, $I_0 = 0.2$ and $R_0 = 0.1$.

For this model we consider the parameter estimates: $\eta = 52$, $\mu = 0.041$, $\gamma = 30$, $\beta_0 = 35$, $\beta_1 = 0.04$ and $\phi = 0.15$ some of them based on what is developed in [8]. We set $\alpha = 0.0003$, $n = 7$ and $k = 2$ to explore the existence of periodic orbits using a non-linear incidence rate. See Figure 1.

Example 2.11. Consider the SIRS model for the transmission of RSV for the country of Singapore given by

$$\begin{gathered} S' = 4.9R + 0.016(1-S) - (70(1+0.12\cos(2\pi t + 0.28)))I(1+0.4I^2)S^1, \\ I' = (70(1+0.12\cos(2\pi t+0.28))I(1+0.4I^2)S^1 - 36.016I, \\ R' = 36I - 4.916R. \end{gathered} \tag{2.14}$$

For this model we consider the parameters: $\eta = 50$, $\mu = 0.016$, $\gamma = 20$, $\beta_0 = 33$, $\beta_1 = 0.06$ and $\phi = 0.28$ some of them based on [8]. We set $\alpha = 0.00005$, $n = 3$ and $k = 1$ to explore the existence of periodic orbits. See Figure 1.

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