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# ASYMPTOTIC BEHAVIOR FOR SECOND-ORDER DIFFERENTIAL EQUATIONS WITH NONLINEAR SLOWLY TIME-DECAYING DAMPING AND INTEGRABLE SOURCE

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Abstract. In this article we establish convergence to the equilibrium of all global and bounded solutions of a gradient-like system of second-order with slow dissipation. Also we estimate the rate of convergence.

## 1. Introduction and main results

In this article we study the asymptotic behaviour of global and bounded solutions of the following gradient like system

$$\begin{gathered} \ddot{x}(t) + a(t)\|\dot{x}(t)\|^{\alpha}\dot{x}(t) + \nabla\Phi(x(t)) = g(t) \\ x(0) = x_0 \in \mathbb{R}^N, \quad \dot{x}(0) = x_1 \in \mathbb{R}^N \end{gathered} \tag{1.1}$$

where $N \in \mathbb{N}^*$, $\alpha \geq 0$, $\Phi \in W^{2,\infty}_{\rm loc}(\mathbb{R}^N, \mathbb{R})$, $g \in L^1(\mathbb{R}_+, \mathbb{R}^N)$, $a \in L^\infty(\mathbb{R}_+)$, $a \geq 0$. We denote by $S$ the set of critical points of $\Phi$:

$$S = \{x \in \mathbb{R}^N : \nabla\Phi(x) = 0\}.$$

Recently, Haraux and Jendoubi [13] studied the asymptotic behavior of global solutions to the nonlinear differential equation

$$\ddot{x}(t) + a(t)\dot{x}(t) + \nabla\Phi(x(t)) = 0. \tag{1.2}$$

They prove among other things that if $a(t) \geq \frac{c}{(1+t)^\beta}$ with $\beta \geq 0$ small enough and $S = \arg\min\Phi$ then the solution converge as $t$ goes to infinity to $S$. Moreover, they proved that if the potential $\Phi$ satisfies an adapted uniform Lojasiewicz gradient inequality then the solution converge to some point $b \in S$. The purpose of this paper is to generalize the results obtained by the authors of [13] to the equation (1.1).

Before stating the results of this paper, recall that equation (1.2) with $a(t) = 1$ has been studied by several authors. When $\Phi$ is analytic, Haraux and Jendoubi [11] (see also [2, 7, 8, 12]) proved convergence to equilibrium of all global and bounded solutions. Now when the potential $\Phi$ is assumed to be convex and still in the case where $a(t) = 1$, Attouch et al [1] proved a similar convergence result.

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Equation (1.2) in the case where $a(t)$ tends to 0 was initiated by Cabot et al [4] in the case where the potential $\Phi$ is convex (see also [5, 14]).

The main results of this paper read as follows.

**Theorem 1.1.** *Let $\Phi \in W^{2,\infty}_{\rm loc}(\mathbb{R}^N,\mathbb{R})$, $a \in L^\infty(\mathbb{R}_+)$ be a positive function, and $x \in W^{2,1}_{\rm loc} \cap L^\infty(\mathbb{R}_+,\mathbb{R}^N)$ be a solution of* (1.1)*. Assume that*

- (H1) $S = \arg\min \Phi$.
- (H2) *There exists $\delta > 0$, $d > 0$ such that $\|g(t)\| \leqslant \frac{d}{(1+t)^{1+\delta}}$.*
- (H3) *There exists $\beta \in ]0,1[$, $c > 0$ for all $t \geqslant 0$, $a(t) \geqslant \frac{c}{(1+t)^\beta}$.*

*Then*
$$\lim_{t\to+\infty} \|\dot{x}(t)\| + \operatorname{dist}(x(t), S) = 0. \tag{1.3}$$

**Theorem 1.2.** *Let $\Phi \in W^{2,\infty}_{\rm loc}(\mathbb{R}^N,\mathbb{R})$, $a \in L^\infty(\mathbb{R}_+)$ be a positive function. Let $x \in W^{2,1}_{\rm loc} \cap W^{1,\infty}(\mathbb{R}_+,\mathbb{R}^N)$ a solution of* (1.1)*. Assume* (H1), (H2) *and that*

- (H4) *There exists $\theta \in ]0, \frac{1}{2}]$ for all $b \in S \exists \sigma_b > 0$, $\exists C_b > 0$ for all $x \in B(b,\sigma_b)$, $\|\nabla\Phi(x)\| \geqslant C_b|\Phi(x) - \Phi(b)|^{1-\theta}$.*
- (H5) *There exists $c > 0$, $\exists \beta \geq 0$: $\alpha + \beta \in ]0, \inf(\frac{\theta}{1-\theta}; \delta)[$ and $a(t) \geq c/(1+t)^\beta$ for all $t \geq 0$.*

*Then there exists $b^* \in S$, $T > 0$ and $M > 0$ such that for every $t > T$*
$$\|x(t) - b^*\| \leqslant M t^{-\lambda}$$
*where*
$$\lambda = \inf\Big(\big[\frac{\theta - (\alpha+\beta)(1-\theta)}{(1-\theta)(\alpha+2) - 1}\big], \big[\frac{\delta - (\alpha+\beta)}{(\alpha+1)}\big]\Big).$$

**Remark 1.3.** (1) If $g = 0$ and $\alpha = 0$, we recover a result previously obtained by Haraux and Jendoubi, see [13, Theorem 2.3]. (2) If $\beta = 0$, we recover a result obtained by Ben Hassen and Chergui, see [3, Theorem 1.6].

**Remark 1.4.** Assumption (H4) is satisfied if one of the following two conditions is satisfied

- $F$ is a polynomial [9], or
- $F$ is analytic and $S$ is compact [6].

**Remark 1.5.** Let us observe that the condition $\alpha + \beta < \delta$ in (H5) is necessary. Here is an example of a nonconvergent solution of the following scalar equation
$$\ddot{x}(t) + |\dot{x}(t)|^\alpha \frac{\dot{x}(t)}{(1+t)^\beta} = g(t). \tag{1.4}$$
Let $x(t) = \cos(\ln(1+t))$ be a solution of (1.4). Then we can easily see that $g$ satisfies assumption (H2) with $\delta = \alpha + \beta$ and that $x$ is a non convergent solution of (1.4) with $\Phi = 0$. Note also that in this case assumption (H4) holds true with $\theta = 1/2$.

**Remark 1.6.** The hypothesis that $\alpha + \beta < \frac{\theta}{1-\theta}$ in (H5) is in some sense optimal. Haraux [10] gives an example of a function $f$ such that the $\omega-$limit set of a global and bounded solution of the following equation
$$\ddot{x}(t) + |\dot{x}(t)|\dot{x}(t) + f(x) = 0$$
is equal to an interval and then this solution does not converge. The nonlinearity can be chosen such that its primitive satisfies assumption $(H4)$ with $\theta = 1/2$.

**Remark 1.7.** Theorems 1.1 and 1.2 remain true if the dissipation term $a(t)\|\dot{x}(t)\|^\alpha\dot{x}(t)$ in the equation 1.1 is replaced by $a(t)\gamma(\dot{x}(t))$ where $\gamma:\mathbb{R}^N\to\mathbb{R}^N$ is a continuous function satisfying

$$\langle\gamma(v),v\rangle\geq\rho_1\|v\|^{\alpha+2},\quad \|\gamma(v)\|\leq\rho_2\|v\|^{\alpha+1}\quad\forall v\in\mathbb{R}^N,$$

with $0<\rho_1<\rho_2<\infty$ and $\alpha$ is as in Theorems 1.1 and 1.2.

## 2. Proof of Theorem 1.1

We define the two functions

$$E(t)=\frac{1}{2}\|\dot{x}(t)\|^2+\Phi(x(t))-\min\Phi,$$

$$K(t)=E(t)+\int_t^{+\infty}\frac{M}{(1+s)^{\frac{-\beta+(1+\delta)(\alpha+2)}{\alpha+1}}}ds \tag{2.1}$$

where

$$M=\big(\frac{c}{2}\big)^{-\frac{1}{\alpha+1}}d^{\frac{\alpha+2}{\alpha+1}},$$

$c$ is as in (H6) and $d$ is as in (H2). Note that by hypotheses (H4)–(H6), we have $\frac{-\beta+(1+\delta)(\alpha+2)}{\alpha+1}>1$ and $K$ is well defined. Now by differentiating $E$ we obtain

$$E'(t)=-a\|\dot{x}(t)\|^{2+\alpha}+\langle g(t);\dot{x}(t)\rangle.$$

By the Cauchy-Schwarz inequality we obtain

$$E'(t)\leq-\frac{c}{(1+t)^\beta}\|\dot{x}(t)\|^{2+\alpha}+\Big(\frac{c}{2(1+t)^\beta}\Big)^{\frac{1}{\alpha+2}}\|\dot{x}(t)\|\Big(\frac{c}{2(1+t)^\beta}\Big)^{-\frac{1}{\alpha+2}}\|g(t)\|.$$

Thanks to Young's inequality we obtain

$$\begin{aligned}E'(t)&\leq-\frac{c}{2(1+t)^\beta}\|\dot{x}(t)\|^{2+\alpha}+\Big(\frac{c}{2(1+t)^\beta}\Big)^{-\frac{1}{\alpha+1}}\|g(t)\|^{\frac{\alpha+2}{\alpha+1}}\\&\leq-\frac{c}{2(1+t)^\beta}\|\dot{x}(t)\|^{2+\alpha}+\frac{M}{(1+t)^{\frac{-\beta+(1+\delta)(\alpha+2)}{\alpha+1}}}.\end{aligned}$$

Now by differentiating $K$ we obtain

$$K'(t)\leq-\frac{c}{2(1+t)^\beta}\|\dot{x}(t)\|^{2+\alpha}.$$

So $K$ is a decreasing and positive function. Hence

$$\dot{x}\in L^\infty(\mathbb{R}_+,\mathbb{R}^N) \tag{2.2}$$

and there exists $l\in\mathbb{R}_+$ such that

$$\lim_{t\to+\infty}K(t)=\lim_{t\to+\infty}E(t)=l.$$

We define the function

$$\mathcal{E}(t)=E(t)+\int_t^{+\infty}\langle g(s),\dot{x}(s)\rangle ds.$$

Using (2.2) and (H2) which implies that $g\in L^1$, we see that $\lim_{t\to\infty}\mathcal{E}(t)=\lim_{t\to+\infty}E(t)$ and $\mathcal{E}'(t)=-a(t)\|\dot{x}(t)\|^{2+\alpha}$. Then we obtain

$$\int_0^\infty a(t)\|\dot{x}(t)\|^{2+\alpha}\,dt<\infty. \tag{2.3}$$

Let $r > 0$ and assume that there exists $\varepsilon > 0$ and $t_0 > 0$ such that for all $t \geqslant t_0$

$$\int_t^{t+r} \|\dot{x}(s)\|^{2+\alpha} ds \geq \varepsilon.$$

Then

$$\int_t^{t+r} a(s)\|\dot{x}(s)\|^{2+\alpha} ds \geq \frac{\varepsilon c}{(1+t+r)^{\beta}} \quad \forall t \geqslant t_0\,.$$

It follows that

$$\begin{aligned} \int_{t_0}^{+\infty} a(s)\|\dot{x}(s)\|^{2+\alpha} ds &\geq \sum_{n=0}^{+\infty} \int_{t_0+nr}^{t_0+(n+1)r} a(s)\|\dot{x}(s)\|^{2*\alpha} ds \\ &\geq \sum_{n=0}^{+\infty} \frac{\varepsilon c}{(1+t_0+(n+1)r)^{\beta}} = \infty \end{aligned}$$

which contradicts (2.3). Then for every $n \in \mathbb{N}^*$, there exists $t_n \geq n$ such that

$$\int_{t_n}^{t_n+r} \|\dot{x}(t)\|^{2+\alpha} dt \leq \frac{1}{n}.$$

Hence there exists a real sequence $(t_n)_n$ such that $\lim_{n\to\infty} t_n = \infty$ and

$$\lim_{n\to+\infty} \int_{t_n}^{t_n+r} \|\dot{x}(t)\|^{2+\alpha} dt = 0. \tag{2.4}$$

By (2.2), $x$ and $\dot{x}$ are bounded, and then by the equation (1.1), $\ddot{x}$ is bounded. Hence $\dot{x}$ is Lipschitz continuous. Thanks to the Cauchy-Schwarz inequality we obtain for all $t \in [t_n, t_n + r]$

$$\begin{aligned} |\|\dot{x}(t_n+t)\|^{2+\alpha} - \|\dot{x}(t_n)\|^{2+\alpha}| &= (2+\alpha)|\int_{t_n}^{t_n+t} \|\dot{x}(s)\|^{\alpha} < \dot{x}(s), \ddot{x}(s) > ds| \\ &\leq (2+\alpha)(\int_{t_n}^{t_n+t} \|\dot{x}(s)\|^{\alpha+1}\|\ddot{x}(s)\| ds) \\ &\leqslant (2+\alpha)\|\dot{x}\|_{\infty}^{\alpha/2}\|\ddot{x}\|_{\infty}(\int_{t_n}^{t_n+r} \|\dot{x}(s)\|^{\frac{\alpha}{2}+1} ds) \\ &\leq (2+\alpha)\|\dot{x}\|_{\infty}^{\alpha/2}\|\ddot{x}\|_{\infty}\sqrt{r}(\int_{t_n}^{t_n+r} \|\dot{x}(s)\|^{\alpha+2} ds)^{1/2}. \end{aligned}$$

Then from (2.4) we obtain

$$\lim_{n\to\infty} \sup_{s\in[0,r]} \|\dot{x}(t_n+s)\| = 0. \tag{2.5}$$

Since $x$ is a bounded function and $\nabla\Phi$ is a Lipschitz continuous function on every bounded domain, then there exists $\lambda > 0$ such that for all $(t,s) \in \mathbb{R}_+^2$

$$\|\nabla\Phi(x(t)) - \nabla\Phi(x(s))\| \leq \lambda\|x(t) - x(s)\|.$$

Then

$$\begin{aligned} \Big\|r\nabla\Phi(x(t_n)) - \int_{t_n}^{t_n+r} \nabla\Phi(x(s))\,ds\Big\| &\le \int_{t_n}^{t_n+r} \|\nabla\Phi(x(t_n)) - \nabla\Phi(x(s))\|ds \\ &\le \lambda\int_{t_n}^{t_n+r} \|x(t_n)-x(s)\|ds \\ &\le \lambda\int_{t_n}^{t_n+r}\int_{t_n}^{s} \|\dot{x}(u)\|du\,ds \\ &\le \lambda r^2 \sup_{s\in[0,r]} \|\dot{x}(t_n+s)\|. \end{aligned} \tag{2.6}$$

Since

$$\int_{t_n}^{t_n+r} \nabla\Phi(x(s))ds = -\int_{t_n}^{t_n+r} \ddot{x}(t)dt - \int_{t_n}^{t_n+r} a(t)\|\dot{x}(t)\|^\alpha \dot{x}(t)dt + \int_{t_n}^{t_n+r} g(t)dt,$$

then

$$\lim_{n\to+\infty}\int_{t_n}^{t_n+r} \nabla\Phi(x(s))ds = 0. \tag{2.7}$$

Combining (2.5), (2.6) and (2.7) yields

$$\lim_{n\to+\infty} \|\nabla\Phi(x(t_n))\| = 0. \tag{2.8}$$

Hence

$$l = \lim_{t\to+\infty} E(t) = \lim_{n\to+\infty} E(t_n) = \lim_{n\to+\infty} \Phi(x(t_n)) - \min\Phi.$$

Since $(x(t_n))_n$ is a bounded sequence, we can extract a subsequence, still denoted by $(x(t_n))_n$ such that $\lim_{n\to\infty} x(t_n) = a$. From (2.8) we obtain

$$\lim_{n\to+\infty} \nabla\Phi(x(t_n)) = 0 = \nabla\Phi(a).$$

Then $a \in S$. By (H1), $S = \arg\min\Phi$, and then it follows that $\lim_{t\to\infty} E(t) = 0$, so $\lim_{t\to\infty} \|\dot{x}(t)\| = 0$ and $\lim_{t\to\infty} \Phi(x(t)) = \min\Phi$. Assume that

$$\lim_{t\to\infty} \operatorname{dist}(x(t), S) \neq 0.$$

Then there exists $\varepsilon > 0$ and $t_n \to \infty$ such that

$$d(x(t_n), S) \geqslant \varepsilon. \tag{2.9}$$

Therefore, we can extract a subsequence still denoted by $(t_n)$ such that

$$\lim_{n\to+\infty} x(t_n) = a.$$

Then $\lim_{n\to\infty} \Phi(x(t_n)) = \Phi(a) = \min\Phi$ that is $a \in S$, which contradicts (2.9).

## 3. Proof of theorem 1.2

To prove Theorem 1.2, we need some lemmas. We begin with the following lemma proved by Alvarez et al [2], here we give a slightly different proof.

**Lemma 3.1.** *Under hypothesis* (H4), *let* $\Gamma$ *be a compact subset of* $\mathbb{R}^N$ *such that*

$$\exists K \in \mathbb{R} : \forall a \in \Gamma\ \Phi(a) = K.$$

*Then there exist* $\sigma, C > 0$ *and* $\theta \in (0, 1/2)$ *such that*

$$[d(u,\Gamma) \le \sigma \Rightarrow \|\nabla\Phi(u)\| \ge C|\Phi(u) - K|^{1-\theta}]. \tag{3.1}$$

*Proof.* Using (H4), there exists $\theta \in ]0,1/2]$ such that for all $a \in \Gamma$ there exists $C_a > 0$, $\sigma_a > 0$ such that

$$\|\nabla\Phi(u)\| \geq C_a|\Phi(u) - \Phi(a)|^{1-\theta} \quad \forall u \in B(a, \sigma_a). \tag{3.2}$$

Since $\Gamma$ is compact, then there exists $(a_1, \dots, a_n) \in \Gamma^n$ such that

$$\Gamma \subset (\cup_{i=1}^n B(a_i, \frac{\sigma_{a_i}}{2})).$$

We choose $\sigma = \inf \sigma_{a_i}/2$ and $C = \inf C_{a_i}$. Let $u \in \mathbb{R}^N$ such that $d(u, \Gamma) \leq \sigma$. Then there exists $a \in \Gamma$ such that $d(u, a) \leq \sigma$ and $i \in \{1, 2, \dots, n\}$ such that $a \in B(a_i, \frac{\sigma_{a_i}}{2})$. Hence we obtain $d(u, a_i) \leq \sigma_{a_i}$. From (3.2) we obtain

$$\|\nabla\Phi(u)\| \geq C_a|\Phi(u) - \Phi(a)|^{1-\theta} \geq C|\Phi(u) - K|^{1-\theta}.$$

□

**Lemma 3.2.** *Let $f$ and $g : \mathbb{R}_+ \to \mathbb{R}_+$ be two continuously differentiable functions, $h : \mathbb{R}_+^2 \to \mathbb{R}$ be continuously differentiable function, and $T \geq 0$ be such that for all $t \geq T$,*

$$f'(t) + h(t, f(t)) \leq g'(t) + h(t, g(t)),$$
$$f(T) \leq g(T).$$

*Then for all $t \geq T$, $f(t) \leq g(t)$.*

*Proof.* Let $k : \mathbb{R}_+ \to \mathbb{R}$ be a function such that

$$k'(t) = \begin{cases} \frac{h(t,g(t))-h(t,f(t))}{g(t)-f(t)} & \text{if } g(t) \neq f(t) \\ \partial_2 h(t, f(t)) & \text{if } g(t) = f(t) \end{cases}$$

and $\phi : \mathbb{R}_+ \to \mathbb{R}; t \longmapsto e^{k(t)}(g - f)(t)$. Then for all $t \geq T$,

$$\phi'(t) = [(g - f)'(t) + k'(t)(g - f)(t)]e^{k(t)} \geq 0.$$

So $\phi$ is an increasing function in $[T, +\infty[$. Finally we see that for all $t \in [T, +\infty[$ we obtain $f(t) \leq g(t)$. □

**Lemma 3.3.** *Let $H \in W^{1,1}_{\rm loc}(\mathbb{R}_+, \mathbb{R}_+)$. Assume that there exist constants $k_1 > 0$, $k_2 \geq 0$, $T > 0$, $\mu > 1 > \beta$ and $\gamma > \beta > 0$ such that for almost every $t \geq T$ we have*

$$H'(t) + \frac{k_1}{(1+t)^\beta}(H(t))^\mu \leq \frac{k_2}{(1+t)^\gamma}.$$

*Then there exists $M > 0$ such that for all $t \geq T$,*

$$H(t) \leq \frac{M}{(1+t)^c}$$

*where*

$$c = \inf\Big(\frac{\gamma - \beta}{\mu}, \frac{1 - \beta}{\mu - 1}\Big).$$

*Proof.* Let $M > 0$ such that $k_1 M^\mu - cM > k_2$ and $M > H(T)(1 + T)^c$. We define the function $\phi : \mathbb{R}_+ \to \mathbb{R}$ by

$$\phi(t) = \frac{M}{(1+t)^c}.$$

Hence for all $t \geq T$, we have

$$\begin{aligned}
\phi'(t) + \frac{k_1}{(1+t)^\beta}(\phi(t))^\mu &= \frac{k_1 M^\mu}{(1+t)^{\beta+c\mu}}\Big(1 - \frac{cM^{1-\mu}}{k_1(1+t)^{1-\beta+c(1-\mu)}}\Big) \\
&\geq \frac{k_1 M^\mu}{(1+t)^{\beta+c\mu}}\Big(1 - \frac{cM^{1-\mu}}{k_1}\Big) \\
&\geq \frac{k_2}{(1+t)^\gamma} \\
&\geq H'(t) + \frac{k_1}{(1+t)^\beta}(H(t))^\mu.
\end{aligned}$$

Since $\phi(T) \geq H(T)$, thanks to Lemma 3.2, for all $t \geq T$, we obtain

$$H(t) \leq \phi(t) = \frac{M}{(1+t)^c}.$$

□

*Proof of Theorem 1.2.* Let $\varepsilon > 0$. We define the function

$$H(t) = \mathcal{E}(t) + \frac{\varepsilon\|\nabla\Phi(x(t))\|^\alpha}{(1+t)^\beta}\langle\nabla\Phi(x(t)), \dot{x}(t)\rangle + \frac{\varepsilon}{2}\int_t^\infty \frac{\|\nabla\Phi(x(s))\|^\alpha\|g(s)\|^2}{(1+s)^\beta}\,ds. \tag{3.3}$$

By differentiating $H$ we obtain

$$\begin{aligned}
H'(t) = &-a(t)\|\dot{x}(t)\|^{\alpha+2} - \frac{\varepsilon\beta}{(1+t)^{\beta+1}}\|\nabla\Phi(x(t))\|^\alpha\langle\nabla\Phi(x(t)), \dot{x}(t)\rangle \\
&+ \frac{\varepsilon}{(1+t)^\beta}\|\nabla\Phi(x(t))\|^\alpha\langle\nabla^2\Phi(x(t))\dot{x}(t), \dot{x}(t)\rangle \\
&+ \frac{\varepsilon}{(1+t)^\beta}\alpha\|\nabla\Phi(x(t))\|^{\alpha-2}\langle\nabla^2\Phi(x(t))\dot{x}(t), \nabla\Phi(x(t))\rangle\langle\nabla\Phi(x(t)), \dot{x}(t)\rangle \\
&- \frac{\varepsilon a(t)}{(1+t)^\beta}\|\dot{x}(t)\|^\alpha\|\nabla\Phi(x(t))\|^\alpha\langle\nabla\Phi(x(t)), \dot{x}(t)\rangle - \frac{\varepsilon}{(1+t)^\beta}\|\nabla\Phi(x)\|^{2+\alpha} \\
&+ \frac{\varepsilon}{(1+t)^\beta}\|\nabla\Phi(x(t))\|^\alpha\langle\nabla\Phi(x(t)), g(t)\rangle - \frac{\varepsilon}{2}\frac{\|\nabla\Phi(x(t))\|^\alpha\|g(t)\|^2}{(1+t)^\beta}.
\end{aligned}$$

By the Cauchy-Schwarz inequality and by setting $M_1 = \|\nabla^2\Phi(x)\|_\infty$ and $M_2 = \|a\|_\infty$ we obtain

$$\begin{aligned}
H'(t) \leq &-a(t)\|\dot{x}(t)\|^{\alpha+2} - \frac{\varepsilon}{(1+t)^\beta}\|\nabla\Phi(x)\|^{2+\alpha} + \frac{\varepsilon\beta}{(1+t)^{\beta+1}}\|\dot{x}(t)\|\|\nabla\Phi(x(t))\|^{\alpha+1} \\
&+ \frac{\varepsilon M_1(\alpha+1)}{(1+t)^\beta}\|\dot{x}(t)\|^2\|\nabla\Phi(x(t))\|^\alpha + \frac{\varepsilon M_2}{(1+t)^\beta}\|\dot{x}(t)\|^{\alpha+1}\|\nabla\Phi(x(t))\|^{\alpha+1} \\
&+ \frac{\varepsilon}{(1+t)^\beta}\|\nabla\Phi(x(t))\|^\alpha\|\nabla\Phi(x(t))\|\|g(t)\| - \frac{\varepsilon}{2}\frac{\|\nabla\Phi(x(t))\|^\alpha\|g(t)\|^2}{(1+t)^\beta}.
\end{aligned}$$

By Young's inequality, there exist $C_1, C_2 > 0$ such that

$$\begin{aligned}
H'(t) \leq &-a(t)\|\dot{x}(t)\|^{2+\alpha} - \frac{\varepsilon}{2(1+t)^\beta}\|\nabla\Phi(x(t))\|^{2+\alpha} \\
&+ \frac{\varepsilon\beta}{(1+t)^{\beta+1}}(\|\dot{x}(t)\|^{\alpha+2} + \|\nabla\Phi(x(t))\|^{\alpha+2})
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{\varepsilon}{(1+t)^\beta}(C_1\|\dot x(t)\|^{\alpha+2} + \frac{1}{8}\|\nabla\Phi(x(t))\|^{\alpha+2})\\
&+ \frac{\varepsilon}{(1+t)^\beta}(C_2\|\dot x(t)\|^{(\alpha+2)(\alpha+1)} + \frac{1}{8}\|\nabla\Phi(x(t))\|^{\alpha+2}).
\end{aligned}
$$

By using (1.3), there exists $T>0$ such that

$$
\|\dot x(t)\| < 1 \quad \forall t \geq T. \tag{3.4}
$$

Then we obtain that for all $t \geq T$ (with $T$ large enough so that $(1/((1+T)^{\beta)} \leq 1/8)$,

$$
H'(t) \leqslant \Big(\frac{-c+\varepsilon(\beta + C_1 + C_2)}{(1+t)^\beta}\Big)\|\dot x(t)\|^{2+\alpha} - \frac{\varepsilon}{8(1+t)^\beta}\|\nabla\Phi(x(t))\|^{2+\alpha}.
$$

By choosing $\varepsilon$ small enough, we obtain that for all $t \geqslant T$,

$$
H'(t) \leqslant -\frac{\varepsilon}{8(1+t)^\beta}(\|\dot x(t)\|^{2+\alpha} + \|\nabla\Phi(x(t))\|^{2+\alpha}). \tag{3.5}
$$

So $H$ is nonincreasing on $[T,\infty)$ and $\lim_{t\to\infty} H(t) = 0$. From (3.3) together with the Cauchy-Schwarz inequality we obtain for all $t > T$,

$$
\begin{aligned}
&[H(t)]^{(1-\theta)(\alpha+2)}\\
&\leq \Big[\mathcal{E}(t) + \frac{\varepsilon\|\nabla\Phi(x(t))\|^{\alpha+1}}{(1+t)^\beta}\|\dot x(t)\| + \frac{\varepsilon}{2}\int_t^\infty \frac{\|\nabla\Phi(x(s))\|^\alpha\|g(s)\|^2}{(1+s)^\beta}\,ds\Big]^{(1-\theta)(\alpha+2)}
\end{aligned}
$$

By using the inequality $\big(\sum_{i=1}^5 a_i\big)^\lambda \leq 5^\lambda \sum_{i=1}^5 a_i^\lambda$ for $a_i$ nonnegative for all $i$ and $0 \leq \lambda \leq 2$, we obtain that for all $t \geq T$,

$$
\begin{aligned}
[H(t)]^{(1-\theta)(\alpha+2)} &\leq C_3\Big[\frac{1}{2}\|\dot x(t)\|^2\Big]^{(1-\theta)(\alpha+2)} + C_3[\Phi(x(t)) - \min\Phi]^{(1-\theta)(\alpha+2)}\\
&\quad + C_3\Big[\int_t^{+\infty}\langle g(s),\dot x(s)\rangle\,ds\Big]^{(1-\theta)(\alpha+2)}\\
&\quad + C_3\Big[\frac{\varepsilon\|\nabla\Phi(x(t))\|^{\alpha+1}}{(1+t)^\beta}\|\dot x(t)\|\Big]^{(1-\theta)(\alpha+2)}\\
&\quad + C_3\Big[\frac{\varepsilon}{2}\int_t^\infty \frac{\|\nabla\Phi(x(s))\|^\alpha\|g(s)\|^2}{(1+s)^\beta}\,ds\Big]^{(1-\theta)(\alpha+2)}.
\end{aligned} \tag{3.6}
$$

where $C_3 = 5^{(1-\theta)(\alpha+2)}$. By using (3.4) and since $2(1-\theta)(\alpha+2) \geq \alpha+2$ we have

$$
[\|\dot x(t)\|^2]^{(1-\theta)(\alpha+2)} \leq \|\dot x(t)\|^{\alpha+2} \tag{3.7}
$$

Now by using (H4) and Lemma 3.1 we obtain that for all $t \geq T$,

$$
[\Phi(x(t)) - \min\Phi]^{(1-\theta)(\alpha+2)} \leq \|\nabla\Phi(x(t))\|^{\alpha+2}. \tag{3.8}
$$

Young's inequality yields

$$
\begin{aligned}
&\Big|\int_t^\infty \langle g(s),\dot x(s)\rangle\,ds\Big|^{(1-\theta)(\alpha+2)}\\
&\leq K(\rho)\Big(\int_t^{+\infty}\|g(s)\|^{\frac{\alpha+2}{\alpha+1}}(1+t)^{\frac{\beta}{\alpha+1}}\,ds\Big)^{(\alpha+2)(1-\theta)}\\
&\quad + \rho(\int_t^{+\infty}\frac{\|\dot x(s)\|^{\alpha+2}}{(1+t)^\beta}\,ds)^{(1-\theta)(\alpha+2)},
\end{aligned} \tag{3.9}
$$

where $\rho$ is a small positive constant which will be fixed in the sequel. Using (H2) we obtain

$$\Big(\int_t^{+\infty}\|g(s)\|^{\frac{\alpha+2}{\alpha+1}}(1+t)^{\frac{\beta}{\alpha+1}}\,ds\Big)^{(\alpha+2)(1-\theta)}\le C_4(1+t)^{-\chi},\tag{3.10}$$

where

$$\chi=\frac{1+\alpha\delta+2\delta-\beta}{1+\alpha}(\alpha+2)(1-\theta).$$

Once again, by applying Young's inequality and using the fact that $(1-\theta)(\alpha+2)\ge 1$, we obtain that for all $t\ge T$,

$$\begin{aligned}
&C_3\Big[\frac{\varepsilon\|\nabla\Phi(x(t))\|^{\alpha+1}}{(1+t)^\beta}\|\dot x(t)\|\Big]^{(1-\theta)(\alpha+2)}\\
&\le C_5\big[\frac{\|\nabla\Phi(x(t))\|^{\alpha+2}+\|\dot x(t)\|^{\alpha+2}}{(1+t)^\beta}\big]^{(1-\theta)(\alpha+2)}\\
&\le C_5(\|\nabla\Phi(x(t))\|^{\alpha+2}+\|\dot x(t)\|^{\alpha+2}).
\end{aligned}\tag{3.11}$$

Now, since $x$ is bounded and by (H2), we obtain

$$C_3\Big[\frac{\varepsilon}{2}\int_t^\infty\frac{\|\nabla\Phi(x(s))\|^\alpha\|g(s)\|^2}{(1+s)^\beta}\,ds\Big]^{(1-\theta)(\alpha+2)}\le C_6(1+t)^{-\eta},\tag{3.12}$$

where $\eta=(1+2\delta+\beta)(1-\theta)(\alpha+2)$.

By combining (3.6), (3.7), (3.8), (3.9), (3.10), (3.11) and (3.12) we obtain

$$\begin{aligned}
&[H(t)]^{(1-\theta)(\alpha+2)}\\
&\le C_7(\|\dot x(t)\|^{\alpha+2}+\|\nabla\Phi(x(t))\|^{\alpha+2})\\
&\quad+C_8(1+t)^{-\chi}+C_9(1+t)^{-\eta}+\rho\Big(\int_t^{+\infty}\frac{\|\dot x(s)\|^{\alpha+2}}{(1+t)^\beta}\,ds\Big)^{(1-\theta)(\alpha+2)}\\
&\le C_7(\|\dot x(t)\|^{\alpha+2}+\|\nabla\Phi(x(t))\|^{\alpha+2})\\
&\quad+C_{10}(1+t)^{-\chi}+\rho\Big(\int_t^{+\infty}\frac{\|\dot x(s)\|^{\alpha+2}}{(1+t)^\beta}\,ds\Big)^{(1-\theta)(\alpha+2)},
\end{aligned}\tag{3.13}$$

where we use the fact that $\eta>\chi$ in the last inequality. On the other hand, by integrating (3.5) over $(t,\infty)$, we obtain

$$\Big(\int_t^\infty\frac{\|\dot x(s)\|^{\alpha+2}}{(1+t)^\beta}\,ds\Big)^{(1-\theta)(\alpha+2)}\le\Big(\frac{8}{\varepsilon}H(t)\Big)^{(1-\theta)(\alpha+2)}.$$

Now by choosing $\rho$ in (3.9) such that $\rho(8/\varepsilon)^{(1-\theta)(\alpha+2)}<1/2$, estimate (3.13) becomes

$$[H(t)]^{(1-\theta)(\alpha+2)}\le C_{11}(\|\dot x(t)\|^{\alpha+2}+\|\nabla\Phi(x(t))\|^{\alpha+2})+C_{12}(1+t)^{-\chi}.\tag{3.14}$$

Now, by combining (3.5) with the above inequality, we obtain that for all $t\ge T$

$$\begin{aligned}
-H'(t)&\ge\frac{\varepsilon}{8(1+t)^\beta}(\|\dot x(t)\|^{2+\alpha}+\|\nabla\Phi(x(t))\|^{2+\alpha})\\
&\ge C_{13}\frac{[H(t)]^{(1-\theta)(\alpha+2)}}{(1+t)^\beta}-C_{14}(1+t)^{-\chi-\beta}.
\end{aligned}$$

We finally obtain the differential inequality

$$H'(t)+\frac{C_{13}}{(1+t)^\beta}[H(t)]^{(2+\alpha)(1-\theta)}\leqslant\frac{C_{14}}{(1+t)^{\chi+\beta}}.$$

By using Lemma 3.3, there exists $M > 0$ such that for all $t \geq T$,

$$H(t) \leq \frac{M}{(1+t)^{\nu}}$$

where

$$\begin{aligned}\nu &= \inf\Big(\frac{\chi}{(2+\alpha)(1-\theta)}, \frac{1-\beta}{(2+\alpha)(1-\theta)-1}\Big)\\ &= \inf\Big(\delta + \frac{1+\delta-\beta}{1+\alpha}, \frac{1-\beta}{(2+\alpha)(1-\theta)-1}\Big).\end{aligned}$$

Once again from (3.5),

$$\frac{\varepsilon}{8(1+t)^{\beta}}\|\dot{x}(t)\|^{2+\alpha} \leq -H'(t).$$

Then for all $t > T$,

$$\int_t^{2t} \frac{\varepsilon}{8(1+s)^{\beta}}\|\dot{x}(s)\|^{2+\alpha} ds \leq H(t) \leq \frac{M}{(1+t)^{\nu}}.$$

Hölder's inequality yields

$$\begin{aligned}\int_t^{2t} \|\dot{x}(s)\| ds &\leq t^{\frac{1+\alpha}{2+\alpha}}\Big(\int_t^{2t} \|\dot{x}(s)\|^{2+\alpha} ds\Big)^{\frac{1}{2+\alpha}}\\ &\leq t^{\frac{1+\alpha}{2+\alpha}}\Big(\frac{8(1+2t)^{\beta}}{\varepsilon}\int_t^{2t} \frac{\varepsilon}{8(1+s)^{\beta}}\|\dot{x}(s)\|^{2+\alpha} ds\Big)^{\frac{1}{2+\alpha}}\\ &\leq t^{\frac{1+\alpha}{2+\alpha}}\Big(\frac{8(1+2t)^{\beta}}{\varepsilon}\frac{M}{(1+t)^{\nu}}\Big)^{\frac{1}{2+\alpha}} \leq \frac{C_{15}}{t^{\lambda}},\end{aligned}$$

where

$$\begin{aligned}\lambda &= \frac{\nu}{2+\alpha} - \frac{\alpha+1+\beta}{2+\alpha}\\ &= \inf\Big(\big[\frac{\theta-(\alpha+\beta)(1-\theta)}{(1-\theta)(\alpha+2)-1}\big], \big[\frac{\delta-(\alpha+\beta)}{(\alpha+1)}\big]\Big) > 0.\end{aligned}$$

Then

$$\begin{aligned}\int_t^{+\infty} \|\dot{x}(s)\| ds &\leq \sum_{n=0}^{+\infty} \int_{2^n t}^{2^{n+1}t} \|\dot{x}(s)\| ds\\ &\leq \sum_{n=0}^{+\infty} \frac{C_{15}}{2^{n\lambda}t^{\lambda}}\\ &\leq \frac{C_{15}}{t^{\lambda}(1-2^{-\lambda})}\end{aligned}$$

and the result follows since

$$\|x(t) - x(\tau)\| \leq \int_t^{\tau} \|\dot{x}(s)\|\, ds \leq \int_t^{\infty} \|\dot{x}(s)\| ds.$$

$\square$

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