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NONLINEAR ELLIPTIC EQUATIONS WITH GENERAL GROWTH IN THE GRADIENT RELATED TO GAUSS MEASURE

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Abstract. In this article, we establish a comparison result through symmetrization for solutions to some problems with general growth in the gradient. This allows to get sharp estimates for the solutions, obtained by comparing them with solutions of simpler problems whose data depend only on the first variable. Furthermore, we use such result to prove the existence of bounded solutions. All the above results are based on the study of a class of nonlinear integral operator of Volterra type.

1. Introduction

Let Ω be an open subset of $\mathbb{R}^n$. We consider the Dirichlet problem whose prototype is

$$\begin{gathered} -\operatorname{div}(\varphi(x)|\nabla u|^{p-2}\nabla u) = H(x,u,\nabla u) \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

where $|H(x,s,\xi)| \le \varphi(x)\left(f(x) + \theta|\xi|^q\right)$ with $p-1 < q \le p$, $1 < p < 2$, $\theta > 0$ and $\varphi(x) = (2\pi)^{-n/2}\exp\left(-\frac{|x|^2}{2}\right)$ is the density of Gauss measure. Problem (1.1) is related to the generator of Ornstein-Uhlenbeck semigroup.

As Ω is bounded, the operator in (1.1) is uniformly elliptic. In this case, it is well known that one can use Schwarz symmetrization to estimate the solutions of elliptic equations in terms of the solutions of radially symmetric problems. This kind of issue has been faced in [8, 24, 19, 2] for linear equations. As regards nonlinear equations, for the case of $p-1$ growth in the gradient, comparison results are obtained in [3, 18]. Optimal summability of solutions are discussed in [1]. For the case of p growth in the gradient, using Schwarz symmetrization, the existence of bounded solutions are obtained in [9, 15, 16, 18]. For the case of $q(p-1 < q \le p)$ growth in the gradient, similar results can be found in [14, 26]. Recently, symmetrization techniques have also been applied to equations involving fractional Laplacian operators (see [28, 29, 13]).

In our case, since Ω maybe unbounded, the degeneracy of the operator does not allow to use the classical approach via Schwarz symmetrization. This leads us to

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consider Gauss symmetrization based on the structure of the problem. By Gauss symmetrization, comparison results for linear equations have been obtained, with a simpler problem which is defined in a half space and has data depending only on the first variable (see [6, 10, 12]). Nonlinear equations with $p-1$ growth in the gradient have also been discussed in [11]. However, the case with other growth in the gradient has not been studied until now. In this paper, we deal with such problem (with $q(p-1<q\le p)$ growth in the gradient).

Our aim is to prove a sharp comparison result which allows us to estimate the solutions of (1.1) in terms of the symmetric solutions of the following "symmetrized" problem

$$\begin{gathered} -D_1(\varphi|D_1v|^{p-2}D_1v)=\varphi f^\sharp+\theta\varphi|D_1v|^q \quad \text{in } \Omega^\sharp, \\ v=0 \quad \text{on } \partial\Omega^\sharp, \end{gathered} \tag{1.2}$$

where $\Omega^\sharp$ is a half space with the same Gauss measure as Ω and $f^\sharp$ is Gauss symmetrization of f. To this end, we first discuss the existence of symmetric solutions to (1.2) and give the regularity results of such solutions, which is a key step for the comparison results. Moreover, by the comparison results, we are able to prove the existence of bounded solutions to (1.1) in weighted Sobolev space $W_0^{1,p}(\varphi,\Omega)$. Note that the assumption $1<p<2$ is necessary to the existence of bounded solutions. We will give an example in the Appendix to show that there may be no solution to (1.1) if $p\ge 2$.

There are two main difficulties in studying (1.1). One is due to the fact that the operator is in general not uniformly elliptic, for instance when Ω is an unbounded domain. Another is due to the presence of general growth in the gradient. Therefore, the present approaches can not be applied to our case. To overcome the above difficulties, based on the properties of the weighted rearrangement, we convert the problems into the study of a class of Volterra integral operator. This class of Volterra integral operator was introduced in [20, 17]. By discussing the existence of fixed point to the Volterra integral operator, we obtained the existence and non-existence of symmetric solution to the "symmetrized" problem (1.2). The sharp comparison results are obtained by proving a new type of comparison principle for the Volterra integral operator. The methods developed in this article can also be used to study the corresponding variational inequalities.

This article is organized as follows: In Section 2 we give some preliminary results. In Section 3, the main results of this paper are stated. In Section 4, three results for a class of Volterra integral operator are proved. In Section 5, we finish the proof of the main results.

2. Notation and preliminary results

In this section, we recall some definitions and results which will be useful in what follows.

Let γ_n be the n-dimensional normalized Gauss measure on $\mathbb{R}^n$ defined as

$$d\gamma_n=\varphi(x)dx=(2\pi)^{-n/2}\exp\big(-\frac{|x|^2}{2}\big)dx, \quad x\in\mathbb{R}^n.$$

We denote by $\Phi(\tau)$ the measure of the half space $\{x\in\mathbb{R}^n: x_1>\tau\}$, i.e.

$$\Phi(\tau)=\gamma_n(\{x\in\mathbb{R}^n: x_1>\tau\})=\frac{1}{\sqrt{2\pi}}\int_\tau^{+\infty}\exp\big(-\frac{t^2}{2}\big)dt, \tau\in\mathbb{R}.$$

Observe that (see [27])

$$\exp\Big(-\frac{\Phi^{-1}(t)^2}{2}\Big) \le \alpha t(1-\log t)^{1/2}, \quad t \in (0,\gamma_n(\Omega)), \tag{2.1}$$

$$\exp\Big(-\frac{\Phi^{-1}(t)^2}{2}\Big) \ge \beta t(1-\log t)^{1/2}, \quad t \in (0,\gamma_n(\Omega)), \tag{2.2}$$

where α and β are positive constants depending on $\gamma_n(\Omega)$. Now we give the notion of rearrangement.

Definition 2.1. If u is a measurable function in Ω and $\mu(t) = \gamma_n(\{x \in \Omega : |u| > t\})$ is the distribution function of u, then we define the decreasing rearrangement of u with respect to Gauss measure as

$$u^\star(s) = \inf\{t \ge 0 : \mu(t) \le s\}, \quad s \in [0, \gamma_n(\Omega)].$$

If $\Omega^\sharp = \{x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n : x_1 > \lambda\}$ is the half-space such that $\gamma_n(\Omega) = \gamma_n(\Omega^\sharp)$, then

$$u^\sharp(x) = u^\star(\Phi(x_1)), \quad x \in \Omega^\sharp$$

denote the increasing Gauss symmetrization of u (or Gauss symmetrization of u).

Similarly, the decreasing Gauss symmetrization of u is

$$u_\sharp(x) = u_\star(\Phi(x_1)), \quad x \in \Omega^\sharp,$$

with

$$u_\star(s) = u^\star(\gamma_n(\Omega) - s), \quad s \in (0, \gamma_n(\Omega)).$$

The properties of rearrangement with respect to Gauss measure or a positive measure have been widely considered, see [7, 5, 21, 22, 23] for instance. Here we just recall that

(a) (Hardy-Little inequality)

$$\begin{aligned}\int_0^{\gamma_n(\Omega)} u_\star(s)v^\star(s)ds &= \int_{\Omega^\sharp} u_\sharp(x)v^\sharp(x)d\gamma_n \le \int_\Omega |u(x)v(x)|d\gamma_n \\ &\le \int_{\Omega^\sharp} u^\sharp(x)v^\sharp(x)d\gamma_n = \int_0^{\gamma_n(\Omega)} u^\star(s)v^\star(s)ds,\end{aligned}$$

where u and v are measurable functions.

(b) (Polya-Szëgo principle) Let $u \in W_0^{1,p}(\varphi,\Omega)$ with $1 < p < +\infty$. Then

$$\|\nabla u^\sharp\|_{L^p(\varphi,\Omega^\sharp)} \le \|\nabla u\|_{L^p(\varphi,\Omega)},$$

and equality holds if and only if $\Omega = \Omega^\sharp$ and $|u| = u^\sharp$ modulo a rotation.

Finally, we recall that the weighted Sobolev space $W_0^{1,p}(\varphi,\Omega)$ is the closure of $C_0^\infty(\Omega)$ under the norm

$$\|u\|_{W^{1,p}(\varphi,\Omega)} = \Big(\int_\Omega |\nabla u(x)|^p\varphi\,dx + \int_\Omega |u(x)|^p\varphi\,dx\Big)^{1/p}.$$

3. Statement of main results

In this article, we consider the problem

$$\begin{gathered} -\operatorname{div}(a(x,u,\nabla u)) = H(x,u,\nabla u) \quad \text{in } \Omega, \\ u \in W_0^{1,p}(\varphi,\Omega) \cap L^\infty(\Omega), \end{gathered} \tag{3.1}$$

where Ω is an open subset of $\mathbb{R}^n (n \geq 2)$ with Gauss measure less than one, $a : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ and $H : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ are Carathéodory functions such that for all $(s,\xi) \in \mathbb{R} \times \mathbb{R}^n$ and for almost every $x \in \Omega$,

- (A1) $a(x,s,\xi)\xi \geq \varphi(x)|\xi|^p$, with $1 < p < 2$;
- (A2) $|a(x,s,\xi)| \leq c_1\varphi(x)\big(k(x) + |s|^{p-1} + |\xi|^{p-1}\big)$, with $c_1 > 0, k(x) \geq 0$ and $k \in L^{p'}(\varphi,\Omega)$;
- (A3) $|H(x,s,\xi)| \leq \varphi(x)\left(f(x) + \theta|\xi|^q\right)$, with $\theta > 0$, $p-1 < q \leq p$, $f(x) \geq 0$ and $f \in L^\infty(\Omega)$.

Definition 3.1. We say that $u \in W_0^{1,p}(\varphi,\Omega) \cap L^\infty(\Omega)$ is a solution of (1.1), if

$$\int_\Omega a(x,u,\nabla u)\nabla\psi\,dx = \int_\Omega H(x,u,\nabla u)\psi\,dx, \quad \forall \psi \in W_0^{1,p}(\varphi,\Omega) \cap L^\infty(\Omega). \tag{3.2}$$

Remark 3.2. The assumption $1 < p < 2$ is necessary. As $p \geq 2$, there may be no solution to (3.1) (see an example in the Appendix).

First, let us turn our attention to the "symmetrized" problem (1.2), discussing the existence and regularity of a unique symmetric solution, which is a key step for the comparison results.

Theorem 3.3. *Let $\gamma = \frac{q}{p-1}$, $\gamma' = \frac{\gamma}{\gamma-1}$ and $M_0 = \theta\big(\sqrt{2\pi}\beta^{-1}(1-\log\gamma_n(\Omega))^{-1/2}\big)^\gamma$. If*

$$\|f\|_{L^\infty(\Omega)} \leq \frac{1}{\gamma'}(\gamma M_0)^{\frac{1}{1-\gamma}}, \tag{3.3}$$

then there exists a unique solution to (1.2) *such that $v(x) = v^\sharp(x)$. Moreover, $v \in C^1(\Omega^\sharp \backslash x_1 = +\infty) \cap W^{1,\infty}(\Omega^\sharp)$ with provides the estimates*

$$\begin{gathered} \|\nabla v\|_{L^\infty(\Omega^\sharp)} \leq C_1(\beta,\gamma,\gamma_n(\Omega)), \\ \|v\|_{L^\infty(\Omega^\sharp)} \leq C_2(\beta,\gamma,\gamma_n(\Omega)), \end{gathered}$$

where C_1, C_2 are constants depending only on β, γ and $\gamma_n(\Omega)$.

In the case $f(x) \equiv f_0$, we have the following nonexistence result for (1.2).

Theorem 3.4. *Let*

$$f_0 \geq A(\gamma_n(\Omega)), \tag{3.4}$$

where

$$A(s) = \Big[\frac{\gamma^{\gamma'}}{\theta(\gamma-1)}\Big(\frac{\alpha}{\sqrt{2\pi}}\Big)^\gamma \frac{s}{F_1(s)}\Big]^{\frac{1}{\gamma-1}}, \quad F_1(s) = \int_0^s (1-\ln\tau)^{-\frac{\gamma}{2}}\,d\tau,$$

for $s \in [0,\gamma_n(\Omega)]$. Then (1.2) *has no symmetric solution.*

Remark 3.5. By computations, it follows that $A(\gamma_n(\Omega)) > \frac{1}{\gamma'}(\gamma M_0)^{\frac{1}{1-\gamma}}$. Thus, the above theorem proposes an example to show that (1.2) has no symmetric solution without the assumption (3.3).

Now, the comparison results can be stated by the following theorem.

Theorem 3.6. *Assume that* (A1)–(A3) *hold. Let u be a solution to* (3.1) *and v be a solution to* (1.2) *such that $v(x) = v^\sharp(x)$. Then*

$$u^\sharp(x) \le v(x), \quad x \in \Omega^\sharp, \tag{3.5}$$

$$\int_\Omega \eta(|\nabla u|^p)\varphi\, dx \le \int_{\Omega^\sharp} \eta(|\nabla v|^p)\varphi\, dx, \tag{3.6}$$

where η is a concave and nondecreasing function on $[0,+\infty)$. Moreover, if $f(x) \not\equiv 0$, the equality in (3.5) *holds if and only if*

$$\begin{gathered}
\Omega = \Omega^\sharp,\\
u(x) = \delta u^\sharp(x), \quad \text{a.e. } x \in \Omega^\sharp,\\
a_1(x,u,\nabla u) = \delta\varphi |D_1 u^\sharp|^{p-2} D_1 u^\sharp, \quad \text{a.e. } x \in \Omega^\sharp,\\
\sum_{i=2}^n D_i a_i(x,u,\nabla u) = 0 \quad \text{in } \mathcal{D}'(\Omega^\sharp),\\
H(x,u,\nabla u) = \delta\varphi(f^\sharp(x) + \theta |D_1 u^\sharp|^q), \quad \text{a.e. } x \in \Omega^\sharp
\end{gathered} \tag{3.7}$$

modulo a rotation with $\delta = \pm 1$.

Remark 3.7. Equality (3.7) implies that the comparison result (3.5) is sharp in the sense that as the equality holds, problem (3.1) is equivalent to its "symmetrized" problem (1.2) modulo a rotation.

The estimates we have found can be applied to prove the existence result by using the well known approximation techniques [4].

Theorem 3.8. *Let* (A1)–(A3) *and* (3.3) *hold. Assume that*

$$[a(x,s,\xi_1) - a(x,s,\xi_2)] \cdot (\xi_1 - \xi_2) > 0, \quad \text{for } \xi_1 \ne \xi_2.$$

Then there exists at least one solution to (3.1).

4. Results for Volterra integral operators

This section is devoted to study a class of Volterra integral operator. We prove three results, i.e. the comparison principle, the existence of fixed point and the nonexistence of fixed point for the Volterra integral operator, which will be useful in proving the main results of this paper.

Assume $h(s,\xi) : [0,T] \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function. Consider the following Volterra integral operator (see [20, 17])

$$K : D(K) \subseteq C([0,T]) \to C([0,T]), \quad K\psi(t) = \int_0^t h(\tau, \psi(\tau))d\tau, \quad \forall \psi \in D(K).$$

Definition 4.1. We say that the operator K has property (m) if for all $\psi_1, \psi_2 \in D(K)$ and $a \in [0,T)$, there exist constants $b \in (a,T]$ and $m(a,b,\psi_1,\psi_2) \in [0,1)$ such that for any $t \in (a,b]$,

$$\|h(\cdot,\psi_1(\cdot)) - h(\cdot,\psi_2(\cdot))\|_{L^1(a,t)} \le t m(a,b,\psi_1,\psi_2) \Big\| \frac{\psi_1 - \psi_2}{s} \Big\|_{L^\infty(a,t)}. \tag{4.1}$$

Lemma 4.2. *Let the operator K satisfy property (m) and $h(t,\cdot)$ be nondecreasing for a.e. $t \in [0,T]$. If $u, v \in D(K)$ are such that $u \le Ku$, $v \ge Kv$, then we have*

$$u \le v. \tag{4.2}$$

In particular, the equation $w = Kw$ possesses at most one solution in $D(K)$.

Proof. We argue by contradiction. If (4.2) does not hold, then there must exist $a \in [0,T)$ and $b_1 \in (a,T]$ such that $u(t) \le v(t)$ for $t \in [0,a]$ and $u(t) > v(t)$ for $t \in (a,b_1]$. Set $b_2 = \min\{b_1, b\}$ with b is the constant of property (m). Then $u(t) > v(t)$ in $(a,b_2]$. Since $u \le Ku$, $v \ge Kv$ and $h(t,\cdot)$ is nondecreasing for a.e. $t \in [0,T]$, it follows that for $\forall t \in (a,b_2]$,

$$\begin{aligned} |u(t)-v(t)| = u(t)-v(t) &\le \int_0^t (h(\tau,u(\tau)) - h(\tau,v(\tau)))\,d\tau \\ &= \int_0^a (h(\tau,u(\tau)) - h(\tau,v(\tau)))\,d\tau + \int_a^t (h(\tau,u(\tau)) - h(\tau,v(\tau)))\,d\tau \\ &\le \int_a^t (h(\tau,u(\tau)) - h(\tau,v(\tau)))\,d\tau \\ &\le \|h(\cdot,u(\cdot)) - h(\cdot,v(\cdot))\|_{L^1(a,t)}. \end{aligned} \tag{4.3}$$

Using property (m), we obtain

$$|u(t)-v(t)| \le tm(a,b,u,v)\|\frac{u-v}{s}\|_{L^\infty(a,t)}, \quad \forall t \in (a,b_2]. \tag{4.4}$$

Taking the maximum over $t \in (a,b_2]$ and noting $m(a,b,u,v) \in [0,1)$, we have

$$\|\frac{u-v}{s}\|_{L^\infty(a,b_2)} \le m(a,b,u,v)\|\frac{u-v}{s}\|_{L^\infty(a,b_2)} < \|\frac{u-v}{s}\|_{L^\infty(a,b_2)},$$

which is a contradiction. Thus

$$u \le v, \forall u,v \in D(K),$$

and the uniqueness claim easily follows. Then the lemma is proved. □

Let

$$K\psi(s) = \int_0^s f^\star(\tau) + \theta\Big(\sqrt{2\pi}\exp\big(\frac{\Phi^{-1}(\tau)^2}{2}\big)\Big)^\gamma \psi^\gamma(\tau)d\tau, \quad s \in [0,\gamma_n(\Omega)].$$

We shall deal with two types of domains

$$\begin{gathered} D_1(K) = \{\psi \in C([0,\gamma_n(\Omega)]) : M \ge 0, 0 \le \psi(s) \le Ms\}, \\ D_2(K) = \{\psi \in C([0,\gamma_n(\Omega)]) : M_\psi \ge 0, 0 \le \psi(s) \le M_\psi s\}, \end{gathered}$$

and let $R_i(K)$ be the range of K on $D_i(K)$, $i = 1,2$.

Lemma 4.3. *Let $M = (\gamma M_0)^{\frac{1}{1-\gamma}}$ in $D_1(K)$. If (3.3) holds, then the equation $w = Kw$ has a unique solution in $D_1(K)$. Furthermore, the solution is also unique in $D_2(K)$.*

Proof. First, we prove that $R_1(K) \subseteq D_1(K)$. For $\forall\psi \in D_1(K)$ and $s \in [0,\gamma_n(\Omega)]$, we have by (2.2) that

$$\begin{aligned} K\psi(s) &= \int_0^s f^\star(\tau) + \theta\Big(\sqrt{2\pi}\exp\big(\frac{\Phi^{-1}(\tau)^2}{2}\big)\Big)^\gamma \psi^\gamma(\tau)d\tau \\ &\le s\|f\|_{L^\infty(\Omega)} + \theta\big(\sqrt{2\pi}\beta^{-1}\big)^\gamma \int_0^s \Big(\frac{1}{\tau(1-\log\tau)^{1/2}}\Big)^\gamma (M\tau)^\gamma d\tau \\ &\le \Big(\|f\|_{L^\infty(\Omega)} + \theta\Big(\sqrt{2\pi}\beta^{-1}M(1-\log\gamma_n(\Omega))^{-1/2}\Big)^\gamma\Big)s \\ &= \big(\|f\|_{L^\infty(\Omega)} + M_0M^\gamma\big)s. \end{aligned} \tag{4.5}$$

Under the assumption (3.3), noting that $M = (\gamma M_0)^{\frac{1}{1-\gamma}}$, we obtain

$$\|f\|_{L^\infty(\Omega)} + M_0 M^\gamma \le M.$$

Thus,

$$0 \le K\psi(s) \le Ms, \quad s \in [0, \gamma_n(\Omega)].$$

This implies that $R_1(K) \subseteq D_1(K)$.

Next, we verify that the operator K is compact with respect to the uniform topology of $C([0,\gamma_n(\Omega)])$. To prove this, let us first show the equicontinuity of $R_1(K)$ in $C([0,\gamma_n(\Omega)])$. Take any $K\psi \in R_1(K)$. For any $0 \le a < b \le \gamma_n(\Omega)$, it follows

$$\begin{aligned}
&|K\psi(b) - K\psi(a)| \\
&\le \int_a^b \Big| f^\star(\tau) + \theta\Big(\sqrt{2\pi}\exp\Big(\frac{\Phi^{-1}(\tau)^2}{2}\Big)\Big)^\gamma \psi^\gamma(\tau)\Big| d\tau \\
&\le \|f\|_{L^\infty(\Omega)}(b-a) + \theta\big(\sqrt{2\pi}\beta^{-1}\big)^\gamma \int_a^b \Big(\frac{1}{\tau(1-\log\tau)^{1/2}}\Big)^\gamma (M\tau)^\gamma d\tau \\
&\le (\|f\|_{L^\infty(\Omega)} + M_0 M^\gamma)(b-a).
\end{aligned} \tag{4.6}$$

Then, $R_1(K)$ is equicontinuous in $C([0,\gamma_n(\Omega)])$. Also, the family of functions from $R_1(K)$ is uniformly bounded (see (4.5)), i.e.

$$0 \le K\psi \le \big(\|f\|_{L^\infty(\Omega)} + M_0 M^\gamma\big)\,\gamma_n(\Omega). \tag{4.7}$$

By Ascoli-Arzela theorem, $R_1(K)$ is relatively compact in $C([0,\gamma_n(\Omega)])$, and hence the operator K is compact.

Since the domain $D_1(K)$ is bounded, closed and convex, by Schauder's fixed point theorem there exists at least one solution $w \in D_1(K)$ to the equation $w = Kw$.

Now, we study the uniqueness of the solution w. By $D_1(K) \subseteq D_2(K)$, it suffices to show that the solution w is unique in $D_2(K)$.

Next we check that K satisfies property (m) in $D_2(K)$. For all $\psi_1, \psi_2 \in D_2(K)$, $0 \le a < b \le \gamma_n(\Omega)$ and $a < t \le b$, we have

$$\begin{aligned}
&\|h(\cdot,\psi_1(\cdot)) - h(\cdot,\psi_2(\cdot))\|_{L^1(a,t)} \\
&\le \int_a^t \theta\Big(\sqrt{2\pi}\exp\Big(\frac{\Phi^{-1}(\tau)^2}{2}\Big)\Big)^\gamma |\psi_1^\gamma(\tau) - \psi_2^\gamma(\tau)| d\tau \\
&\le \theta\gamma\big(\sqrt{2\pi}\beta^{-1}\big)^\gamma \int_a^t \Big(\frac{1}{\tau(1-\log\tau)^{1/2}}\Big)^\gamma \max\{\psi_1^{\gamma-1}(\tau), \psi_2^{\gamma-1}(\tau)\} \\
&\quad \times |\psi_1(\tau) - \psi_2(\tau)| d\tau \\
&\le \theta\gamma\Big(\sqrt{2\pi}\beta^{-1}\Big)^\gamma \max\{M_{\psi_1}, M_{\psi_2}\}^{\gamma-1} \int_a^t \Big(\frac{1}{\tau(1-\log\tau)^{1/2}}\Big)^\gamma \tau^{\gamma-1} \\
&\quad \times |\psi_1(\tau) - \psi_2(\tau)| d\tau \\
&\le \theta\gamma\Big(\sqrt{2\pi}\beta^{-1}(1-\log\gamma_n(\Omega))^{-1/2}\Big)^\gamma \widetilde{M}^{\gamma-1}(t-a)\|\frac{\psi_1-\psi_2}{\tau}\|_{L^\infty(a,t)} \\
&\le t\gamma M_0 \widetilde{M}^{\gamma-1}\big(1 - \frac{a}{b}\big)\|\frac{\psi_1-\psi_2}{\tau}\|_{L^\infty(a,t)} \\
&= t m(a,b,\psi_1,\psi_2)\|\frac{\psi_1-\psi_2}{\tau}\|_{L^\infty(a,t)},
\end{aligned} \tag{4.8}$$

where
$$m(a,b,\psi_1,\psi_2)=\gamma M_0\widetilde{M}^{\gamma-1}(1-\frac{a}{b})$$
with $\widetilde{M}=\max\{M_{\psi_1},M_{\psi_2}\}$.

From $\lim_{b\to a^+} m(a,b,\psi_1,\psi_2)=0$, it follows that there exists $b_0>a$ such that $0\le m(a,b_0,\psi_1,\psi_2)<1$. Thus (4.8) implies that K satisfies property (m) in $D_2(K)$. Since $h(t,\psi)$ is nondecreasing for $t\in[0,\gamma_n(\Omega)]$, we have by Lemma 4.2 that the solution of $w=Kw$ is unique in $D_2(K)$. Thus the proof is complete. □

For $f(x)\equiv f_0$, we have the following nonexistence result.

Lemma 4.4. *Let* (3.4) *hold. Then the equation $z=Kz$ has no solution in $D_1(K)$.*

Proof. We argue by contradiction. Assume that there exists a solution $z\in D_1(K)$ of $z=Kz$. Let $z_0(s)=f_0 s$ and
$$z_m(s)=\theta\int_0^s\Big(\sqrt{2\pi}\exp\Big(\frac{\Phi^{-1}(\tau)^2}{2}\Big)\Big)^{\gamma}z_{m-1}^{\gamma}(\tau)d\tau,\quad s\in[0,\gamma_n(\Omega)]. \tag{4.9}$$
First, we claim that
$$z_m(s)\ge sA(s)\Big(\frac{f_0}{A(s)}\Big)^{\gamma^m}\gamma^{\frac{m}{\gamma-1}},\quad m\ge 1 \tag{4.10}$$
where
$$A(s)=\Big[\frac{\gamma^{\gamma'}}{\theta(\gamma-1)}\Big(\frac{\alpha}{\sqrt{2\pi}}\Big)^{\gamma}\frac{s}{F_1(s)}\Big]^{\frac{1}{\gamma-1}},\quad F_1(s)=\int_0^s(1-\ln\tau)^{-\frac{\gamma}{2}}d\tau.$$
To prove (4.10), set
$$F_{m+1}(s)=\int_0^s(1-\ln\tau)^{-\gamma/2}F_m^{\gamma}(\tau)d\tau.$$
Thus,
$$z_m(s)\ge\theta^{\frac{\gamma^m-1}{\gamma-1}}\Big(\frac{\sqrt{2\pi}}{\alpha}\Big)^{\frac{\gamma^{m+1}-\gamma}{\gamma-1}}f_0^{\gamma^m}s^{-\frac{\gamma^m-\gamma}{\gamma-1}}F_m(s). \tag{4.11}$$
Indeed, by (2.1) and (4.9), we have
$$\begin{aligned}
z_1(s)&=\theta\int_0^s\Big(\sqrt{2\pi}\exp\Big(\frac{\Phi^{-1}(\tau)^2}{2}\Big)\Big)^{\gamma}z_0^{\gamma}(\tau)d\tau\\
&\ge\theta\Big(\frac{\sqrt{2\pi}}{\alpha}\Big)^{\gamma}f_0^{\gamma}\int_0^s\Big(\frac{1}{\tau(1-\ln\tau)^{1/2}}\Big)^{\gamma}\tau^{\gamma}d\tau\\
&=\theta\Big(\frac{\sqrt{2\pi}}{\alpha}\Big)^{\gamma}f_0^{\gamma}F_1(s),
\end{aligned}$$
which proves (4.11) for $m=1$. Assume it holds for $m-1$. Then (4.11) can be established by induction on m. Next, treat the term $F_m(s)$ in (4.11). Actually,
$$F_m(s)\ge\prod_{\delta=1}^{m}\Big(\frac{\gamma-1}{\gamma^{\delta}-1}\Big)^{\gamma^{m-\delta}}F_1^{\frac{\gamma^m-1}{\gamma-1}}(s), m\ge 1. \tag{4.12}$$
The case $m=1$ is obvious. Suppose (4.12) holds for some m. Then
$$F_{m+1}(s)=\int_0^s(1-\ln\tau)^{-\gamma/2}F_m^{\gamma}(\tau)d\tau$$

$$
\begin{aligned}
&\geq \prod_{\delta=1}^{m} \Big(\frac{\gamma-1}{\gamma^\delta-1}\Big)^{\gamma^{m+1-\delta}} \int_0^s (1-\ln\tau)^{-\gamma/2} F_1^{\frac{\gamma^{m+1}-\gamma}{\gamma-1}}(\tau)d\tau \\
&= \prod_{\delta=1}^{m} \Big(\frac{\gamma-1}{\gamma^\delta-1}\Big)^{\gamma^{m+1-\delta}} \int_0^s F_1^{\frac{\gamma^{m+1}-\gamma}{\gamma-1}}(\tau)dF_1(\tau) \\
&= \prod_{\delta=1}^{m+1} \Big(\frac{\gamma-1}{\gamma^\delta-1}\Big)^{\gamma^{m+1-\delta}} F_1^{\frac{\gamma^{m+1}-1}{\gamma-1}}(s),
\end{aligned}
$$

which proves (4.12). Moreover,

$$
\begin{aligned}
\prod_{\delta=1}^{m} \Big(\frac{\gamma-1}{\gamma^\delta-1}\Big)^{\gamma^{m-\delta}} &\geq \prod_{\delta=1}^{m} \Big(\frac{\gamma-1}{\gamma^\delta}\Big)^{\gamma^{m-\delta}} = (\gamma-1)^{\sum_{\delta=1}^m \gamma^{m-\delta}} \gamma^{-\sum_{\delta=1}^m \delta\gamma^{m-\delta}} \\
&= (\gamma-1)^{\frac{\gamma^m-1}{\gamma-1}} \gamma^{\frac{m}{\gamma-1}-\frac{\gamma(\gamma^m-1)}{(\gamma-1)^2}}.
\end{aligned} \tag{4.13}
$$

Combining (4.11), (4.12) and (4.13), we know that (4.10) holds.

On the other hand, $A(s)$ is a decreasing continuous function on $(0,\gamma_n(\Omega)]$ and $\lim_{s\to 0^+} A(s) = +\infty$. Then the range of A is $[A(\gamma_n(\Omega)),+\infty)$. Recalling that $f_0 \geq A(\gamma_n(\Omega)))$, thus there must exist a constant $s^* \in (0,\gamma_n(\Omega)]$ such that $f_0 = A(s^*)$. As $s \geq s^*$, it follows that $\frac{f_0}{A(s)} \geq 1$ and

$$
z_m(s) \geq sA(s)\gamma^{\frac{m}{\gamma-1}}.
$$

Note that $\lim_{m\to\infty} \gamma^{\frac{m}{\gamma-1}} = +\infty$. We conclude that $\sum_{m=0}^\infty z_m(s) = +\infty$ for $s \in [s^*,\gamma_n(\Omega)]$. On the other hand, by inducing on k, it is easy to prove that

$$
z(s) \geq \sum_{m=0}^{k} z_m(s), \quad \forall k \in \mathbb{N}. \tag{4.14}
$$

Indeed, since $z(s) = Kz(s) \geq f_0 s = z_0(s)$, we obtain

$$
z(s) = f_0 s + \theta \int_0^s \Big(\sqrt{2\pi}\exp\Big(\frac{\Phi^{-1}(\tau)^2}{2}\Big)\Big)^\gamma z^\gamma(\tau)d\tau \geq z_0(s) + z_1(s),
$$

which proves the claim for $k=1$. Now assume (4.14) holds for some $k \in N$. Then

$$
\begin{aligned}
z(s) &\geq z_0(s) + \theta \int_0^s \Big(\sqrt{2\pi}\exp\Big(\frac{\Phi^{-1}(\tau)^2}{2}\Big)\Big)^\gamma \Big(\sum_{m=0}^{k} z_m(\tau)\Big)^\gamma d\tau \\
&\geq z_0(s) + \theta \sum_{m=0}^{k} \int_0^s \Big(\sqrt{2\pi}\exp\Big(\frac{\Phi^{-1}(\tau)^2}{2}\Big)\Big)^\gamma z_m^\gamma(\tau)d\tau \\
&= z_0(s) + \sum_{m=0}^{k} z_{m+1}(s) = \sum_{m=0}^{k+1} z_m(s).
\end{aligned}
$$

Thus for $k \in \mathbb{N}$, (4.14) holds. However, recalling that $z \in D_1(K)$, we obtain

$$
Ms \geq z(s) \geq \sum_{m=0}^{\infty} z_m(s) = +\infty, \quad s \in [s^*,\gamma_n(\Omega)],
$$

which is a contradiction. Thus the lemma is proved. □

5. Proofs of the main results

First, let us enunciate the following lemma, the proof of which is not supplied here since it follows the same lines as in [14, 18].

Lemma 5.1. *Let u be a solution of* (1.1) *and v be a solution of* (1.2) *such that $v(x) = v^\sharp(x)$. Then*

$$\begin{aligned} &\big(-u^{\star\prime}(s)\big)^{p-1}\Big(\frac{1}{\sqrt{2\pi}}\exp\Big(-\frac{\Phi^{-1}(s)^2}{2}\Big)\Big)^p \\ &\le \int_0^s f^\star(\tau)\exp\Big[\theta\int_\tau^s\Big(\sqrt{2\pi}\exp\Big(\frac{\Phi^{-1}(\sigma)^2}{2}\Big)\Big)^{p-q}\Big(-u^{\star\prime}(\sigma)\Big)^{q-p+1}d\sigma\Big]d\tau \end{aligned} \tag{5.1}$$

and

$$\begin{aligned} &\big(-v^{\star\prime}(s)\big)^{p-1}\Big(\frac{1}{\sqrt{2\pi}}\exp\Big(-\frac{\Phi^{-1}(s)^2}{2}\Big)\Big)^p \\ &= \int_0^s f^\star(\tau)\exp\Big[\theta\int_\tau^s\Big(\sqrt{2\pi}\exp\Big(\frac{\Phi^{-1}(\sigma)^2}{2}\Big)\Big)^{p-q}\big(-v^{\star\prime}(\sigma)\big)^{q-p+1}d\sigma\Big]d\tau, \end{aligned} \tag{5.2}$$

a.e. $s \in (0, \gamma_n(\Omega))$.

Proof of Theorem 3.3. Let

$$v(x) = V(\Phi(x_1)) = \int_{\Phi(x_1)}^{\gamma_n(\Omega)}\Big(\sqrt{2\pi}\exp\Big(\frac{\Phi^{-1}(\tau)^2}{2}\Big)\Big)^{p'}w^{\frac{1}{p-1}}(\tau)d\tau, x\in\Omega^\sharp, \tag{5.3}$$

where w is the unique solution of $w = Kw$ obtained in Lemma 4.3. Clearly, $v(x) = v^\sharp(x)$, $x \in \Omega^\sharp$. By (5.3) and (2.2),

$$\begin{aligned} |\nabla v(x)| = D_1 v(x) &= -\frac{1}{\sqrt{2\pi}}V'(\Phi(x_1))\exp\big(-\frac{x_1^2}{2}\big) \\ &= \Big(\sqrt{2\pi}\exp\big(\frac{x_1^2}{2}\big)\Big)^{\frac{1}{p-1}}w^{\frac{1}{p-1}}(\Phi(x_1)) \\ &\le \big(\sqrt{2\pi}\big)^{\frac{1}{p-1}}\Big(\beta\Phi(x_1)(1-\log\Phi(x_1))^{1/2}\Big)^{-\frac{1}{p-1}}(M\Phi(x_1))^{\frac{1}{p-1}} \\ &\le \Big(\sqrt{2\pi}\beta^{-1}M(1-\log\gamma_n(\Omega))^{-1/2}\Big)^{\frac{1}{p-1}}. \end{aligned}$$

Moreover, since $1 < p < 2$,

$$\begin{aligned} \|v\|_{L^\infty(\Omega^\sharp)} = V(0) &= \int_0^{\gamma_n(\Omega)}\Big(\sqrt{2\pi}\exp\big(\frac{\Phi^{-1}(\tau)^2}{2}\big)\Big)^{p'}w^{\frac{1}{p-1}}(\tau)d\tau \\ &\le \big(\sqrt{2\pi}\beta^{-1}\big)^{p'}M^{\frac{1}{p-1}}\int_0^{\gamma_n(\Omega)}\Big(\frac{1}{\tau(1-\log\tau)^{1/2}}\Big)^{p'}\tau^{\frac{1}{p-1}}d\tau \\ &= \big(\sqrt{2\pi}\beta^{-1}\big)^{p'}M^{\frac{1}{p-1}}\int_0^{\gamma_n(\Omega)}(1-\log\tau)^{-\frac{p'}{2}}\tau^{-1}d\tau \\ &= \frac{2}{p'-2}\big(\sqrt{2\pi}\beta^{-1}\big)^{p'}M^{\frac{1}{p-1}}(1-\log\gamma_n(\Omega))^{-\frac{p'}{2}+1}. \end{aligned}$$

Then $v \in W_0^{1,p}(\varphi, \Omega^\sharp) \cap L^\infty(\Omega^\sharp)$.

On the other hand, for all $\psi \in W_0^{1,p}(\varphi, \Omega^\sharp) \cap L^\infty(\Omega^\sharp)$, by (5.3) and the fact that $w = Kw$ in $D_1(K)$, we have

$$\begin{aligned}
&\int_{\Omega^\sharp} \varphi (D_1 v)^{p-1} D_1\psi\, dx \\
&= \int_{\Omega^\sharp} \varphi\Big(\frac{1}{\sqrt{2\pi}} \exp\big(-\frac{x_1^2}{2}\big)\Big)^{p-1} \left(-V'\left(\Phi(x_1)\right)\right)^{p-1} D_1\psi\, dx \\
&= \int_{\Omega^\sharp} w(\Phi(x_1)) D_1\psi\, dx \\
&= \int_{\Omega^\sharp} \int_0^{\Phi(x_1)} \Big[f^\star(\tau) + \theta\Big(\sqrt{2\pi}\exp\Big(\frac{\Phi^{-1}(\tau)^2}{2}\Big)\Big)^{\gamma} w^\gamma(\tau)\Big] d\tau\, D_1\psi\, dx.
\end{aligned}$$

Integrating by part on the right-hand side of the third equality to obtain

$$\begin{aligned}
&\int_{\Omega^\sharp} \varphi\left(D_1 v\right)^{p-1} D_1\psi\, dx \\
&= \int_{\Omega^\sharp} \Big[f^\sharp(x) + \theta\Big(\sqrt{2\pi}\exp\big(\frac{x_1^2}{2}\big)\Big)^{\gamma} w^\gamma(\Phi(x_1))\Big] \psi\varphi\, dx \\
&= \int_{\Omega^\sharp} f^\sharp \psi\varphi\, dx + \int_{\Omega^\sharp} \theta\Big(-V'(\Phi(x_1))\frac{1}{\sqrt{2\pi}}\exp\Big(-\frac{x_1^2}{2}\Big)\Big)^{q} \psi\varphi\, dx \\
&= \int_{\Omega^\sharp} f^\sharp \psi\varphi\, dx + \int_{\Omega^\sharp} \theta |D_1 v|^q \psi\varphi\, dx.
\end{aligned}$$

Hence, v is a symmetric solution to (1.2). $\square$

Next, we show that v is the unique symmetric solutions to (1.2). Indeed, assume that there exists another symmetric solution v_1. Let

$$\begin{aligned}
w_1(s) = \int_0^s f^\star(\tau) \exp\Big[\theta \int_\tau^s \Big(\sqrt{2\pi}\exp\Big(\frac{\Phi^{-1}(\sigma)^2}{2}\Big)\Big)^{p-q} \\
\times\big(-v_1^{\star\prime}(\sigma)\big)^{q-p+1} d\sigma\Big] d\tau.
\end{aligned} \tag{5.4}$$

It follows from (5.2) that

$$-v_1^{\star\prime}(s) = \Big(\sqrt{2\pi}\exp\Big(\frac{\Phi^{-1}(s)^2}{2}\Big)\Big)^{p'} w_1^{\frac{1}{p-1}}(s), \quad \text{a.e. } s \in (0, \gamma_n(\Omega)). \tag{5.5}$$

Furthermore, a simple computation shows that

$$w_1'(s) = f^\star(s) + \theta\Big(\sqrt{2\pi}\exp\Big(\frac{\Phi^{-1}(s)^2}{2}\Big)\Big)^{\gamma} w_1^\gamma, \quad \text{a.e. } s \in (0, \gamma_n(\Omega)),$$

which gives $w_1 = Kw_1$. By (5.4), (2.2) and Hölder inequality,

$$\begin{aligned}
w_1(s) &\le \int_0^s f^\star(\tau) \exp\Big[\theta\big(\sqrt{2\pi}\beta^{-1}\big)^{p-q} \Big(\int_\tau^s \frac{1}{\sigma(1-\log\sigma)^{1/2}} d\sigma\Big)^{p-q} \\
&\quad \times \Big(-\int_\tau^s v_1^{\star\prime}(\sigma) d\sigma\Big)^{q-p+1}\Big] d\tau \\
&\le \int_0^s f^\star(\tau) \exp\Big[\theta\big(\sqrt{2\pi}\beta^{-1}\big)^{p-q} \|v_1\|_{L^\infty(\Omega^\sharp)}^{q-p+1} \Big(\int_\tau^s \frac{1}{\sigma(1-\log\sigma)^{1/2}} d\sigma\Big)^{p-q}\Big] d\tau \\
&\le \int_0^s f^\star(\tau) \exp\Big[C_1 \Big(\int_\tau^s \sigma^{-1}(1-\log\sigma)^{-\frac{3}{2}} d\sigma\Big)^{\frac{p-q}{3}} \Big(\int_\tau^s \sigma^{-1} d\sigma\Big)^{\frac{2(p-q)}{3}}\Big] d\tau,
\end{aligned}$$

where $C_1 = \theta\big(\sqrt{2\pi}\beta^{-1}\big)^{p-q}\|v_1\|^{q-p+1}_{L^\infty(\Omega^\sharp)}$. Since

$$\int_\tau^s \sigma^{-1}(1-\log\sigma)^{-\frac{3}{2}}d\sigma \le 2(1-\log\gamma_n(\Omega))^{-1/2},$$

by Young's inequality with ε, the above inequality becomes

$$\begin{aligned}
w_1(s) &\le \int_0^s f^\star(\tau)\exp\Big[C_2\Big(\int_\tau^s \sigma^{-1}d\sigma\Big)^{\frac{2(p-q)}{3}}\Big]d\tau \\
&\le \int_0^s f^\star(\tau)\exp\Big(C_{1\varepsilon}C_2 + \varepsilon C_2\int_\tau^s \sigma^{-1}d\sigma\Big)d\tau \\
&= C_{2\varepsilon}\int_0^s f^\star(\tau)(\frac{s}{\tau})^{\varepsilon C_2}d\tau,
\end{aligned}$$

where

$$C_2 = \theta\big(\sqrt{2\pi}\beta^{-1}\big)^{p-q}\|v_1\|^{q-p+1}_{L^\infty(\Omega^\sharp)}\big(2(1-\log\gamma_n(\Omega))^{-1/2}\big)^{\frac{p-q}{3}},$$

$C_{1\varepsilon}$ and $C_{2\varepsilon}$ are positive constants depending on ε. Take ε small enough such that $\varepsilon C_2 < 1$. Then

$$0 \le w_1(s) \le C_{3\varepsilon}\|f\|_{L^\infty(\Omega)}s.$$

Thus, $w_1 \in D_2(K)$ satisfying $w_1 = Kw_1$. By Lemma 4.3, we know $w_1 = w$. Noting $v_1^\star(\gamma_n(\Omega)) = 0$, from (5.5) we have

$$v_1^\star(s) = \int_s^{\gamma_n(\Omega)}\Big(\sqrt{2\pi}\exp\Big(\frac{\Phi^{-1}(\tau)^2}{2}\Big)\Big)^{p'}w^{\frac{1}{p-1}}(\tau)d\tau.$$

Hence $v_1 = v$ and then the uniqueness is proved.

Remark 5.2. *From the above proof, we find that $v^\star$ and w can be expressed by each other via the following equations:*

$$v^\star(s) = \int_s^{\gamma_n(\Omega)}\Big(\sqrt{2\pi}\exp\Big(\frac{\Phi^{-1}(\tau)^2}{2}\Big)\Big)^{p'}w^{\frac{1}{p-1}}(\tau)d\tau \tag{5.6}$$

and

$$w(s) = \int_0^s f^\star(\tau)\exp\Big[\theta\int_\tau^s\Big(\sqrt{2\pi}\exp\Big(\frac{\Phi^{-1}(\sigma)^2}{2}\Big)\Big)^{p-q}\big(-{v^\star}'(\sigma)\big)^{q-p+1}d\sigma\Big]d\tau. \tag{5.7}$$

Proof of Theorem 3.4. Assume that there exists a symmetric solution v to (1.2). From Remark 5.2, we see that w defined by (5.7) is a solution of the equation

$$z = Kz, \quad z \in D_1(K),$$

which contradict with Lemma 4.4. Thus the proof is complete. $\square$

Proof of Theorem 3.6.

Step 1. First we verify (3.5). Take

$$\rho(s) = \int_0^s f^\star(\tau)\exp\Big[\theta\int_\tau^s\Big(\sqrt{2\pi}\exp\Big(\frac{\Phi^{-1}(\sigma)^2}{2}\Big)\Big)^{p-q}\big(-{u^\star}'(\sigma)\big)^{q-p+1}d\sigma\Big]d\tau.$$

Then $\rho \in D_2(K)$ and $\rho(s) \le K\rho(s)$ for $s \in [0,\gamma_n(\Omega)]$. Now remembering that K has property (m) (see the proof of Lemma 4.3) and $w = Kw$ in $D_2(K)$, by Lemma 4.2 we obtain

$$\rho(s) \le w(s), \quad s \in [0,\gamma_n(\Omega)].$$

Moreover, (5.1) and (5.2) imply that

$$\big(-{u^{\star}}'(s)\big)^{p-1}\Big(\frac{1}{\sqrt{2\pi}}\exp\Big(-\frac{\Phi^{-1}(s)^2}{2}\Big)\Big)^p \le \rho(s) \quad \text{a.e. } s\in(0,\gamma_n(\Omega))$$

and

$$\big(-{v^{\star}}'(s)\big)^{p-1}\Big(\frac{1}{\sqrt{2\pi}}\exp\Big(-\frac{\Phi^{-1}(s)^2}{2}\Big)\Big)^p = w(s) \quad \text{a.e. } s\in(0,\gamma_n(\Omega)).$$

Thus, $-{u^{\star}}'(s)\le -{v^{\star}}'(s)$, a.e. $s\in(0,\gamma_n(\Omega))$. Note that $u^\star(\gamma_n(\Omega))=v^\star(\gamma_n(\Omega))=0$. We have

$$u^\star(s)\le v^\star(s), s\in[0,\gamma_n(\Omega)].$$

The proof of (3.6) can be done by repeating the proof of [14, Theorem 4.1].

Step 2. The equality case of (3.5) will be studied. Sufficiency is obvious. Let us prove the necessity. Assume that $u^\star(s)=v^\star(s)$ for $s\in[0,\gamma_n(\Omega)]$. By Polya-Szëgo principle and (3.6), we have

$$\int_{\Omega^\sharp}|\nabla v|^p\varphi\;dx=\int_{\Omega}|\nabla u^\sharp|^p\varphi\,dx\le\int_{\Omega}|\nabla u|^p\varphi\;dx\le\int_{\Omega^\sharp}|\nabla v|^p\varphi\,dx.$$

Thus,

$$\int_{\Omega^\sharp}|\nabla u^\sharp|^p\varphi\,dx=\int_{\Omega}|\nabla u|^p\varphi\,dx$$

and then $\Omega=\Omega^\sharp$ and $|u|=u^\sharp$ modulo a rotation, which implies $u=\delta u^\sharp$ and $\Omega^\sharp$ modulo a rotation with $\delta=\pm1$. Now for $h>0$, we take

$$\psi(x)=\begin{cases}\operatorname{sign}u(x) & \text{if } |u(x)|>t+h,\\ \frac{(|u(x)|-t)\operatorname{sign}u(x)}{h} & \text{if } t<|u(x)|\le t+h,\\ 0 & \text{otherwise}\end{cases}$$

as the test function in (3.2) and let $h\to0$. Since $\delta u=|u|=u^\sharp=v^\sharp=v$, we have by Hardy-Littlewood inequality that

$$\begin{aligned}
&-\frac{d}{dt}\int_{u^\sharp>t}|\nabla u^\sharp|^p\varphi\,dx\\
&=-\frac{d}{dt}\int_{|u|>t}|\nabla u|^p\varphi\,dx\le-\frac{d}{dt}\int_{|u|>t}a(x,u,\nabla u)\nabla u\,dx\\
&=\int_{|u|>t}H(x,u,\nabla u)udx\le\int_{|u|>t}fu\varphi\,dx+\theta\int_{|u|>t}|\nabla u|^q u\varphi\,dx\\
&\le\int_{u^\sharp>t}f^\sharp u^\sharp\varphi\,dx+\theta\int_{u^\sharp>t}|\nabla u^\sharp|^q u^\sharp\varphi\,dx\\
&=\int_{v>t}f^\sharp v\varphi\,dx+\theta\int_{v>t}|\nabla v|^q v\varphi\,dx\\
&=-\frac{d}{dt}\int_{v>t}|\nabla v|^p\varphi\,dx=-\frac{d}{dt}\int_{u^\sharp>t}|\nabla u^\sharp|^p\varphi\,dx.
\end{aligned}\tag{5.8}$$

Thus, the above equalities hold. In particular,

$$-\frac{d}{dt}\int_{|u|>t}a(x,u,\nabla u)\nabla u\,dx=-\frac{d}{dt}\int_{u^\sharp>t}|\nabla u^\sharp|^p\varphi\,dx,\tag{5.9}$$

$$\int_{|u|>t}H(x,u,\nabla u)udx=\int_{u^\sharp>t}f^\sharp u^\sharp\varphi\,dx+\theta\int_{u^\sharp>t}|\nabla u^\sharp|^q u^\sharp\varphi\,dx,\tag{5.10}$$

$$\int_{u^\sharp>t} fu^\sharp\varphi\,dx=\int_{u^\sharp>t} f^\sharp u^\sharp\varphi\,dx. \tag{5.11}$$

By [25, Lemma 4.3], from (5.11) on has $f=f^\sharp$ a.e. $x\in\Omega^\sharp$. Then combine (5.10) and assumption (iii) to discover that

$$H(x,u,\nabla u)=\delta(f^\sharp\varphi+\theta|D_1u^\sharp|^q\varphi),\quad \text{a.e. } x\in\Omega^\sharp$$

Similarly, from (5.9) and assumption (i) we have

$$\delta a(x,u,\nabla u)\nabla u^\sharp=|\nabla u^\sharp|^p\varphi.$$

Recalling (5.6), we have

$$\begin{gathered}
D_1u^\sharp(x)=D_1v^\sharp(x)=\Big(\sqrt{2\pi}\exp\big(\frac{x_1^2}{2}\big)\Big)^{\frac{1}{p-1}}w^{\frac{1}{p-1}}(\Phi(x_1))>0,\quad \text{a.e. } x\in\Omega^\sharp,\\
D_iu^\sharp(x)=0,\quad i=2,3,\dots,n.
\end{gathered}$$

Hence,

$$a_1(x,u,\nabla u)=\delta|D_1u^\sharp|^{p-2}D_1u^\sharp\varphi,\quad \text{a.e. } x\in\Omega^\sharp.$$

From the definition of solution it follows that for all $\psi\in W_0^{1,p}(\varphi,\Omega^\sharp)\cap L^\infty(\Omega^\sharp)$,

$$\begin{aligned}
&\int_{\Omega^\sharp}\delta|D_1u^\sharp|^{p-2}D_1u^\sharp D_1\psi\varphi\,dx+\int_{\Omega^\sharp}\sum_{i=2}^n a_i(x,u,\nabla u)D_i\psi\,dx\\
&=\int_\Omega a(x,u,\nabla u)\nabla\psi\,dx=\int_\Omega H(x,u,\nabla u)\psi\,dx\\
&=\int_{\Omega^\sharp}\delta(f^\sharp\varphi+\theta|D_1u^\sharp|^q\varphi)\psi\,dx=\int_{\Omega^\sharp}\delta(f^\sharp\varphi+\theta|D_1v|^q\varphi)\psi\,dx\\
&=\int_{\Omega^\sharp}\delta|D_1v|^{p-2}D_1vD_1\psi\varphi\,dx=\int_{\Omega^\sharp}\delta|D_1u^\sharp|^{p-2}D_1u^\sharp D_1\psi\varphi\,dx.
\end{aligned}$$

Then

$$\int_{\Omega^\sharp}\sum_{i=2}^n a_i(x,u,\nabla u)D_i\phi\,dx=0,\quad \forall\psi\in W_0^{1,p}(\varphi,\Omega^\sharp)\cap L^\infty(\Omega^\sharp),$$

which completes the proof. □

6. Appendix

In this section give an example to Remark 3.2. Assume that v is a solution of the problem

$$\begin{gathered}
-D_1(\varphi|D_1v|^{p-2}D_1v)=\varphi+\theta|D_1v|^q\varphi\quad \text{in } \Omega^\sharp,\\
v\in W_0^{1,p}(\varphi,\Omega^\sharp)\cap L^\infty(\Omega^\sharp)
\end{gathered}\tag{6.1}$$

and Y is the solution of the problem

$$\begin{gathered}
-D_1(\varphi|D_1Y|^{p-2}D_1Y)=\varphi\quad \text{in } \Omega^\sharp,\\
Y=0\quad \text{on } \partial\Omega^\sharp.
\end{gathered}\tag{6.2}$$

For any given $k>0$, let

$$T_k(s)=\begin{cases} k & \text{if } s>k,\\ s & \text{if } |s|\le k,\\ -k & \text{if } s<-k.\end{cases}$$

Take $T_k((v-Y)^-)$ as the test function in (6.1) and (6.2), and subtract the two results. We obtain

$$0 \geq \int_{\{0<(v-Y)^- \leq k\}} \varphi(|D_1 v|^{p-2} D_1 v - |D_1 Y|^{p-2} D_1 Y)(Y-v)dx \geq 0.$$

Thus, $\gamma_n(\{0 < (v-Y)^- \leq k\}) = 0$. Let $k \to +\infty$. Then $\gamma_n(\{(v-Y)^- > 0\}) = 0$, which implies $v \geq Y$ a.e. in $\Omega^\sharp$. However, if $p \geq 2$,

$$\begin{aligned}
\|v\|_{L^\infty(\Omega^\sharp)} &\geq \|Y\|_{L^\infty(\Omega^\sharp)} = Y^\star(0) \\
&= \int_0^{\gamma_n(\Omega)} \Big(\sqrt{2\pi}\exp\Big(\frac{\Phi^{-1}(\tau)^2}{2}\Big)\Big)^{p'} \tau^{\frac{1}{p-1}} d\tau \\
&\geq C\int_0^{\gamma_n(\Omega)} \Big(\frac{1}{\tau(1-\log\tau)^{1/2}}\Big)^{p'} \tau^{\frac{1}{p-1}} d\tau \\
&= C\int_0^{\gamma_n(\Omega)} (1-\log\tau)^{-\frac{p'}{2}} \tau^{-1} d\tau = +\infty.
\end{aligned}$$

This contradicts with $v \in L^\infty(\Omega^\sharp)$. As $p \geq 2$, problem (6.1) has no solution.

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