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FIRST-ORDER PRODUCT-TYPE SYSTEMS OF DIFFERENCE EQUATIONS SOLVABLE IN CLOSED FORM

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ABSTRACT. We show that the first-order system of difference equations

$$z_{n+1} = \alpha z_n^a w_n^b, \quad w_{n+1} = \beta z_n^c w_n^d, \quad n \in \mathbb{N}_0,$$

where $a, b, c, d \in \mathbb{Z}$, $\alpha, \beta \in \mathbb{C} \setminus \{0\}$, $z_0, w_0 \in \mathbb{C} \setminus \{0\}$, is solvable in closed form, by finding closed form formulas of its solutions.

1. Introduction

The study of nonlinear difference equations and systems is of a great recent interest (see, for example, [1]-[6], [8], [10]-[23]). The classical area of solving difference equations and systems has re-attracted a quite recent attention (see, for example, [1]-[3], [6], [12], [18]-[21], [23] and the related references therein). Our recent idea of transforming some complicated difference equations and systems into simpler solvable ones, used for the first time in explaining the solvability of the equation appearing in [6], has been employed recently in several papers (see, for example, [1, 3, 12, 18, 21, 23] and the references therein). Another area of some recent interest, essentially initiated by Papaschinopoulos and Schinas, is studying symmetric and close to symmetric systems of difference equations (see, for example, [3, 5, 10, 11, 15, 16], [19]-[23]). Our important observation in some of above quoted papers on solvability of difference equations was that suitable changes of variables transform relatively complicated equations considered therein into special cases of the linear first-order difference equation

$$x_n = a_n x_{n-1} + b_n, \quad n \in \mathbb{N}, \tag{1.1}$$

which is a basic solvable difference equation (for a nice presentation of some methods for solving equation (1.1) and some related ones see, for example, monograph [9]). This was also essentially the case with some of the equations in [1, 12, 18]. Actually, in this or that way, many equations and systems are related to equation (1.1) or to the corresponding difference inequalities or to the corresponding system of linear difference equations (for example, some of the equations, inequalities and systems in [3]-[5], [8, 9, 21, 23] are of this type). For some results on general theory of difference equations and systems or on some other types of results on various classes

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of difference equations and systems see, for example, [4, 5], [7]-[9], [13, 14], and the references therein.

On the other hand, the present author also essentially triggered a systematic study of non-rational concrete difference equations and systems, from one side those obtained by the scalar translation operator (see, for example, [17] and the references therein) and from the other side those obtained by using some max-type operators (see, for example, [22]). It can be noticed that behavior of only positive solutions of the difference equations and systems in [17, 22], are investigated. As we have mentioned in [20], in [22] was studied the boundedness character of positive solutions to the system

$$x_{n+1} = \max\Big\{a, \frac{y_n^p}{x_{n-1}^q}\Big\}, \quad y_{n+1} = \max\Big\{a, \frac{x_n^p}{y_{n-1}^q}\Big\}, \quad n \in \mathbb{N}_0,$$

with $\min\{a, p, q\} > 0$, which is obtained from the product-type one

$$x_{n+1} = \frac{y_n^p}{x_{n-1}^q}, \quad y_{n+1} = \frac{x_n^p}{y_{n-1}^q}, \quad n \in \mathbb{N}_0, \tag{1.2}$$

by acting with a max-type operator onto the right-hand sides of both equations in (1.2) ([17] deals with a related scalar equation). If initial values of system (1.2) are positive, then it can be solved by taking the logarithm to the both sides of both equations (this transforms the system to a solvable linear second-order system of difference equations, whose general theory can be found in [7]). However, the method is not possible if initial values are not positive, due to the fact that the logarithm of a complex number is not uniquely defined. Another reason for a detailed study of product-type difference equations and systems is found in the fact that behavior of their solutions are not so rarely related to the ones of the equations and systems obtained from them by acting with the translation, max-type or some other natural operators.

These observations lead us to the investigation of some product-type difference equations and systems with real and/or complex initial values. Namely, in [19] and [20], the present author and his collaborators started studying such systems by modifying methods and ideas from above mentioned papers, not only those on solving difference equations and systems, but also using some ideas on non-rational difference equations and systems appearing, for example, in [17] and [22].

In this article we continue our investigation of solvability of nonlinear difference equations and systems by studying the solvability of the following product-type system of difference equations

$$z_{n+1} = \alpha z_n^a w_n^b, \quad w_{n+1} = \beta z_n^c w_n^d, \quad n \in \mathbb{N}_0, \tag{1.3}$$

where $a, b, c, d \in \mathbb{Z}$, $\alpha, \beta \in \mathbb{C}$ and initial values $z_0, w_0 \in \mathbb{C}$.

The reason why the parameters a, b, c and d are chosen to be integers is that solutions to system (1.3) are uniquely defined.

Note that if any of numbers a, b, c, d is a negative integer then the domain of undefinable solutions to system (1.3) is a subset of the set

$$\mathcal{U} = \{(z_0, w_0) \in \mathbb{C}^2 : z_0 = 0 \text{ or } w_0 = 0\}.$$

Otherwise, if additionally $\alpha, \beta \in \mathbb{C} \setminus \{0\}$, system (1.3) is defined on the whole complex space $\mathbb{C}^2$. Hence, from now on we will assume that our initial values belong to the set $\mathbb{C}^2 \setminus \mathcal{U}$.

Throughout the paper we use the following standard convention $\sum_{j=k}^{k-1} a_j = 0$, if $k \in \mathbb{Z}$.

2. Main result

In this section we formulate and prove the main results of this paper. Before this we quote two lemmas which will be frequently used in the rest of the paper. The following lemma was essentially proved in [20], so we will omit its proof.

Lemma 2.1. *Let $\eta \in \mathbb{C} \setminus \{0\}$, $f \in \mathbb{Z} \setminus \{0\}$ and $u_0 \in \mathbb{C} \setminus \{0\}$. Then the difference equation*

$$u_n = u_{n-1}^f \eta, \quad n \in \mathbb{N}, \tag{2.1}$$

is solvable and

$$u_n = u_0^{f^n} \eta^{\sum_{j=0}^{n-1} f^j}, \quad n \in \mathbb{N}_0. \tag{2.2}$$

Remark 2.2. *Note that if $\eta = 0$, then from (2.1) we have that $u_n = 0$, $n \in \mathbb{N}$, while if $f = 0$ then $u_n = \eta$, $n \in \mathbb{N}$. If $u_0 = 0$, then $u_n = 0$, $n \in \mathbb{N}$, if $f \in \mathbb{N}$, while if $f < 0$ then such a solution is not defined.*

The following elementary lemma is well-know (see, e.g., [9]).

Lemma 2.3. *Let $S_k(z) = 1 + 2z + 3z^2 + \cdots + kz^{k-1}$. Then*

$$S_k(z) = \frac{1 - (k+1)z^k + kz^{k+1}}{(1-z)^2},$$

for $z \in \mathbb{C} \setminus \{1\}$.

Now, first note that if $\alpha = 0$, then from the first equation in (1.3) we have $z_n = 0$, $n \in \mathbb{N}$, from which along with the second equation in (1.3) it follows that $w_n = 0$ for $n \geq 2$, if $c > 0$, while if $c = 0$ we have

$$w_n = w_{n-1}^d \beta, \quad n \in \mathbb{N},$$

so by Lemma 2.1, if $\beta \neq 0$ and $w_0 \neq 0$, we have

$$w_n = w_0^{d^n} \beta^{\frac{1-d^n}{1-d}},$$

if $d \neq 1$, and

$$w_n = w_0 \beta^n, \quad n \in \mathbb{N}_0,$$

if $d = 1$.

Similarly, if $\beta = 0$, then from the second equation in (1.3) we have $w_n = 0$, $n \in \mathbb{N}$, from which along with the first equation in (1.3) it follows that $z_n = 0$ for $n \geq 2$, if $b > 0$, while if $b = 0$ we have

$$z_n = z_{n-1}^a \alpha, \quad n \in \mathbb{N},$$

so by Lemma 2.1, if $\alpha \neq 0 \neq z_0$ we have that

$$z_n = z_0^{a^n} \alpha^{\frac{1-a^n}{1-a}},$$

if $a \neq 1$, and

$$z_n = z_0 \alpha^n, \quad n \in \mathbb{N}_0,$$

if $a = 1$. Hence, from now on we may also assume that $\alpha \neq 0 \neq \beta$.

Our first result deals with the case when all the parameters a, b, c and d are integers different from zero.

Theorem 2.4. *Assume that* $a, b, c, d \in \mathbb{Z} \setminus \{0\}$, $\alpha, \beta \in \mathbb{C} \setminus \{0\}$, *and* $z_0, w_0 \in \mathbb{C} \setminus \{0\}$. *Then system* (1.3) *is solvable in closed form.*

Proof. First note that the assumption $\alpha, \beta, z_0, w_0 \in \mathbb{C} \setminus \{0\}$ along with a simple inductive argument shows that all such solutions are well-defined.

From the first equation in (1.3), we have that for every well-defined solution of the system

$$w_n^b = \alpha^{-1} z_{n+1} z_n^{-a}, \quad n \in \mathbb{N}_0. \tag{2.3}$$

Taking the second equation in (1.3) to the power b (the condition $b \neq 0$ is essential here), we obtain

$$w_{n+1}^b = \beta^b z_n^{bc} w_n^{bd}, \quad n \in \mathbb{N}_0. \tag{2.4}$$

Employing equality (2.3) into (2.4), we obtain

$$\alpha^{-1} z_{n+2} z_{n+1}^{-a} = \beta^b z_n^{bc} \alpha^{-d} z_{n+1}^d z_n^{-ad}, \quad n \in \mathbb{N}_0,$$

from which it follows that

$$z_{n+2} = z_{n+1}^{a+d} z_n^{bc-ad} \alpha^{1-d} \beta^b, \quad n \in \mathbb{N}_0. \tag{2.5}$$

Note also that

$$z_0 \in \mathbb{C} \setminus \{0\}, \quad z_1 = z_0^a w_0^b \alpha. \tag{2.6}$$

Let $\gamma = \alpha^{1-d} \beta^b$ and

$$a_1 = a + d, \quad b_1 = bc - ad, \quad c_1 = 1. \tag{2.7}$$

Then equation (2.5) can be written in the following form

$$z_n = z_{n-1}^{a_1} z_{n-2}^{b_1} \gamma^{c_1}, \quad n \geq 2. \tag{2.8}$$

By using the equality

$$z_{n-1} = z_{n-2}^{a_1} z_{n-3}^{b_1} \gamma^{c_1}, \quad n \geq 3,$$

in (2.8), it follows that

$$\begin{aligned} z_n &= (z_{n-2}^{a_1} z_{n-3}^{b_1} \gamma^{c_1})^{a_1} z_{n-2}^{b_1} \gamma^{c_1} \\ &= z_{n-2}^{a_1 a_1 + b_1} z_{n-3}^{a_1 b_1} \gamma^{a_1 c_1 + c_1} \\ &= z_{n-2}^{a_2} z_{n-3}^{b_2} \gamma^{c_2}, \end{aligned} \tag{2.9}$$

for $n \geq 3$, where

$$a_2 := a_1 a_1 + b_1, \quad b_2 := a_1 b_1, \quad c_2 := a_1 c_1 + c_1. \tag{2.10}$$

If we use the equality

$$z_{n-2} = z_{n-3}^{a_1} z_{n-4}^{b_1} \gamma^{c_1}, \quad n \geq 4,$$

in (2.9), we obtain

$$z_n = (z_{n-3}^{a_1} z_{n-4}^{b_1} \gamma^{c_1})^{a_2} z_{n-3}^{b_2} \gamma^{c_2} = z_{n-3}^{a_1 a_2 + b_2} z_{n-4}^{b_1 a_2} \gamma^{c_1 a_2 + c_2} = z_{n-3}^{a_3} z_{n-4}^{b_3} \gamma^{c_3}, \tag{2.11}$$

for $n \geq 4$, where

$$a_3 := a_1 a_2 + b_2, \quad b_3 := b_1 a_2, \quad c_3 := c_1 a_2 + c_2. \tag{2.12}$$

Now, assume that the following equality was proven,

$$z_n = z_{n-k}^{a_k} z_{n-k-1}^{b_k} \gamma^{c_k}, \tag{2.13}$$

for some $k \in \mathbb{N}$ such that $n \geq k + 1$, and that

$$a_k = a_1 a_{k-1} + b_{k-1}, \quad b_k = b_1 a_{k-1}, \quad c_k = c_1 a_{k-1} + c_{k-1}. \tag{2.14}$$

Then, employing the equality

$$z_{n-k} = z_{n-k-1}^{a_1} z_{n-k-2}^{b_1} \gamma^{c_1},$$

for $n \geq k+2$, into (2.13) we obtain

$$\begin{aligned} z_n &= (z_{n-k-1}^{a_1} z_{n-k-2}^{b_1} \gamma^{c_1})^{a_k} z_{n-k-1}^{b_k} \gamma^{c_k} \\ &= z_{n-k-1}^{a_1 a_k + b_k} z_{n-k-2}^{b_1 a_k} \gamma^{c_1 a_k + c_k} \\ &= z_{n-k-1}^{a_{k+1}} z_{n-k-2}^{b_{k+1}} \gamma^{c_{k+1}}, \end{aligned} \tag{2.15}$$

for $n \geq k+2$, where

$$a_{k+1} := a_1 a_k + b_k, \quad b_{k+1} := b_1 a_k, \quad c_{k+1} := c_1 a_k + c_k. \tag{2.16}$$

From (2.9), (2.10), (2.15), (2.16) and by using the method of induction it follows that (2.13) and (2.14) hold for all natural numbers k and n such that $2 \leq k \leq n-1$.

Plugging $k = n-1$ into (2.13), we obtain

$$\begin{aligned} z_n &= z_1^{a_{n-1}} z_0^{b_{n-1}} \gamma^{c_{n-1}} && (2.17) \\ &= \big(z_0^a w_0^b \alpha\big)^{a_{n-1}} z_0^{b_{n-1}} \gamma^{c_{n-1}} \\ &= z_0^{a a_{n-1} + b_{n-1}} w_0^{b a_{n-1}} \alpha^{a_{n-1}} \gamma^{c_{n-1}} \\ &= z_0^{a a_{n-1} + b_{n-1}} w_0^{b a_{n-1}} \alpha^{a_{n-1} + (1-d) c_{n-1}} \beta^{b c_{n-1}}, \quad n \in \mathbb{N}_0. && (2.18) \end{aligned}$$

From the first two equations in (2.14) we see that

$$a_k = a_1 a_{k-1} + b_1 a_{k-2}, \quad k \geq 3,$$

which by (2.7) can be rewritten as

$$a_{k+2} - (a+d) a_{k+1} + (ad - bc) a_k = 0, \quad k \in \mathbb{N}. \tag{2.19}$$

Case $ad \neq bc$. To calculate a_k more easily in this case, note that from the first two relations in (2.16) with $k = 0$, we have

$$a_1 = a_1 a_0 + b_0, \quad b_1 = b_1 a_0,$$

from which along with the assumption $b_1 = bc - ad \neq 0$, it follows that

$$a_0 = 1, \quad b_0 = 0. \tag{2.20}$$

The characteristic polynomial associated to equation (2.19) is

$$P(\lambda) = \lambda^2 - (a+d)\lambda + ad - bc,$$

from which it follows that the characteristic values are

$$\lambda_{1,2} = \frac{a + d \pm \sqrt{(a+d)^2 - 4(ad - bc)}}{2},$$

which implies that

$$a_k = \hat{c}_1 \lambda_1^k + \hat{c}_2 \lambda_2^k, \quad k \in \mathbb{N}_0,$$

for some constants $\hat{c}_1$ and $\hat{c}_2$, if $\Delta := (a+d)^2 - 4(ad - bc) \neq 0$, while

$$a_k = (\hat{c}_3 + k \hat{c}_4) \lambda_1^k, \quad k \in \mathbb{N}_0,$$

for some constants $\hat{c}_3$ and $\hat{c}_4$, if $\Delta = 0$ (in this case $\lambda_1 = \lambda_2 = (a+d)/2$).

Using the initial conditions $a_0 = 1$ and $a_1 = a + d$, after some calculations, it is not difficult to see that

$$a_k = \frac{\lambda_1^{k+1} - \lambda_2^{k+1}}{\lambda_1 - \lambda_2}, \quad k \in \mathbb{N}_0, \tag{2.21}$$

when $\Delta \neq 0$, while

$$a_k = (k+1)\lambda_1^k, \quad k \in \mathbb{N}_0, \tag{2.22}$$

when $\Delta = 0$.

From (2.21), (2.22) and the second equation in (2.14), we have

$$b_k = (bc - ad)\frac{\lambda_1^k - \lambda_2^k}{\lambda_1 - \lambda_2}, \quad k \in \mathbb{N}_0, \tag{2.23}$$

when $\Delta \neq 0$, while

$$b_k = (bc - ad)k\lambda_1^{k-1}, \quad k \in \mathbb{N}_0, \tag{2.24}$$

when $\Delta = 0$.

From the third equation in (2.14) and since $c_1 = 1$, we obtain

$$c_k = c_{k-1} + a_{k-1}, \quad k \geq 2,$$

from which it follows that

$$c_k = 1 + \sum_{j=1}^{k-1} a_j, \quad k \geq 2. \tag{2.25}$$

Using (2.21) in (2.25) we have

$$c_k = \frac{(\lambda_2 - 1)(\lambda_1^{k+1} - 1) - (\lambda_1 - 1)(\lambda_2^{k+1} - 1)}{(\lambda_1 - \lambda_2)(\lambda_1 - 1)(\lambda_2 - 1)}, \quad k \in \mathbb{N}, \tag{2.26}$$

if $\Delta \neq 0$, while by Lemma 2.1

$$c_k = \frac{1 - (k+1)\lambda_1^k + k\lambda_1^{k+1}}{(1 - \lambda_1)^2}, \quad k \in \mathbb{N}, \tag{2.27}$$

if $\Delta = 0$.

Hence, if $\Delta \neq 0$ using (2.21), (2.23) and (2.26) in (2.18), we obtain

$$\begin{aligned} z_n = {} & z_0^{a\frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2} + (bc-ad)\frac{\lambda_1^{n-1} - \lambda_2^{n-1}}{\lambda_1 - \lambda_2}} w_0^{b\frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2}} \\ & \times \alpha^{\frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2} + (1-d)\left(\frac{(\lambda_2 - 1)(\lambda_1^n - 1) - (\lambda_1 - 1)(\lambda_2^n - 1)}{(\lambda_1 - \lambda_2)(\lambda_1 - 1)(\lambda_2 - 1)}\right)} \\ & \times \beta^{b\frac{(\lambda_2 - 1)(\lambda_1^n - 1) - (\lambda_1 - 1)(\lambda_2^n - 1)}{(\lambda_1 - \lambda_2)(\lambda_1 - 1)(\lambda_2 - 1)}}, \end{aligned} \tag{2.28}$$

for $n \in \mathbb{N}_0$, while if $\Delta = 0$ using (2.22), (2.24) and (2.27) in (2.18), we obtain

$$\begin{aligned} z_n = {} & z_0^{an\lambda_1^{n-1} + (bc-ad)(n-1)\lambda_1^{n-2}} w_0^{bn\lambda_1^{n-1}} \\ & \times \alpha^{n\lambda_1^{n-1} + (1-d)\frac{1 - n\lambda_1^{n-1} + (n-1)\lambda_1^n}{(1-\lambda_1)^2}} \beta^{b\frac{1 - n\lambda_1^{n-1} + (n-1)\lambda_1^n}{(1-\lambda_1)^2}}, \end{aligned} \tag{2.29}$$

for $n \in \mathbb{N}_0$.

On the other hand, from the second equation in (1.3), we have that for every well-defined solution of the system

$$z_n^c = \beta^{-1} w_n^{-d} w_{n+1}, \quad n \in \mathbb{N}_0. \tag{2.30}$$

Taking the first equation in (1.3) to the cth power (the condition $c \neq 0$ is essential here), we obtain

$$z_{n+1}^c = z_n^{ac} w_n^{bc} \alpha^c, \quad n \in \mathbb{N}_0. \tag{2.31}$$

Employing (2.30) in (2.31), we obtain

$$\beta^{-1} w_{n+1}^{-d} w_{n+2} = \beta^{-a} w_n^{-ad} w_{n+1}^a w_n^{bc} \alpha^c, \quad n \in \mathbb{N}_0,$$

which can be written as

$$w_{n+2} = w_{n+1}^{a+d} w_n^{bc-ad} \beta^{1-a} \alpha^c, \quad n \in \mathbb{N}_0. \tag{2.32}$$

Let $\delta = \beta^{1-a}\alpha^c$ and a_1, b_1 and c_1 are given by (2.7). Then (2.32) can be written as

$$w_{n+2} = w_{n+1}^{a_1} w_n^{b_1} \delta^{c_1}, \quad n \in \mathbb{N}_0. \tag{2.33}$$

Note also that

$$w_0 \in \mathbb{C} \setminus \{0\}, \quad w_1 = z_0^c w_0^d \beta. \tag{2.34}$$

Since (2.33) has the same form as equation (2.8), where only γ is replaced by δ, we have that the recurrent relations in (2.14) hold, and consequently formulas (2.21)-(2.24), (2.26) and (2.27). Also we have that (2.17) holds with γ replaced by δ, and z replaced by w. From this, by using (2.34) and the definition of δ, we have

$$\begin{aligned} w_n &= w_1^{a_{n-1}} w_0^{b_{n-1}} \delta^{c_{n-1}} \\ &= (z_0^c w_0^d \beta)^{a_{n-1}} w_0^{b_{n-1}} \delta^{c_{n-1}} \\ &= z_0^{ca_{n-1}} w_0^{da_{n-1}+b_{n-1}} \beta^{a_{n-1}+(1-a)c_{n-1}} \alpha^{cc_{n-1}}, \quad n \in \mathbb{N}_0. \end{aligned} \tag{2.35}$$

Thus, if $\Delta \neq 0$ using (2.21), (2.23) and (2.26) in (2.35), we obtain

$$\begin{aligned} w_n &= z_0^{c\frac{\lambda_1^n-\lambda_2^n}{\lambda_1-\lambda_2}} w_0^{d\frac{\lambda_1^n-\lambda_2^n}{\lambda_1-\lambda_2}+(bc-ad)\frac{\lambda_1^{n-1}-\lambda_2^{n-1}}{\lambda_1-\lambda_2}} \\ &\quad \times \beta^{\frac{\lambda_1^n-\lambda_2^n}{\lambda_1-\lambda_2}+(1-a)\frac{(\lambda_2-1)(\lambda_1^n-1)-(\lambda_1-1)(\lambda_2^n-1)}{(\lambda_1-\lambda_2)(\lambda_1-1)(\lambda_2-1)}} \\ &\quad \times \alpha^{c\frac{(\lambda_2-1)(\lambda_1^n-1)-(\lambda_1-1)(\lambda_2^n-1)}{(\lambda_1-\lambda_2)(\lambda_1-1)(\lambda_2-1)}}, \end{aligned} \tag{2.36}$$

for $n \in \mathbb{N}_0$, while if $\Delta = 0$ using (2.22), (2.24) and (2.27) in (2.35), we obtain

$$\begin{aligned} w_n &= z_0^{cn\lambda_1^{n-1}} w_0^{dn\lambda_1^{n-1}+(bc-ad)(n-1)\lambda_1^{n-2}} \\ &\quad \times \beta^{n\lambda_1^{n-1}+(1-a)\frac{1-n\lambda_1^{n-1}+(n-1)\lambda_1^n}{(1-\lambda_1)^2}} \alpha^{c\frac{1-n\lambda_1^{n-1}+(n-1)\lambda_1^n}{(1-\lambda_1)^2}}, \end{aligned} \tag{2.37}$$

for $n \in \mathbb{N}_0$.

Case $ad = bc$. Since $b \neq 0 \neq c$, we see that (2.5) and (2.32) hold, from which along with the assumption we have that

$$z_{n+1} = z_n^{a+d} \alpha^{1-d} \beta^b = z_n^{a_1} \gamma, \tag{2.38}$$

$$w_{n+1} = w_n^{a+d} \beta^{1-a} \alpha^c = w_n^{a_1} \delta, \tag{2.39}$$

for $n \in \mathbb{N}$. By Lemma 2.1 we have

$$z_n = z_1^{a_1^{n-1}} \gamma^{\sum_{j=0}^{n-2} a_1^j} = (\alpha z_0^a w_0^b)^{a_1^{n-1}} \gamma^{\sum_{j=0}^{n-2} a_1^j}, \tag{2.40}$$

for $n \in \mathbb{N}$. Hence, if $a_1 \neq 1$, then from (2.40) we obtain

$$\begin{aligned} z_n &= (\alpha z_0^a w_0^b)^{a_1^{n-1}} \gamma^{\frac{1-a_1^{n-1}}{1-a_1}} \\ &= z_0^{a(a+d)^{n-1}} w_0^{b(a+d)^{n-1}} \alpha^{\frac{1-d-a(a+d)^{n-1}}{1-a-d}} \beta^{b\frac{1-(a+d)^{n-1}}{1-a-d}}, \end{aligned} \tag{2.41}$$

for $n \in \mathbb{N}$, while if $a_1 = 1$, from (2.40) we obtain

$$z_n = \alpha z_0^a w_0^b \gamma^{n-1} = z_0^a w_0^b \alpha^{(1-d)n+d} \beta^{b(n-1)}, \tag{2.42}$$

for $n \in \mathbb{N}$.

On the other hand, from (2.39) and by using Lemma 2.1, we obtain

$$w_n = w_1^{a_1^{n-1}} \delta^{\sum_{j=0}^{n-2} a_1^j} = (\beta z_0^c w_0^d)^{a_1^{n-1}} \delta^{\sum_{j=0}^{n-2} a_1^j}, \tag{2.43}$$

for $n \in \mathbb{N}$. Hence, if $a_1 \neq 1$, then from (2.43) we obtain

$$\begin{aligned} w_n &= (\beta z_0^c w_0^d)^{a_1^{n-1}} \delta^{\frac{1-a_1^{n-1}}{1-a_1}} \\ &= z_0^{c(a+d)^{n-1}} w_0^{d(a+d)^{n-1}} \alpha^{c\frac{1-(a+d)^{n-1}}{1-a-d}} \beta^{\frac{1-a-d(a+d)^{n-1}}{1-a-d}}, \end{aligned} \tag{2.44}$$

for $n \in \mathbb{N}$, while if $a_1 = 1$, from (2.43) we obtain

$$w_n = \beta z_0^c w_0^d \delta^{n-1} = z_0^c w_0^d \alpha^{c(n-1)} \beta^{(1-a)n+a}, \tag{2.45}$$

for $n \in \mathbb{N}$.

It is easy to see that formulas (2.28) and (2.36) in the case $ad \neq bc$ and $\Delta \neq 0$, (2.29) and (2.37) in the case $ad \neq bc$ and $\Delta = 0$, (2.41) and (2.44) in the case $ad = bc$ and $a + d \neq 1$, and (2.42) and (2.45) in the case $ad = bc$ and $a + d = 1$, annihilate system (1.3). So, they are solutions to the system, and it is solvable indeed, as claimed. □

Remark 2.5. In the case $ad \neq bc$, this condition enables to prolong solutions to system (2.14) for every non-positive integer k. For example, using (2.20) in the third equation in (2.14) it is obtained $c_0 = 0$. From this along with (2.20) and (2.14) with $k = 0$ it is further easily obtained $a_{-1} = 0 = c_{-1}$ and $b_{-1} = 1$. This fact, among others, shows that equalities (2.18) and (2.35) really hold for every $n \in \mathbb{N}_0$.

The following corollary is a consequence of Theorem 2.4.

Corollary 2.6. *Consider system* (1.3) *with* $a, b, c, d \in \mathbb{Z} \setminus \{0\}$ *and* $\alpha, \beta \in \mathbb{C} \setminus \{0\}$. *Assume that* $z_0, w_0 \in \mathbb{C} \setminus \{0\}$. *Then the following statements are true.*

(a) *If* $ad \neq bc$ *and* $\Delta \neq 0$, *then the general solution to system* (1.3) *is given by* (2.28) *and* (2.36).
(b) *If* $ad \neq bc$ *and* $\Delta = 0$, *then the general solution to system* (1.3) *is given by* (2.29) *and* (2.37).
(c) *If* $ad = bc$ *and* $a + d \neq 1$, *then the general solution to system* (1.3) *is given by* (2.41) *and* (2.44).
(d) *If* $ad = bc$ *and* $a + d = 1$, *then the general solution to system* (1.3) *is given by* (2.42) *and* (2.45).

Now we consider the cases when some of the coefficients a, b, c, d are equal to zero.

Case $a = 0$. Since $a = 0$, then the first equation in (1.3) becomes

$$z_{n+1} = \alpha w_n^b, \quad n \in \mathbb{N}_0. \tag{2.46}$$

By substituting (2.46) into the second equation in (1.3), we obtain

$$w_{n+1} = \alpha^c \beta w_n^d w_{n-1}^{bc}, \quad n \in \mathbb{N}. \tag{2.47}$$

Equation (2.47) is nothing but equation (2.33) with $a = 0$. Hence, we can apply formulas for w_n obtained in the proof of Theorem 2.4 along with (2.46) and get the following result.

Theorem 2.7. *Consider system* (1.3) *with* $b, c, d \in \mathbb{Z}$, $a = 0$ *and* $\alpha, \beta \in \mathbb{C} \setminus \{0\}$. *Assume that* $z_0, w_0 \in \mathbb{C} \setminus \{0\}$. *Then the following statements are true.*

(a) *If* $bc \neq 0$ *and* $\Delta \neq 0$, *then the general solution to system* (1.3) *is*

$$\begin{aligned} z_n &= z_0^{bc\frac{\lambda_1^{n-1}-\lambda_2^{n-1}}{\lambda_1-\lambda_2}} w_0^{bd\frac{\lambda_1^{n-1}-\lambda_2^{n-1}}{\lambda_1-\lambda_2}+b^2c\frac{\lambda_1^{n-2}-\lambda_2^{n-2}}{\lambda_1-\lambda_2}} \\ &\quad\times \beta^{b\frac{\lambda_1^{n-1}-\lambda_2^{n-1}}{\lambda_1-\lambda_2}+b\frac{(\lambda_2-1)(\lambda_1^{n-1}-1)-(\lambda_1-1)(\lambda_2^{n-1}-1)}{(\lambda_1-\lambda_2)(\lambda_1-1)(\lambda_2-1)}} \\ &\quad\times \alpha^{1+bc\frac{(\lambda_2-1)(\lambda_1^{n-1}-1)-(\lambda_1-1)(\lambda_2^{n-1}-1)}{(\lambda_1-\lambda_2)(\lambda_1-1)(\lambda_2-1)}}, \end{aligned}$$

$$\begin{aligned} w_n &= z_0^{c\frac{\lambda_1^{n}-\lambda_2^{n}}{\lambda_1-\lambda_2}} w_0^{d\frac{\lambda_1^{n}-\lambda_2^{n}}{\lambda_1-\lambda_2}+bc\frac{\lambda_1^{n-1}-\lambda_2^{n-1}}{\lambda_1-\lambda_2}} \beta^{\frac{\lambda_1^{n}-\lambda_2^{n}}{\lambda_1-\lambda_2}+\frac{(\lambda_2-1)(\lambda_1^{n}-1)-(\lambda_1-1)(\lambda_2^{n}-1)}{(\lambda_1-\lambda_2)(\lambda_1-1)(\lambda_2-1)}} \\ &\quad\times \alpha^{c\frac{(\lambda_2-1)(\lambda_1^{n}-1)-(\lambda_1-1)(\lambda_2^{n}-1)}{(\lambda_1-\lambda_2)(\lambda_1-1)(\lambda_2-1)}}, \end{aligned}$$

for $n \in \mathbb{N}$, *where*

$$\lambda_{1,2} = \frac{d \pm \sqrt{d^2 + 4bc}}{2}.$$

(b) *If* $bc \neq 0$ *and* $\Delta = 0$, *then the general solution to system* (1.3) *is*

$$\begin{aligned} z_n &= z_0^{bc(n-1)\lambda_1^{n-2}} w_0^{bd(n-1)\lambda_1^{n-2}+b^2c(n-2)\lambda_1^{n-3}} \\ &\quad\times \beta^{b(n-1)\lambda_1^{n-2}+b\frac{1-(n-1)\lambda_1^{n-2}+(n-2)\lambda_1^{n-1}}{(1-\lambda_1)^2}} \alpha^{1+bc\frac{1-(n-1)\lambda_1^{n-2}+(n-2)\lambda_1^{n-1}}{(1-\lambda_1)^2}}, \end{aligned}$$

$$\begin{aligned} w_n &= z_0^{cn\lambda_1^{n-1}} w_0^{dn\lambda_1^{n-1}+bc(n-1)\lambda_1^{n-2}} \\ &\quad\times \beta^{n\lambda_1^{n-1}+\frac{1-n\lambda_1^{n-1}+(n-1)\lambda_1^{n}}{(1-\lambda_1)^2}} \alpha^{c\frac{1-n\lambda_1^{n-1}+(n-1)\lambda_1^{n}}{(1-\lambda_1)^2}}, \end{aligned}$$

for $n \in \mathbb{N}$, *where* $\lambda_1 = d/2$.

(c) *If* $bc = 0$ *and* $d \neq 1$, *then the general solution to system* (1.3) *is*

$$\begin{aligned} z_n &= w_0^{bd^{n-1}} \alpha \beta^{b\frac{1-d^{n-1}}{1-d}} \\ w_n &= z_0^{cd^{n-1}} w_0^{d^n} \alpha^{c\frac{1-d^{n-1}}{1-d}} \beta^{\frac{1-d^n}{1-d}}, \end{aligned}$$

for $n \in \mathbb{N}$.

(d) *If* $bc = 0$ *and* $d = 1$, *then the general solution to system* (1.3) *is*

$$\begin{aligned} z_n &= w_0^b \alpha \beta^{b(n-1)} \\ w_n &= z_0^c w_0 \alpha^{c(n-1)} \beta^n, \end{aligned}$$

for $n \in \mathbb{N}$.

Case $b = 0$**.** Since $b = 0$, then the first equation in (1.3) becomes

$$z_{n+1} = \alpha z_n^a, \quad n \in \mathbb{N}_0. \tag{2.48}$$

By using Lemma 2.1 see that

$$z_n = z_0^{a^n} \alpha^{\sum_{j=0}^{n-1} a^j}, \quad n \in \mathbb{N}_0. \tag{2.49}$$

From (2.49) we have that

$$z_n = z_0^{a^n} \alpha^{\frac{1-a^n}{1-a}}, \quad n \in \mathbb{N}_0, \tag{2.50}$$

when $a \neq 1$, while

$$z_n = z_0 \alpha^n, \quad n \in \mathbb{N}_0, \tag{2.51}$$

when $a = 1$.

Hence, if $a \neq 1$, substituting (2.50) into the second equation in (1.3), we obtain

$$w_n = w_{n-1}^d \beta z_0^{ca^{n-1}} \alpha^{c\frac{1-a^{n-1}}{1-a}}, \quad n \in \mathbb{N}. \tag{2.52}$$

By repeating use of (2.52) we obtain that for $n \geq 3$,

$$\begin{aligned} w_n &= \Big(w_{n-2}^d \beta z_0^{ca^{n-2}} \alpha^{c\frac{1-a^{n-2}}{1-a}}\Big)^d \beta z_0^{ca^{n-1}} \alpha^{c\frac{1-a^{n-1}}{1-a}} \\ &= w_{n-2}^{d^2} \beta^{1+d} z_0^{ca^{n-1}+cda^{n-2}} \alpha^{c\frac{1-a^{n-1}}{1-a}+cd\frac{1-a^{n-2}}{1-a}} \\ &= \Big(w_{n-3}^d \beta z_0^{ca^{n-3}} \alpha^{c\frac{1-a^{n-3}}{1-a}}\Big)^{d^2} \beta^{1+d} z_0^{ca^{n-1}+cda^{n-2}} \alpha^{c\frac{1-a^{n-1}}{1-a}+cd\frac{1-a^{n-2}}{1-a}} \\ &= w_{n-3}^{d^3} \beta^{1+d+d^2} z_0^{ca^{n-1}+cda^{n-2}+cd^2a^{n-3}} \alpha^{c\frac{1-a^{n-1}}{1-a}+cd\frac{1-a^{n-2}}{1-a}+cd^2\frac{1-a^{n-3}}{1-a}}. \end{aligned} \tag{2.53}$$

An inductive argument shows that

$$w_n = w_0^{d^n} \beta^{\sum_{j=0}^{n-1} d^j} z_0^{c\sum_{j=0}^{n-1} a^{n-1-j} d^j} \alpha^{c\sum_{j=0}^{n-1} \frac{1-a^{n-1-j}}{1-a} d^j}, \quad n \in \mathbb{N}. \tag{2.54}$$

Hence, from (2.54) we obtain

$$w_n = w_0^{d^n} \beta^{\frac{1-d^n}{1-d}} z_0^{c\frac{a^n-d^n}{a-d}} \alpha^{c\left(\frac{1-d^n}{(1-a)(1-d)} - \frac{a^n-d^n}{(1-a)(a-d)}\right)}, \quad n \in \mathbb{N}_0, \tag{2.55}$$

if $d \neq 1$ and $a \neq d$,

$$w_n = w_0^{d^n} \beta^{\frac{1-d^n}{1-d}} z_0^{cnd^{n-1}} \alpha^{c\left(\frac{1-d^n}{(1-d)^2} - \frac{nd^{n-1}}{1-d}\right)}, \quad n \in \mathbb{N}_0, \tag{2.56}$$

if $d \neq 1$ and $a = d$, and

$$w_n = w_0 \beta^n z_0^{c\frac{1-a^n}{1-a}} \alpha^{c\left(\frac{n}{1-a} - \frac{1-a^n}{(1-a)^2}\right)}, \quad n \in \mathbb{N}_0, \tag{2.57}$$

if $d = 1$ and $a \neq 1$.

If $a = 1$, substituting (2.51) into the second equation in (1.3), we obtain

$$w_n = w_{n-1}^d \beta z_0^c \alpha^{c(n-1)}, \quad n \in \mathbb{N}. \tag{2.58}$$

By repeated use of (2.58) we obtain

$$\begin{aligned} w_n &= \big(w_{n-2}^d \beta z_0^c \alpha^{c(n-2)}\big)^d \beta z_0^c \alpha^{c(n-1)} \\ &= w_{n-2}^{d^2} \beta^{1+d} z_0^{c+cd} \alpha^{c(n-1)+cd(n-2)} \\ &= \big(w_{n-3}^d \beta z_0^c \alpha^{c(n-3)}\big)^{d^2} \beta^{1+d} z_0^{c+cd} \alpha^{c(n-1)+cd(n-2)} \\ &= w_{n-3}^{d^3} \beta^{1+d+d^2} z_0^{c+cd+cd^2} \alpha^{c(n-1)+cd(n-2)+cd^2(n-3)}, \quad n \geq 3. \end{aligned} \tag{2.59}$$

An inductive argument shows that

$$w_n = w_0^{d^n} \beta^{\sum_{j=0}^{n-1} d^j} z_0^{c\sum_{j=0}^{n-1} d^j} \alpha^{c\sum_{j=0}^{n-1} d^j(n-1-j)}, \tag{2.60}$$

for $n \in \mathbb{N}$. Hence, from (2.60) and by using Lemma 2.3, we obtain

$$w_n = w_0^{d^n} \beta^{\frac{1-d^n}{1-d}} z_0^{c\frac{1-d^n}{1-d}} \alpha^{c\left((n-1)\frac{1-d^n}{1-d} - d\frac{1-nd^{n-1}+(n-1)d^n}{(1-d)^2}\right)}, \quad n \in \mathbb{N}_0, \tag{2.61}$$

if $d \neq 1$, and

$$w_n = w_0 \beta^n z_0^{cn} \alpha^{c\frac{n(n-1)}{2}}, \quad n \in \mathbb{N}_0, \tag{2.62}$$

if $d = 1$. From the above considerations we see that the following result holds.

Theorem 2.8. *Consider system (1.3) with $a, c, d \in \mathbb{Z}$, $b = 0$ and $\alpha, \beta \in \mathbb{C} \setminus \{0\}$. Assume that $z_0, w_0 \in \mathbb{C} \setminus \{0\}$. Then the following statements are true.*

(a) *If $a \neq 1$, $d \neq 1$ and $a \neq d$, then the general solution to system (1.3) is given by (2.50) and (2.55).*
(b) *If $a \neq 1$, $d \neq 1$ and $a = d$, then the general solution to system (1.3) is given by (2.50) and (2.56).*
(c) *If $a \neq 1$ and $d = 1$, then the general solution to system (1.3) is given by (2.50) and (2.57).*
(d) *If $a = 1$ and $d \neq 1$, then the general solution to system (1.3) is given by (2.51) and (2.61).*
(e) *If $a = d = 1$, then the general solution to system (1.3) is given by (2.51) and (2.62).*

Case $c = 0$. Since $c = 0$, then the second equation in (1.3) becomes

$$w_{n+1} = \beta w_n^d, \quad n \in \mathbb{N}_0. \tag{2.63}$$

By using Lemma 2.1 in (2.63) we obtain

$$w_n = w_0^{d^n} \beta^{\sum_{j=0}^{n-1} d^j}, \quad n \in \mathbb{N}_0. \tag{2.64}$$

Then we have

$$w_n = w_0^{d^n} \beta^{\frac{1-d^n}{1-d}}, \quad n \in \mathbb{N}_0, \tag{2.65}$$

when $d \neq 1$, while

$$w_n = w_0 \beta^n, \quad n \in \mathbb{N}_0, \tag{2.66}$$

when $d = 1$. Hence, if $d \neq 1$, substituting (2.65) into the first equation in (1.3), we obtain

$$z_n = z_{n-1}^a \alpha w_0^{bd^{n-1}} \beta^{b\frac{1-d^{n-1}}{1-d}}, \quad n \in \mathbb{N}. \tag{2.67}$$

By repeated use of (2.67) we have that for $n \geq 3$,

$$\begin{aligned} z_n &= \Big(z_{n-2}^a \alpha w_0^{bd^{n-2}} \beta^{b\frac{1-d^{n-2}}{1-d}}\Big)^a \alpha w_0^{bd^{n-1}} \beta^{b\frac{1-d^{n-1}}{1-d}} \\ &= z_{n-2}^{a^2} \alpha^{1+a} w_0^{bd^{n-1}+bad^{n-2}} \beta^{b\frac{1-d^{n-1}}{1-d}+ba\frac{1-d^{n-2}}{1-d}} \\ &= \Big(z_{n-3}^a \alpha w_0^{bd^{n-3}} \beta^{b\frac{1-d^{n-3}}{1-d}}\Big)^{a^2} \alpha^{1+a} w_0^{bd^{n-1}+bad^{n-2}} \beta^{b\frac{1-d^{n-1}}{1-d}+ba\frac{1-d^{n-2}}{1-d}} \\ &= z_{n-3}^{a^3} \alpha^{1+a+a^2} w_0^{bd^{n-1}+bad^{n-2}+ba^2d^{n-3}} \beta^{b\frac{1-d^{n-1}}{1-d}+ba\frac{1-d^{n-2}}{1-d}+ba^2\frac{1-d^{n-3}}{1-a}}. \end{aligned} \tag{2.68}$$

An inductive argument shows that

$$z_n = z_0^{a^n} \alpha^{\sum_{j=0}^{n-1} a^j} w_0^{b\sum_{j=0}^{n-1} d^{n-1-j} a^j} \beta^{b\sum_{j=0}^{n-1} \frac{1-d^{n-1-j}}{1-d} a^j}, \tag{2.69}$$

for $n \in \mathbb{N}$. Hence, from (2.69) we obtain

$$z_n = z_0^{a^n} \alpha^{\frac{1-a^n}{1-a}} w_0^{b\frac{a^n-d^n}{a-d}} \beta^{b\left(\frac{1-a^n}{(1-a)(1-d)} - \frac{a^n-d^n}{(1-d)(a-d)}\right)}, \quad n \in \mathbb{N}_0, \tag{2.70}$$

if $a \neq 1$ and $a \neq d$,

$$z_n = z_0^{d^n} \alpha^{\frac{1-d^n}{1-d}} w_0^{bnd^{n-1}} \beta^{b\left(\frac{1-d^n}{(1-d)^2} - \frac{nd^{n-1}}{1-d}\right)}, \quad n \in \mathbb{N}_0, \tag{2.71}$$

if $a \neq 1$ and $a = d$, and

$$z_n = z_0 \alpha^n w_0^{b\frac{1-d^n}{1-d}} \beta^{b\left(\frac{n}{1-d} - \frac{1-d^n}{(1-d)^2}\right)}, \quad n \in \mathbb{N}_0, \tag{2.72}$$

if $a = 1$ and $d \neq 1$.

If $d = 1$, substituting (2.66) into the first equation in (1.3), we obtain

$$z_n = z_{n-1}^a \alpha w_0^b \beta^{b(n-1)}, \quad n \in \mathbb{N}. \tag{2.73}$$

By repeated use of (2.73) we obtain

$$\begin{aligned} z_n &= \left(z_{n-2}^a \alpha w_0^b \beta^{b(n-2)}\right)^a \alpha w_0^b \beta^{b(n-1)} \\ &= z_{n-2}^{a^2} \alpha^{1+a} w_0^{b+ba} \beta^{b(n-1)+ba(n-2)} \\ &= \left(z_{n-3}^a \alpha w_0^b \beta^{b(n-3)}\right)^{a^2} \alpha^{1+a} w_0^{b+ba} \beta^{b(n-1)+ba(n-2)} \\ &= z_{n-3}^{a^3} \alpha^{1+a+a^2} w_0^{b+ba+ba^2} \beta^{b(n-1)+ba(n-2)+ba^2(n-3)}, \end{aligned} \tag{2.74}$$

for $n \geq 3$. An inductive argument shows that

$$z_n = z_0^{a^n} \alpha^{\sum_{j=0}^{n-1} a^j} w_0^{b\sum_{j=0}^{n-1} a^j} \beta^{b\sum_{j=0}^{n-1} a^j (n-1-j)}, \tag{2.75}$$

for $n \in \mathbb{N}$. Hence, from (2.75) we obtain

$$z_n = z_0^{a^n} \alpha^{\frac{1-a^n}{1-a}} w_0^{b\frac{1-a^n}{1-a}} \beta^{b\left((n-1)\frac{1-a^n}{1-a} - a\frac{1-na^{n-1}+(n-1)a^n}{(1-a)^2}\right)}, \quad n \in \mathbb{N}_0, \tag{2.76}$$

if $a \neq 1$, and

$$z_n = z_0 \alpha^n w_0^{bn} \beta^{b\frac{n(n-1)}{2}}, \quad n \in \mathbb{N}_0, \tag{2.77}$$

if $a = 1$. From the above considerations we see that the following result holds.

Theorem 2.9. *Consider system* (1.3) *with* $a, b, d \in \mathbb{Z}$, $c = 0$ *and* $\alpha, \beta \in \mathbb{C} \setminus \{0\}$. *Assume that* $z_0, w_0 \in \mathbb{C} \setminus \{0\}$. *Then the following statements are true.*

(a) *If* $d \neq 1$, $a \neq 1$ *and* $a \neq d$, *then the general solution to system* (1.3) *is given by* (2.65) *and* (2.70).
(b) *If* $d \neq 1$, $a \neq 1$ *and* $a = d$, *then the general solution to system* (1.3) *is given by* (2.65) *and* (2.71).
(c) *If* $d \neq 1$ *and* $a = 1$, *then the general solution to system* (1.3) *is given by* (2.65) *and* (2.72).
(d) *If* $d = 1$ *and* $a \neq 1$, *then the general solution to system* (1.3) *is given by* (2.66) *and* (2.76).
(e) *If* $a = d = 1$, *then the general solution to system* (1.3) *is given by* (2.66) *and* (2.77).

Case $d = 0$. Since $d = 0$, then the second equation in (1.3) becomes

$$w_{n+1} = \beta z_n^c, \quad n \in \mathbb{N}_0. \tag{2.78}$$

By substituting (2.78) into the first equation in (1.3), we obtain

$$z_{n+1} = \alpha \beta^b z_n^a z_{n-1}^{bc}, \quad n \in \mathbb{N}. \tag{2.79}$$

This equation is nothing but (2.5) with $d = 0$. Hence, we can use the formulas for z_n obtained in the proof of Theorem 2.4 along with (2.78) and get the following result.

Theorem 2.10. *Consider system* (1.3) *with* $a, b, c \in \mathbb{Z}$, $d = 0$ *and* $\alpha, \beta \in \mathbb{C} \setminus \{0\}$. *Assume that* $z_0, w_0 \in \mathbb{C} \setminus \{0\}$. *Then the following statements are true.*

(a) *If* $bc \neq 0$ *and* $\Delta \neq 0$, *then the general solution to system* (1.3) *is*

$$z_n = z_0^{a\frac{\lambda_1^n-\lambda_2^n}{\lambda_1-\lambda_2}+bc\frac{\lambda_1^{n-1}-\lambda_2^{n-1}}{\lambda_1-\lambda_2}} w_0^{b\frac{\lambda_1^n-\lambda_2^n}{\lambda_1-\lambda_2}} \alpha^{\frac{\lambda_1^n-\lambda_2^n}{\lambda_1-\lambda_2}+\frac{(\lambda_2-1)(\lambda_1^n-1)-(\lambda_1-1)(\lambda_2^n-1)}{(\lambda_1-\lambda_2)(\lambda_1-1)(\lambda_2-1)}} \times \beta^{b\frac{(\lambda_2-1)(\lambda_1^n-1)-(\lambda_1-1)(\lambda_2^n-1)}{(\lambda_1-\lambda_2)(\lambda_1-1)(\lambda_2-1)}},$$

$$w_n = z_0^{ac\frac{\lambda_1^{n-1}-\lambda_2^{n-1}}{\lambda_1-\lambda_2}+bc^2\frac{\lambda_1^{n-2}-\lambda_2^{n-2}}{\lambda_1-\lambda_2}} w_0^{bc\frac{\lambda_1^{n-1}-\lambda_2^{n-1}}{\lambda_1-\lambda_2}} \times \alpha^{c\frac{\lambda_1^{n-1}-\lambda_2^{n-1}}{\lambda_1-\lambda_2}+c\frac{(\lambda_2-1)(\lambda_1^{n-1}-1)-(\lambda_1-1)(\lambda_2^{n-1}-1)}{(\lambda_1-\lambda_2)(\lambda_1-1)(\lambda_2-1)}} \times \beta^{1+cb\frac{(\lambda_2-1)(\lambda_1^{n-1}-1)-(\lambda_1-1)(\lambda_2^{n-1}-1)}{(\lambda_1-\lambda_2)(\lambda_1-1)(\lambda_2-1)}},$$

for $n \in \mathbb{N}$, *where*

$$\lambda_{1,2} = \frac{a+\sqrt{a^2+4bc}}{2}.$$

(b) *If* $bc \neq 0$ *and* $\Delta = 0$, *then the general solution to system* (1.3) *is*

$$z_n = z_0^{an\lambda_1^{n-1}+bc(n-1)\lambda_1^{n-2}} w_0^{bn\lambda_1^{n-1}} \alpha^{n\lambda_1^{n-1}+\frac{1-n\lambda_1^{n-1}+(n-1)\lambda_1^n}{(1-\lambda_1)^2}} \beta^{b\frac{1-n\lambda_1^{n-1}+(n-1)\lambda_1^n}{(1-\lambda_1)^2}}$$

$$w_n = z_0^{ac(n-1)\lambda_1^{n-2}+bc^2(n-2)\lambda_1^{n-3}} w_0^{bc(n-1)\lambda_1^{n-2}} \times \alpha^{c(n-1)\lambda_1^{n-2}+c\frac{1-(n-1)\lambda_1^{n-2}+(n-2)\lambda_1^{n-1}}{(1-\lambda_1)^2}} \beta^{1+bc\frac{1-(n-1)\lambda_1^{n-2}+(n-2)\lambda_1^{n-1}}{(1-\lambda_1)^2}}$$

for $n \in \mathbb{N}$, *where* $\lambda_1 = a/2$.

(c) *If* $bc = 0$ *and* $a \neq 1$, *then the general solution to system* (1.3) *is*

$$z_n = z_0^{a^n} w_0^{ba^{n-1}} \alpha^{\frac{1-a^n}{1-a}} \beta^{b\frac{1-a^{n-1}}{1-a}}$$
$$w_n = z_0^{ca^{n-1}} \alpha^{c\frac{1-a^{n-1}}{1-a}} \beta,$$

for $n \in \mathbb{N}$.

(d) *If* $bc = 0$ *and* $a = 1$, *then the general solution to system* (1.3) *is*

$$z_n = z_0 w_0^b \alpha^n \beta^{b(n-1)}$$
$$w_n = z_0^c \alpha^{c(n-1)} \beta,$$

for $n \in \mathbb{N}$.

Remark 2.11. The formulaes obtained in this article can be used in describing the long-term behavior of solutions to system (1.3) in many cases. The formulations and proofs of the results we leave to the reader as some exercises.

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