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EXACT CONTROLLABILITY PROBLEM OF A WAVE EQUATION IN NON-CYLINDRICAL DOMAINS

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Abstract. Let $\alpha : [0,\infty) \to (0,\infty)$ be a twice continuous differentiable function which satisfies that $\alpha(0) = 1$, α' is monotone and $0 < c_1 \le \alpha'(t) \le c_2 < 1$ for some constants c_1, c_2. The exact controllability of a one-dimensional wave equation in a non-cylindrical domain is proved. This equation characterizes small vibrations of a string with one of its endpoint fixed and the other moving with speed $\alpha'(t)$. By using the Hilbert Uniqueness Method, we obtain the exact controllability results of this equation with Dirichlet boundary control on one endpoint. We also give an estimate on the controllability time that depends only on c_1 and c_2.

1. Introduction and main results

Suppose $\alpha : [0,\infty) \to (0,\infty)$ is a twice continuous differentiable function satisfying the following assumptions:

(A1) $0 < c_1 \le \alpha'(t) \le c_2 < 1$ for all $0 \le t < \infty$;
(A2) α' is monotone;
(A3) $\alpha(0) = 1$.

Let $T > 0$. We define the non-cylindrical domain $\widehat{Q}_T^\alpha$ by

$$\widehat{Q}_T^\alpha = \{(y,t) \in \mathbb{R}^2 : 0 < y < \alpha(t), t \in (0,T)\}.$$

This article concerns the exact controllability of the one-dimensional wave equation

$$\begin{gathered} u_{tt}(y,t) - u_{yy}(y,t) = 0, \quad (y,t) \in \widehat{Q}_T^\alpha, \\ u(0,t) = 0, \quad u(\alpha(t),t) = v(t), \quad t \in (0,T), \\ u(y,0) = u^0(y), \quad u_t(y,0) = u^1(y), \quad y \in (0,1), \end{gathered} \tag{1.1}$$

where $v \in L^2(0,T)$ and $(u^0,u^1) \in L^2(0,1) \times H^{-1}(0,1)$. Since $\sup_{t\in(0,T)} |\alpha'(t)| < 1$, by [9], the system of (1.1) admits a unique solution in the sense of transposition. Here, as in [10], $u \in L^\infty(0,T;L^2(0,\alpha(t)))$ is called a solution by transposition of

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problem (1.1) if u verifies

$$\begin{aligned} &\int_0^T \int_0^{\alpha(t)} u(y,t)\hat{h}(y,t)\,dy\,dt \\ &= \int_0^1 [u^1(y)\theta(y,0) - u^0(y)\theta_t(y,0)]dy - \int_0^T v(t)\theta_y(\alpha(t),t)dt, \end{aligned} \tag{1.2}$$

for all $\hat{h} \in L^1(0,T;L^2(0,\alpha(t))$, where θ is the weak solution of the problem

$$\begin{gathered} \theta_{tt}(y,t) - \theta_{yy}(y,t) = \hat{h}, \quad (y,t) \in \widehat{Q}_T^\alpha, \\ \theta(0,t) = \theta(\alpha(t),0) = 0, \quad t \in (0,T), \\ \theta(T) = \theta'(T) = 0, \quad x \in (0,1). \end{gathered} \tag{1.3}$$

The exact controllability problem of system (1.1) is stated as follows.

Definition 1.1. We say system (1.1) is exactly controllable at time T, if for any $(u^0,u^1) \in L^2(0,1) \times H^{-1}(0,1)$, $(u_d^0,u_d^1) \in L^2(0,\alpha(T)) \times H^{-1}(0,\alpha(T))$, there exists $v \in L^2(0,T)$ such that the solution by transposition u of (1.1) satisfies $u(T) = u_d^0$ and $u_t(T) = u_d^1$.

For a function α satisfying conditions (A1)–(A3), we define

$$T^* = \frac{1}{c_2}\big\{ \exp\big(\frac{2c_2^2(1-c_1)(1+c_2)}{c_1(1-c_2)^2}\big) - 1\big\}, \tag{1.4}$$

$$T_1^* = \frac{1}{c_2}\big\{ \exp\big(\frac{2c_2^2(1-c_1)}{c_1(1-c_2)^3}\big) - 1\big\}. \tag{1.5}$$

One of the main results of this article as follows.

Theorem 1.2. *For any given $T > T^*$, (1.1) is exactly controllable at time T.*

Similarly, for the exact controllability problem, when the control is acting on the fixed endpoint,

$$\begin{gathered} u_{tt}(y,t) - u_{yy}(y,t) = 0, \quad (y,t) \in \widehat{Q}_T^\alpha, \\ u(0,t) = v(t),\ u(\alpha(t),t) = 0, \quad t \in (0,T), \\ u(y,0) = u^0(y), \quad u_t(y,0) = u^1(y), \quad y \in (0,1), \end{gathered} \tag{1.6}$$

we have the following result.

Theorem 1.3. *For any given $T > T_1^*$, (1.6) is exactly controllable at time T.*

Remark 1.4. When $\alpha(t) = 1 + kt$ for some constant $k \in (0,1)$, T^* is reduced to T_k^* defined in [4], and Theorem 1.2 is reduced to [4, Theorem 1.1].

Remark 1.5. Theorem 1.3 extends the results in [5] and [6]. In fact, when $\alpha(t) = 1 + kt$, an exact controllability result of system (1.6) has been proved for $0 < k < 1 - \frac{1}{\sqrt{e}}$ in [5] and for $0 < k < 1 - \frac{2}{1+e^2}$ in [6]. We also note that the controllability time T_1^* given here is better than the constants T_k^* in [5] and [6] in this case.

Remark 1.6. We note that there are many functions $\alpha(t)$ satisfying conditions (A1)–(A3) but are not the form $1 + kt$, for example $\alpha(t) = 1 + (t + \arctan t)/c$ where c is any constant that is greater than 2.

Recently, several works on the controllability problems of wave equations in non-cylindrical domains have been published. The existence of solutions of the initial boundary value problem for the nonlinear wave equation in non-cylindrical domains has been studied in [3, 8]. The controllability problem for a multi-dimensional wave equation in a non-cylindrical domain has been investigated in [2, 9, 10]. About the one-dimension cases, there have been extensive study of the controllability problem in a non-cylindrical domain. We refer the reader to [1, 4, 5, 6].

When $\alpha(t) = 1 + kt$ for some constant $0 < k < 1$, in [4], the exact controllability of the system (1.1) has been acquired. When $\alpha(t) = 1 + kt$, Cui and Song obtained that the system (1.6) is exactly controllable for $0 < k < 1 - \frac{1}{\sqrt{e}}$ in [5] and is exactly controllable for $0 < k < 1 - \frac{2}{1+e^2}$ in [6].

There are also other results on the exact controllability problem for wave equations of variable coefficients in cylindrical domains, see [7, 10, 11, 12] and the references therein. So, our first aim is to transform (1.1) and (1.6) into wave equations with variable coefficients in a cylindrical domain.

Let $x = \frac{y}{\alpha(t)}$ and $w(x,t) = u(y,t) = u(\alpha(t)x,t)$ for $(y,t) \in \widehat{Q}_T^{\alpha}$. Then, it is straightforward to show that (x,t) varies in $Q_T := (0,1) \times (0,T)$ and (1.1) is transformed into the wave equation with variable coefficients,

$$
\begin{gathered}
w_{tt} - \big[\frac{\beta(x,t)}{\alpha(t)} w_x\big]_x + \frac{\gamma(x,t)}{\alpha(t)} w_{tx} + \frac{\tau(x,t)}{\alpha(t)} w_x = 0, \quad \text{in } Q_T, \\
w(0,t) = 0, \quad w(1,t) = v(t) \quad t \in (0,T), \\
w(x,0) = w^0(x), \quad w_t(x,0) = w^1(x), \quad x \in (0,1),
\end{gathered}
\tag{1.7}
$$

where $\beta(x,t) = \frac{1-\alpha'^2(t)x^2}{\alpha(t)}$, $\gamma(x,t) = -2\alpha'(t)x$, $\tau(x,t) = -\alpha''(t)x$, $w^0 = u^0$, $w^1 = u^1 + \alpha'(0)xu_x^0$.

From [10], we know that for $(u^0, u^1) \in L^2(0,1) \times H^{-1}(0,1)$ and $v \in L^2(0,T)$, (1.7) admits a unique solution $w \in C([0,T]; L^2(0,1)) \cap C^1([0,T]; H^{-1}(0,1))$ in the sense of transposition, where w is called a solution by transposition of problem (1.7) if

$$
\begin{aligned}
&\int_0^T \int_0^1 wh\,dx\,dt \\
&= \int_0^1 [-w^0(x)z_t(x,0) + \alpha'(0)w^0(x)z(x,0) + w'(x)z(x,0)]dx \\
&\quad - \int_0^T \beta(1,t)z_x(1,t)v(t)dt + \int_0^1 [\gamma_x(x,0)w^0(x)z(x,0) + \gamma(x,0)w_x^0(x)z(x,0)]dx,
\end{aligned}
$$

for every $h \in L^1(0,T; L^2(0,1))$ and z is the weak solution of the problem

$$
\begin{gathered}
L^* z = h, \quad \text{in } Q_T, \\
z(0,t) = z(1,t) = 0, \quad t \in (0,T), \\
z(x,T) = z_t(x,T) = 0, \quad x \in (0,1),
\end{gathered}
\tag{1.8}
$$

where the formal adjoint L^* of L is defined by

$$
L^* z = \alpha(t) z_{tt} - [\beta(x,t) z_x]_x + \gamma(x,t) z_{xt} + \tau(x,t) z_x. \tag{1.9}
$$

Thus, Theorem 1.2 can be restated as the following exact controllability result for equation (1.7).

Theorem 1.7. *For any $T > T^*$ where T^* is given by* (1.4)*, any $(w^0, w^1) \in L^2(0,1) \times H^{-1}(0,1)$ and $(w_d^0, w_d^1) \in L^2(0,1) \times H^{-1}(0,1)$, we can always find a control $v \in L^2(0,T)$ such that the corresponding solution by transposition w of* (1.7) *satisfies $w(T) = w_d^0$, $w_t(T) = w_d^1$.*

Similarly, (1.6) can be transformed into the wave equation with variable coefficients,

$$
\begin{gathered}
w_{tt} - \big[\frac{\beta(x,t)}{\alpha(t)} w_x\big]_x + \frac{\gamma(x,t)}{\alpha(t)} w_{tx} + \frac{\tau(x,t)}{\alpha(t)} w_x = 0, \quad \text{in } Q_T, \\
w(0,t) = v(t), \quad w(1,t) = 0, \quad t \in (0,T), \\
w(x,0) = w^0(x), \quad w_t(x,0) = w^1(x), \quad x \in (0,1),
\end{gathered}
\tag{1.10}
$$

and Theorem 1.3 can be restated as the following exact controllability result for equation(1.10).

Theorem 1.8. *For any $T > T_1^*$ where T_1^* is given by* (1.5)*, any $(w^0, w^1) \in L^2(0,1) \times H^{-1}(0,1)$ and $(w_d^0, w_d^1) \in L^2(0,1) \times H^{-1}(0,1)$, we can always find a control $v \in L^2(0,T)$ such that the corresponding solution by transposition w of* (1.10) *satisfies $w(T) = w_d^0$, $w_t(T) = w_d^1$.*

2. Description of the Hilbert uniqueness method

In this section, we describe the Hilbert uniqueness method which is used in the proof of Theorems 1.7 and 1.8. Next, we consider Theorem 1.7 in detail.

Firstly, for any $(w_d^0, w_d^1) \in L^2(0,1) \times H^{-1}(0,1)$, the system

$$
\begin{gathered}
\alpha(t)\xi_{tt} - \big[\beta(x,t)\xi_x\big]_x + \gamma(x,t)\xi_{tx} + \tau(x,t)\xi_x = 0, \quad \text{in } Q_T, \\
\xi(0,t) = 0, \quad \xi(1,t) = 0, \quad t \in (0,T), \\
\xi(x,T) = w_d^0(x), \quad \xi_t(x,T) = w_d^1(x), \quad x \in (0,1)
\end{gathered}
\tag{2.1}
$$

has a unique solution $\xi \in C([0,T]; L^2(0,1)) \cap C^1([0,T]; H^{-1}(0,1))$ in the sense of transportation.

Secondly, for any $(z^0, z^1) \in H_0^1(0,1) \times L^2(0,1)$, we solve

$$
\begin{gathered}
\alpha(t)z_{tt} - [\beta(x,t)z_x]_x + \gamma(x,t)z_{xt} + \tau(x,t)z_x = 0, \quad \text{in } Q_T, \\
z(0,t) = z(1,t) = 0, \quad t \in (0,T), \\
z(x,0) = z^0(x), \quad z_t(x,0) = z^1(x), \quad x \in (0,1),
\end{gathered}
\tag{2.2}
$$

and

$$
\begin{gathered}
\alpha(t)\eta_{tt} - \big[\beta(x,t)\eta_x\big]_x + \gamma(x,t)\eta_{tx} + \tau(x,t)\eta_x = 0, \quad \text{in } Q_T, \\
\eta(0,t) = 0, \quad \eta(1,t) = z_x(1,t), \quad t \in (0,T), \\
\eta(x,T) = 0, \quad \eta_t(x,T) = 0, \quad x \in (0,1).
\end{gathered}
\tag{2.3}
$$

Then we define a linear operator $\Lambda : H_0^1(0,1) \times L^2(0,1) \to H^{-1}(0,1) \times L^2(0,1)$, by

$$
(z^0, z^1) \mapsto (\eta_t(\cdot,0) + \gamma(\cdot,0)\eta_x(\cdot,0) - \alpha'(0)\eta(\cdot,0), -\eta(\cdot,0)),
$$

Lastly, the problem is reduced to prove the existence of some $(z^0, z^1) \in H_0^1(0,1) \times L^2(0,1)$ such that

$$
\Lambda(z^0, z^1) = ([w^1 - \xi_t(0)] - \alpha'(0)[w^0 - \xi(0)] + \gamma(0)[w_x^0 - \xi_x(0)] - [w^0 - \xi(0)]). \tag{2.4}
$$

To solve (2.4), we observe that

$$\int_0^1 \beta(1,t)|z_x(1,t)|^2 dt = \langle \Lambda(z^0,z^1),(z^0,z^1)\rangle_{H^{-1}(0,1)\times L^2(0,1),H_0^1(0,1)\times L^2(0,1)}. \quad (2.5)$$

In section 3, we prove the following observability inequality for system (2.2): there exists a constant $C>0$ such that

$$\int_0^T \beta(1,t)|z_x(1,t)|^2 dt \geq C\big(\|z^0\|^2_{H_0^1(0,1)} + \|z^1\|^2_{L^2(0,1)}\big). \quad (2.6)$$

Also, we prove that Λ is a bounded linear operator; i.e., there exists a constant $C>0$ such that

$$\int_0^T \beta(1,t)|z_x(1,t)|^2 dt \leq C(||z^0||^2_{H_0^1(0,1)} + ||z^1||^2_{L^2(0,1)}). \quad (2.7)$$

Combining (2.6), (2.7) and the Lax-Milgram Theorem, we can show that Λ is an isomorphism.

Then, the equation(2.4) has a unique solution $(z^0,z^1)\in H_0^1(0,1)\times L^2(0,1)$, and the function $z_x(1,t)$ is the desired control such that the solution w of (1.7) satisfies $w(T)=w_d^0$, $w_t(T)=w_d^1$.

For the proof of Theorem 1.8, the steps are similar to those of Theorem 1.7. In this case, instead of (2.3), we consider the following homogeneous wave equation

$$\begin{gathered}
\alpha(t)\eta_{tt} - \big[\beta(x,t)\eta_x\big]_x + \gamma(x,t)\eta_{tx} + \tau(x,t)\eta_x = 0, \quad \text{in } Q_T,\\
\eta(0,t) = z_x(0,t), \quad \eta(1,t)=0, \quad t\in(0,T),\\
\eta(x,T)=0, \quad \eta_t(x,T)=0, \quad x\in(0,1),
\end{gathered} \quad (2.8)$$

and define a linear operator Λ just same as (2.4), then we observe that

$$\int_0^1 \beta(0,t)|z_x(0,t)|^2 dt = -\langle \Lambda(z^0,z^1),(z^0,z^1)\rangle_{H^{-1}(0,1)\times L^2(0,1),H_0^1(0,1)\times L^2(0,1)}. \quad (2.9)$$

We omit the details of the proof here.

3. Observability estimates

The main purpose of this section is to prove the observability inequalities for system (2.2). To prove those estimates, we need some technical lemmas.

From [10], we know that: for any $(z^0,z^1)\in H_0^1(0,1)\times L^2(0,1)$, the equation (2.2) has a unique weak solution $z\in C([0,T];H_0^1(0,1))\cap C^1([0,T];L^2(0,1))$ in the sense of transportation.

The energy for (2.2) is defined as

$$E(t) = \frac{1}{2}\int_0^1 [\alpha(t)|z_t(x,t)|^2 + \beta(x,t)|z_x(x,t)|^2]dx, \quad \text{for } t\geq 0, \quad (3.1)$$

where z is the solution of (2.2). Since $\alpha(0)=1$, we have

$$E_0 := E(0) = \frac{1}{2}\int_0^1 \big[|z^1(x)|^2 + \beta(x,0)|z_x^0(x)|^2\big]dx. \quad (3.2)$$

First, we prove a lemma which is related to the decay rate of the energy $E(t)$.

Lemma 3.1. *If $\alpha(0)=1$, $0<c_1\le\alpha'(t)\le c_2<1$ and α' is monotone, then*
$$\frac{c_3E_0}{\alpha(t)}\le E(t)\le\frac{c_4E_0}{\alpha(t)}, \tag{3.3}$$
where
$$(c_3,c_4)=\begin{cases}\big(\frac{1-c_2}{1-c_1},\frac{c_2}{c_1}\big), & \text{if } \alpha' \text{ is increasing},\\ \big(\frac{c_1}{c_2},\frac{1-c_1}{1-c_2}\big), & \text{if } \alpha' \text{ is decreasing}.\end{cases} \tag{3.4}$$

Proof. For any $0<t\le T$, through multiplying the first equation of (2.2) by z_t and integrating the result on $(0,1)\times(0,t)$, we conclude that
$$\begin{aligned}0&=\int_0^t\int_0^1\big\{\alpha(s)z_{tt}(x,s)z_t(x,s)-[\beta(x,s)z_x(x,s)]_xz_t(x,s)\\&\quad+\gamma(x,s)z_{xt}(x,s)z_t(x,s)+\tau(x,s)z_x(x,s)z_t(x,s)\big\}\,dx\,ds\\&:=I_1+I_2+I_3+I_4,\end{aligned}$$
where
$$\begin{gathered}I_1=\frac12\int_0^1\alpha(s)|z_t(x,s)|^2dx\big|_0^t-\frac12\int_0^t\int_0^1\alpha'(s)|z_t(x,s)|^2\,dx\,ds,\\ \begin{aligned}I_2&=\frac12\int_0^1\beta(x,s)|z_x(x,s)|^2dx\big|_0^t-\frac12\int_0^t\int_0^1\beta_t(x,s)|z_x(x,s)|^2\,dx\,ds\\&=\frac12\int_0^1\beta(x,s)|z_x(x,s)|^2dx\big|_0^t+\frac12\int_0^t\int_0^1\frac{\alpha'(s)}{\alpha(s)}\beta(x,s)|z_x(x,s)|^2\,dx\,ds\\&\quad+\int_0^t\int_0^1\frac{\alpha'(s)\alpha''(s)}{\alpha(s)}x^2|z_x(x,s)|^2\,dx\,ds,\end{aligned}\\ I_3=\int_0^t\int_0^1\alpha'(s)|z_t(x,s)|^2\,dx\,ds,\\ I_4=-\int_0^t\int_0^1x\alpha''(s)z_x(x,s)z_t(x,s)\,dx\,ds.\end{gathered}$$
We thereby obtain:
$$\begin{gathered}\begin{aligned}E(t)&=E_0-\int_0^t\frac{\alpha'(s)}{\alpha(s)}E(s)ds-\int_0^t\int_0^1\frac{\alpha'(s)\alpha''(s)}{\alpha(s)}x^2|z_x(x,s)|^2\,dx\,ds\\&\quad+\int_0^t\int_0^1\alpha''(s)xz_x(x,s)z_t(x,s)\,dx\,ds.\end{aligned}\\ E'(t)=-\frac{\alpha'(t)}{\alpha(t)}E(t)-\int_0^1\frac{\alpha'(t)\alpha''(t)}{\alpha(t)}x^2|z_x(x,t)|^2dx+\int_0^1\alpha''(t)xz_x(x,t)z_t(x,t)dx.\end{gathered} \tag{3.5}$$
We subdivide the proof into two cases:

(1) α' is increasing; that is, $\alpha''(t)\ge0$. By using the inequalities
$$\begin{aligned}&-\frac{\alpha'(t)\alpha''(t)}{2\epsilon(t)\alpha(t)}x^2|z_x(x,t)|^2-\frac{\epsilon(t)\alpha(t)\alpha''(t)}{2\alpha'(t)}|z_t(x,t)|^2\\&\le\alpha''(t)xz_x(x,t)z_t(x,t)\\&\le\frac{\alpha'(t)\alpha''(t)}{2\alpha(t)}x^2|z_x(x,t)|^2+\frac{\alpha(t)\alpha''(t)}{2\alpha'(t)}|z_t(x,t)|^2,\end{aligned}$$

where $\epsilon(t) = \frac{\alpha'(t)}{1-\alpha'(t)}$, we easily obtain

$$-(\frac{\alpha'(t)}{\alpha(t)} + \frac{\alpha''(t)}{1-\alpha'(t)})E(t) \le E'(t) \le -(\frac{\alpha'(t)}{\alpha(t)} - \frac{\alpha''(t)}{\alpha'(t)})E(t), \tag{3.6}$$

so

$$\frac{(1-\alpha'(t))E_0}{(1-\alpha'(0))\alpha(t)} \le E(t) \le \frac{\alpha'(t)E_0}{\alpha'(0)\alpha(t)}. \tag{3.7}$$

Using $0 < c_1 \le \alpha'(t) \le c_2 < 1$, we conclude that

$$\frac{c_3 E_0}{\alpha(t)} \le E(t) \le \frac{c_4 E_0}{\alpha(t)}, \tag{3.8}$$

where $c_3 = \frac{1-c_2}{1-c_1}$, $c_4 = \frac{c_2}{c_1}$.

(2) α' is decreasing; that is, $\alpha''(t) \le 0$. By using the inequalities

$$\begin{aligned}
&\frac{\alpha'(t)\alpha''(t)}{2\alpha(t)} x^2 |z_x(x,t)|^2 + \frac{\alpha(t)\alpha''(t)}{2\alpha'(t)} |z_t(x,t)|^2 \\
&\le \alpha''(t) x z_x(x,t) z_t(x,t) \\
&\le -\frac{\alpha'(t)\alpha''(t)}{2\epsilon(t)\alpha(t)} x^2 |z_x(x,t)|^2 - \frac{\epsilon(t)\alpha(t)\alpha''(t)}{2\alpha'(t)} |z_t(x,t)|^2,
\end{aligned}$$

where $\epsilon(t) = \frac{\alpha'(t)}{1-\alpha'(t)}$, we easily get

$$-(\frac{\alpha'(t)}{\alpha(t)} - \frac{\alpha''(t)}{\alpha'(t)})E(t) \le E'(t) \le -(\frac{\alpha'(t)}{\alpha(t)} + \frac{\alpha''(t)}{1-\alpha'(t)})E(t), \tag{3.9}$$

so

$$\frac{\alpha'(t)E_0}{\alpha'(0)\alpha(t)} \le E(t) \le \frac{(1-\alpha'(t))E_0}{(1-\alpha'(0))\alpha(t)}. \tag{3.10}$$

Using $0 < c_1 \le \alpha'(t) \le c_2 < 1$, we conclude that

$$\frac{c_3 E_0}{\alpha(t)} \le E(t) \le \frac{c_4 E_0}{\alpha(t)}, \tag{3.11}$$

where $c_3 = \frac{c_1}{c_2}$, $c_4 = \frac{1-c_1}{1-c_2}$. $\square$

Remark 3.2. When $\alpha''(t) \equiv 0$, that is, $\alpha(t) = 1 + kt$ for some constant $k \in (0,1)$, then $c_3 = c_4 = 1$, Lemma 3.1 is reduced to Lemma 3.1 in [4].

Next, similar to the proof of [4, Lemma 3.2], we can get the following estimate for each weak solution z of (2.2) by the multiplier method.

Lemma 3.3. *For any function $q \in C^1([0,1])$, the solution z of (2.2) satisfies the estimate*

$$\begin{aligned}
&\frac{1}{2}\int_0^T \beta(x,t) q(x) |z_x(x,t)|^2 dt\Big|_0^1 \\
&= \frac{1}{2}\int_0^T \int_0^1 q'(x)[\alpha(t)|z_t(x,t)|^2 + \beta(x,t)|z_x(x,t)|^2]\,dx\,dt \\
&\quad - \int_0^T \int_0^1 \alpha'(t) q(x) z_x(x,t) z_t(x,t)\,dx\,dt - \frac{1}{2}\int_0^T \int_0^1 \beta_x(x,t) q(x) |z_x(x,t)|^2\,dx\,dt \\
&\quad + \int_0^1 [\alpha(t) q(x) z_x(x,t) z_t(x,t) - x\alpha'(t) q(x) |z_x(x,t)|^2] dx\Big|_0^T .
\end{aligned} \tag{3.12}$$

Finally, we derive the continuity estimate.

Theorem 3.4. *Assume $T>0$, for any $(z^0,z^1)\in H_0^1(0,1)\times L^2(0,1)$, there exists a constant $C>0$ such that the solution of (2.2) satisfies the following two estimates:*

$$\int_0^T \beta(1,t)|z_x(1,t)|^2dt \le C(||z^0||^2_{H_0^1(0,1)}+||z^1||^2_{L^2(0,1)}), \tag{3.13}$$

$$\int_0^T \beta(0,t)|z_x(0,t)|^2dt \le C(||z^0||^2_{H_0^1(0,1)}+||z^1||^2_{L^2(0,1)}); \tag{3.14}$$

so $z_x(0,\cdot)\in L^2(0,T)$ and $z_x(1,\cdot)\in L^2(0,T)$.

Proof. First, we prove inequality (3.13). Let $q(x)=x$ for $x\in[0,1]$ in (3.12) and noticing that $\beta_x(x,t)=-\frac{2\alpha'^2(t)x}{\alpha(t)}$, $\gamma(x,t)=-2\alpha'(t)x$, it follows that

$$\begin{aligned}
&\frac{1}{2}\int_0^T \beta(1,t)|z_x(1,t)|^2dt\\
&=\int_0^T E(t)dt-\int_0^T\int_0^1 \alpha'(t)xz_t(x,t)z_x(x,t)\,dx\,dt\\
&\quad+\int_0^T\int_0^1 \frac{\alpha'^2(t)}{\alpha(t)}x^2|z_x(x,t)|^2\,dx\,dt\\
&\quad+\int_0^1[\alpha(t)xz_t(x,t)z_x(x,t)-\alpha'(t)x^2|z_x(x,t)|^2]dx\Big|_0^T.
\end{aligned}\tag{3.15}$$

We estimate every terms on the right side of (3.15). By the assumption for α, we have $1\le\alpha(t)\le 1+c_2T$ and $0<\frac{1-c_2^2}{1+c_2T}\le\beta(x,t)\le 1$ for any $(x,t)\in Q_T$, these inequalities together with (3.3) and the boundedness of $\alpha'(t)$ imply

$$\begin{aligned}
&\int_0^T E(t)dt-\int_0^T\int_0^1 \alpha'(t)xz_t(x,t)z_x(x,t)\,dx\,dt\\
&+\int_0^T\int_0^1 \frac{\alpha'^2(t)}{\alpha(t)}x^2|z_x(x,t)|^2\,dx\,dt\\
&\le\int_0^T E(t)dt+C\int_0^T\int_0^1[|z_t(x,t)|^2+|z_x(x,t)|^2]\,dx\,dt\\
&\le\int_0^T E(t)dt+C\int_0^T\int_0^1[\alpha(t)|z_t(x,t)|^2+\beta(x,t)|z_x(x,t)|^2]\,dx\,dt\\
&\le CE_0.
\end{aligned}\tag{3.16}$$

For each $t\in[0,T]$ and $\epsilon(t)>0$, it holds that

$$\begin{aligned}
&\Big|\int_0^1[\alpha(t)xz_t(x,t)z_x(x,t)-\alpha'(t)x^2|z_x(x,t)|^2]dx\Big|\\
&\le\int_0^1[\alpha(t)|z_t(x,t)||z_x(x,t)|+\alpha'(t)|z_x(x,t)|^2]dx\\
&\le\frac{1}{2\epsilon(t)}\int_0^1\alpha^2(t)|z_t(x,t)|^2dx+\frac{\epsilon(t)}{2}\int_0^1|z_x(x,t)|^2dx+\int_0^1\alpha'(t)|z_x(x,t)|^2dx\\
&\le\frac{\alpha(t)}{2\epsilon(t)}\int_0^1\alpha(t)|z_t(x,t)|^2dx+[\frac{\epsilon(t)}{2}+\alpha'(t)]\frac{\alpha(t)}{1-\alpha'^2(t)}\int_0^1\beta(x,t)|z_x(x,t)|^2dx.
\end{aligned}$$

Choosing $\epsilon(t) = 1 - \alpha'(t)$, then it is easy to see

$$\epsilon(t) > 0 \text{ and } \frac{\alpha(t)}{\epsilon} = [\frac{\epsilon}{2} + \alpha'(t)]\frac{2\alpha(t)}{1-\alpha'^2(t)} = \frac{\alpha(t)}{1-\alpha'(t)}.$$

This implies that

$$\Big|\int_0^1 [\alpha(t)xz_t(x,t)z_x(x,t) - \alpha'(t)x^2|z_x(x,t)|^2]dx\Big| \le \frac{\alpha(t)}{1-\alpha'(t)}E(t) \le \frac{\alpha(t)}{1-c_2}E(t).$$

Then, using (3.3), it follows that

$$\Big|\int_0^1 [\alpha(t)xz_t(x,t)z_x(x,t) - \alpha'(t)x^2|z_x(x,t)|^2]dx\Big|_0^T\Big| \le c_5 E_0, \tag{3.17}$$

where $c_5 = \frac{2c_4}{1-c_2}$. Therefore, combining (3.15), (3.16) and (3.17), it follows that

$$\int_0^T \beta(1,t)|z_x(1,t)|^2 dt \le CE_0 \le C\big(\|z^0\|^2_{H_0^1(0,1)} + \|z^1\|^2_{L^2(0,1)}\big).$$

Next, we prove the inequality (3.14). Let $q(x) = x - 1$ for $x \in [0,1]$ in (3.12) and noticing that $\beta_x(x,t) = -\frac{2\alpha'^2(t)x}{\alpha(t)}$, $\gamma(x,t) = -2\alpha'(t)x$, it follows that

$$\begin{aligned}
&\frac{1}{2}\int_0^T \beta(0,t)|z_x(0,t)|^2 dt \\
&= \int_0^T E(t)dt - \int_0^T\int_0^1 \alpha'(t)(x-1)z_t(x,t)z_x(x,t)\,dx\,dt \\
&\quad + \int_0^T\int_0^1 \frac{\alpha'^2(t)}{\alpha(t)}x(x-1)|z_x(x,t)|^2\,dx\,dt \\
&\quad + \int_0^1 [\alpha(t)(x-1)z_t(x,t)z_x(x,t) - \alpha'(t)x(x-1)|z_x(x,t)|^2]dx\Big|_0^T.
\end{aligned} \tag{3.18}$$

Through estimating every terms on the right side of (3.18), similar to the derive of (3.16), it follows that

$$\begin{aligned}
&\int_0^T E(t)dt - \int_0^T\int_0^1 \alpha'(t)(x-1)z_t(x,t)z_x(x,t)\,dx\,dt \\
&+ \int_0^T\int_0^1 \frac{\alpha'^2(t)}{\alpha(t)}x(x-1)|z_x(x,t)|^2\,dx\,dt \le CE_0.
\end{aligned} \tag{3.19}$$

Since

$$\begin{aligned}
&\Big|\int_0^1 [\alpha(t)(x-1)z_t(x,t)z_x(x,t) - \alpha'(t)x(x-1)|z_x(x,t)|^2]dx\Big| \\
&\le \int_0^1 [\alpha(t)|z_t(x,t)||z_x(x,t)| + \alpha'(t)|z_x(x,t)|^2]dx,
\end{aligned}$$

similar to the derive of (3.17), it follows that

$$\Big|\int_0^1 [\alpha(t)(x-1)z_t(x,t)z_x(x,t) - \alpha'(t)x(x-1)|z_x(x,t)|^2]dx\Big|_0^T\Big| \le c_5 E(0), \tag{3.20}$$

where $c_5 = \frac{2c_4}{1-c_2}$.

From (3.18), (3.19) and (3.20), it follows that

$$\int_0^T \beta(0,t)|z_x(0,t)|^2 dt \le CE_0 \le (||z^0||^2_{H_0^1(0,1)} + ||z^1||^2_{L^2(0,1)}).$$

□

Now, we give the proof of the observability inequalities.

Theorem 3.5. *For $T > T^*$ where T^* satisfies* (1.4) *and any $(z^0, z^1) \in H_0^1(0,1) \times L^2(0,1)$, there exists a constant $C > 0$ such that the corresponding solution of* (2.2) *satisfies*

$$\int_0^T \beta(1,t)|z_x(1,t)|^2 dt \ge C\big(\|z^0\|^2_{H_0^1(0,1)} + \|z^1\|^2_{L^2(0,1)}\big). \tag{3.21}$$

Proof. If we choose $\epsilon(t) = \frac{\alpha'(t)}{1+\alpha'(t)}$, then it is obvious that

$$0 < \epsilon(t) < \frac{1}{2}, \quad \text{and} \quad 1 - \epsilon(t) = 1 + \Big(2 - \frac{1}{\epsilon(t)}\Big)\frac{\alpha'^2(t)}{1-\alpha'^2(t)} = \frac{1}{1+\alpha'(t)}.$$

Thus, using

$$x^2 = \frac{\alpha(t)x^2}{1-\alpha'^2(t)x^2}\beta(x,t) \le \frac{\alpha(t)}{1-\alpha'^2(t)}\beta(x,t)$$

and (3.3), it follows that

$$\begin{aligned}
&\int_0^T E(t)dt - \int_0^T\int_0^1 \alpha'(t)xz_t(x,t)z_x(x,t)\,dx\,dt \\
&+ \int_0^T\int_0^1 \frac{\alpha'^2(t)}{\alpha(t)}x^2|z_x(x,t)|^2\,dx\,dt \\
&\ge \int_0^T\int_0^1 \frac{1-\epsilon(t)}{2}\alpha(t)|z_t(x,t)|^2\,dx\,dt \\
&\quad + \int_0^T\int_0^1 \big\{\frac{1}{2}\beta(x,t) + (1 - \frac{1}{2\epsilon(t)})\frac{\alpha'^2(t)}{\alpha(t)}x^2\big\}|z_x(x,t)|^2\,dx\,dt \\
&\ge \int_0^T\int_0^1 \frac{1-\epsilon(t)}{2}\alpha(t)|z_t(x,t)|^2\,dx\,dt \\
&\quad + \int_0^T\int_0^1 \big[1 + \frac{(2-\frac{1}{\epsilon(t)})\alpha'^2(t)}{1-\alpha'^2(t)}\big]\frac{1}{2}\beta(x,t)|z_x(x,t)|^2\,dx\,dt \\
&= \int_0^T \frac{1}{1+\alpha'(t)}E(t)dt \\
&\ge c_6E_0\int_0^T \frac{1}{\alpha(t)}dt,
\end{aligned} \tag{3.22}$$

where $c_6 = c_3/(1+c_2)$. By (3.15), (3.17) and (3.22), we obtain

$$\begin{aligned}
\frac{1}{2}\int_0^T \beta(1,t)|z_x(1,t)|^2 dt &\ge c_6E_0\int_0^T \frac{1}{1+c_2t}dt - c_5E_0 \\
&= \big(\frac{c_6}{c_2}\log(1+c_2T) - c_5\big)E_0.
\end{aligned}$$

If we choose T^* as in (1.4), then it is easy to see that

$$T^* = \frac{1}{c_2}\big\{ \exp\big(\frac{c_2 c_5}{c_6}\big) - 1\big\}, \tag{3.23}$$

so for $T > T^*$,

$$\int_0^T \beta(1,t)|z_x(1,t)|^2 dt \geq C\big(\|z^0\|^2_{H_0^1(0,1)} + \|z^1\|^2_{L^2(0,1)}\big),$$

holds for $C = \frac{2c_6}{c_2}\log(1 + c_2 T) - 2c_5 > 0$. □

Theorem 3.6. *For $T > T_1^*$ where T_1^* satisfies* (1.5) *and any $(z^0, z^1) \in H_0^1(0,1) \times L^2(0,1)$, there exists a constant $C > 0$ such that the corresponding solution of* (2.2) *satisfies*

$$\int_0^T \beta(0,t)|z_x(0,t)|^2 dt \geq C(||z^0||^2_{H_0^1(0,1)} + ||z^1||^2_{L^2(0,1)}). \tag{3.24}$$

Proof. Choosing $\epsilon(x,t) = \frac{\alpha'(t)(1-\alpha'(t)x)}{\alpha(t)}$, it is easy to see that

$$\begin{aligned}
&|\alpha'(t) z_x(x,t) z_t(x,t)| \\
&\leq \frac{\alpha'^2(t)}{2\epsilon(x,t)}|z_t(x,t)|^2 + \frac{\epsilon(x,t)}{2}|z_x(x,t)|^2 \\
&= \frac{\alpha'(t)}{1-\alpha'(t)x}\frac{\alpha(t)}{2}|z_t(x,t)|^2 + \frac{\alpha'(t)}{1+\alpha'(t)x}\frac{\beta(x,t)}{2}|z_x(x,t)|^2.
\end{aligned}$$

Since $x - 1 \leq 0$ for $x \in [0,1]$, we have

$$\begin{aligned}
&\int_0^T E(t)dt - \int_0^T\int_0^1 \alpha'(t)(x-1) z_t(x,t) z_x(x,t)\,dx\,dt \\
&\quad + \int_0^T\int_0^1 \frac{\alpha'^2(t)}{\alpha(t)} x(x-1)|z_x(x,t)|^2\,dx\,dt \\
&\geq \int_0^T E(t)dt + \int_0^T\int_0^1 \frac{\alpha'(t)(x-1)}{1-\alpha'(t)x}\frac{\alpha(t)}{2}|z_t(x,t)|^2\,dx\,dt \\
&\quad + \frac{\alpha'(t)(x-1)}{1+\alpha'(t)x}\frac{\beta(x,t)}{2}|z_x(x,t)|^2\,dx\,dt \\
&\quad + \int_0^T\int_0^1 \frac{2\alpha'^2(t)x(x-1)}{1-\alpha'^2(t)x^2}\frac{\beta(x,t)}{2}|z_x(x,t)|^2\,dx\,dt \\
&= \frac{1}{2}\int_0^T\int_0^1 \frac{1-\alpha'(t)}{1-\alpha'(t)x}[\alpha(t)|z_t(x,t)|^2 + \beta(x,t)|z_x(x,t)|^2]\,dx\,dt \\
&\geq \int_0^T (1-\alpha'(t))E(t)dt \\
&\geq c_6^* E_0 \int_0^T \frac{1}{\alpha(t)}dt,
\end{aligned} \tag{3.25}$$

where $c_6^* = (1 - c_2)c_3$.

From (3.18), (3.20) and (3.25), we arrive at

$$\frac{1}{2}\int_0^T \beta(0,t)|z_x(0,t)|^2 dt \geq c_6^* E_0 \int_0^T \frac{1}{\alpha(t)}dt - c_5 E_0$$

$$\geq c_6^* E_0 \int_0^T \frac{1}{1+c_2 t} dt - c_5 E_0 = (\frac{c_6^*}{c_2} \log(1+c_2 T) - c_5) E_0.$$

If we choose T_1^* as in (1.5), then it is easy to see that

$$T_1^* = \frac{1}{c_2} \big\{ \exp(\frac{c_2 c_5}{c_6^*}) - 1 \big\};$$

thus, when $T > T_1^*$,

$$\int_0^T \beta(0,t) |z_x(0,t)|^2 dt \geq C \big(\|z^0\|^2_{H_0^1(0,1)} + \|z^1\|^2_{L^2(0,1)} \big).$$

holds for $C = \frac{2c_6^*}{c_2} \log(1+c_2 T) - 2c_5 > 0$. $\square$

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