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QUENCHING BEHAVIOR OF SEMILINEAR HEAT EQUATIONS WITH SINGULAR BOUNDARY CONDITIONS

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Abstract. In this article, we study the quenching behavior of solution to the semilinear heat equation

$$v_t = v_{xx} + f(v),$$

with $f(v) = -v^{-r}$ or $(1-v)^{-r}$ and

$$v_x(0,t) = v^{-p}(0,t), \quad v_x(a,t) = (1-v(a,t))^{-q}.$$

For this, we utilize the quenching problem $u_t = u_{xx}$ with $u_x(0,t) = u^{-p}(0,t)$, $u_x(a,t) = (1-u(a,t))^{-q}$. In the second problem, if u_0 is an upper solution (a lower solution) then we show that quenching occurs in a finite time, the only quenching point is $x = 0$ ($x = a$) and u_t blows up at quenching time. Further, we obtain a local solution by using positive steady state. In the first problem, we first obtain a local solution by using monotone iterations. Finally, for $f(v) = -v^{-r}$ ($(1-v)^{-r}$), if v_0 is an upper solution (a lower solution) then we show that quenching occurs in a finite time, the only quenching point is $x = 0$ ($x = a$) and v_t blows up at quenching time.

1. Introduction

In this article, we study the quenching behavior of solutions to the semilinear heat equation with singular boundary conditions:

$$\begin{gathered} v_t = v_{xx} + f(v), \quad 0 < x < a, \ 0 < t < T, \\ v_x(0,t) = v^{-p}(0,t), \quad v_x(a,t) = (1-v(a,t))^{-q}, \quad 0 < t < T, \\ v(x,0) = v_0(x), \quad 0 \le x \le a, \end{gathered} \tag{1.1}$$

where $p, q, r > 0, T \le \infty$, $f(u) = -v^{-r}$ or $f(u) = (1-v)^{-r}$. The initial function $v_0 : [0,a] \to (0,1)$ satisfies the compatibility conditions

$$v_0'(0) = v_0^{-p}(0), \quad v_0'(a) = (1-v_0(a))^{-q}.$$

Our main purpose is to examine the quenching behavior of the solutions of the problem (1.1) having two singular heat sources. A solution $v(x,t)$ of the problem (1.1) is said to quench if there exists a finite time T such that

$$\lim_{t\to T^-} \max\{v(x,t) : 0 \le x \le a\} \to 1 \quad \text{or} \quad \lim_{t\to T^-} \min\{v(x,t) : 0 \le x \le a\} \to 0.$$

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For the rest of this article, we denote the quenching time of (1.1) with T.

To study Problem (1.1), we utilize the following problem

$$\begin{gathered} u_t = u_{xx}, \quad 0 < x < a,\ 0 < t < T, \\ u_x(0,t) = u^{-p}(0,t), \quad u_x(a,t) = (1-u(a,t))^{-q}, \quad 0 < t < T, \\ u(x,0) = u_0(x), \quad 0 \le x \le a, \end{gathered} \tag{1.2}$$

where p, q are positive constants and $T \le \infty$. The initial function $u_0 : [0,a] \to (0,1)$ satisfies the compatibility conditions

$$u_0'(0) = u_0^{-p}(0), \quad u_0'(a) = (1-u_0(a))^{-q}.$$

Since 1975, quenching problems with various boundary conditions have been studied extensively. Recently, the quenching problems which have been studied with two nonlinear heat sources can be seen in [3, 7, 8, 10, 11]. For example, Chan and Yuen [3] considered the problem

$$\begin{gathered} u_t = u_{xx}, \quad \text{in } \Omega, \\ u_x(0,t) = (1-u(0,t))^{-p}, \quad u_x(a,t) = (1-u(a,t))^{-q}, \quad 0 < t < T, \\ u(x,0) = u_0(x), \quad 0 \le u_0(x) < 1, \quad \text{in } \bar{D}, \end{gathered}$$

where $a, p, q > 0$, $T \le \infty$, $D = (0,a)$, $\Omega = D \times (0,T)$. They showed that $x = a$ is the unique quenching point in finite time if u_0 is a lower solution, and u_t blows up at quenching time. Further, they obtained criteria for nonquenching and quenching by using the positive steady states. Selcuk and Ozalp [10] considered the problem

$$\begin{gathered} u_t = u_{xx} + (1-u)^{-p}, \quad 0 < x < 1,\ 0 < t < T, \\ u_x(0,t) = 0, \quad u_x(1,t) = -u^{-q}(1,t), \quad 0 < t < T, \\ u(x,0) = u_0(x), \quad 0 < u_0(x) < 1,\ 0 \le x \le 1. \end{gathered}$$

They showed that $x = 0$ is the quenching point in finite time, $\lim_{t\to T^-} u(0,t) \to 1$, if $u(x,0)$ satisfies $u_{xx}(x,0)+(1-u(x,0))^{-p} \ge 0$ and $u_x(x,0) \le 0$. Further they showed that u_t blows up at quenching time. Furthermore, they obtained a quenching rate and a lower bound for the quenching time.

Problems (1.1) and (1.2) have two type of singularity terms $(1-u)^{-q}$ and u^{-p} on the boundaries. We discuss these two situations in this article, $\lim_{t\to T^-} u(0,t) \to 0$ or $\lim_{t\to T^-} u(a,t) \to 1$. This article is organized as follows. In Section 2, we consider the problem (1.2). Firstly, if u_0 is an upper solution (a lower solution) then we show that quenching occurs in a finite time, the only quenching point is $x = 0$ ($x = a$) and u_t blows up at quenching time. Further, we obtain a local existence result by using positive steady state. In Section 3, we consider problem (1.1). Firstly, we obtain local existence of (1.1) by using monotone iterations. Further, for $f(v) = -v^{-r}$ ($(1-v)^{-r}$), if v_0 is an upper solution (a lower solution) then we show that quenching occurs in a finite time, the only quenching point is $x = 0$ ($x = a$) and v_t blows up at quenching time.

2. Problem (1.2)

2.1. **Quenching on the boundary.** The proofs of the following lemma and theorem are analogous to those by Chan and Yuen [3, Section 2].

Definition 2.1. μ is called a lower solution of (1.2) if $\mu \in C([0,a]\times[0,T)) \cap C^{2,1}((0,a)\times(0,T))$ satisfies the following conditions:

$$\begin{gathered}
\mu_t \le \mu_{xx}, \quad 0<x<a,\ 0<t<T,\\
\mu_x(0,t) \ge \mu^{-p}(0,t), \quad \mu_x(a,t) \le (1-\mu(a,t))^{-q}, \quad <t<T,\\
\mu(x,0) \le u_0(x), \quad 0 \le x \le a.
\end{gathered}$$

It is an upper solution when the inequalities are reversed.

Theorem 2.2. *Let $u(x,t,u_0)$ and $h(x,t,h_0)$ be solutions of problem* (1.2) *with data given by $u_0(x)$ and $h_0(x)$, respectively. If $u_0 \le h_0 < 1$, then $u(x,t,u_0) \le h(x,t,h_0)$ on $[0,a]\times[0,T)$.*

Proof. For any $\tau < T$, let w be a solution of the problem

$$\begin{gathered}
w_{xx} - w + w_t = 0 \quad \text{in } (0,a)\times(0,\tau),\\
w(x,\tau) = g(x) \quad \text{on } [0,a],\\
w_x(0,t) = r(t)w(0,t), \quad w_x(a,t) = s(t)w(a,t), \quad 0<t<\tau,
\end{gathered}$$

where $g \in C^2(\overline{D})$ has compact support in D, $0 \le g \le 1$, and r and s are smooth functions to be determined. By Lieberman [6], w exists. By Andersen [1], there exists a constant k (depending on the length of the interval D) such that $0 \le w \le k$. Now,

$$\begin{aligned}
&\int_0^a [(u(x,\tau)-h(x,\tau))w(x,\tau) - (u_0(x)-h_0(x))w(x,0)]dx\\
&= \int_0^\tau \int_0^a \frac{\partial}{\partial\sigma}[(u(x,\sigma)-h(x,\sigma))w(x,\sigma)]\,dx\,d\sigma\\
&= \int_0^\tau \int_0^a [w(x,\sigma)\frac{\partial}{\partial\sigma}(u(x,\sigma)-h(x,\sigma)) + (u(x,\sigma)-h(x,\sigma))\frac{\partial}{\partial\sigma}w(x,\sigma)]\,dx\,d\sigma\\
&= \int_0^\tau \int_0^a \Big[w(x,\sigma)\frac{\partial^2}{\partial x^2}(u(x,\sigma)-h(x,\sigma)) + (u(x,\sigma)-h(x,\sigma))\frac{\partial}{\partial\sigma}w(x,\sigma)\Big]\,dx\,d\sigma\\
&= \int_0^\tau \{w(a,\sigma)[(1-u(a,\sigma))^{-q} - (1-h(a,\sigma))^{-q}] - w(0,\sigma)[u^{-p}(0,\sigma) - h^{-p}(0,\sigma)]\\
&\quad - s(\sigma)[u(a,\sigma)-h(a,\sigma)]w(a,\sigma) + r(\sigma)[u(0,\sigma)-h(0,\sigma)]w(0,\sigma)\}d\sigma\\
&\quad + \int_0^\tau \int_0^a (u(x,\sigma)-h(x,\sigma))(w_\sigma(x,\sigma) + w_{xx}(x,\sigma))\,dx\,d\sigma.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\int_0^a [(u(x,\tau)-h(x,\tau))g(x) - (u_0(x)-h_0(x))w(x,0)]dx\\
&= \int_0^\tau \{w(a,\sigma)\left[(1-u(a,\sigma))^{-q} - (1-h(a,\sigma))^{-q} - s(\sigma)\left[u(a,\sigma)-h(a,\sigma)\right]\right]\\
&\quad - w(0,\sigma)\left[u^{-p}(0,\sigma) - h^{-p}(0,\sigma) - r(\sigma)\left[u(0,\sigma)-h(0,\sigma)\right]\right]\}d\sigma\\
&\quad + \int_0^\tau \int_0^a (u(x,\sigma)-h(x,\sigma))w(x,\sigma)\,dx\,d\sigma.
\end{aligned}$$

Let $r(\sigma)$ and $s(\sigma)$ be given by

$$r(\sigma)(u(0,\sigma)-h(0,\sigma)) = u^{-p}(0,\sigma) - h^{-p}(0,\sigma),$$

$$s(\sigma)(u(a,\sigma)-h(a,\sigma))=(1-u(a,\sigma))^{-q}-(1-h(a,\sigma))^{-q}.$$

Since $u_0 \le h_0$ and $w(x,0)\ge 0$, we have

$$\int_0^a (u(x,\tau)-h(x,\tau))g(x)dx \le \int_0^\tau \int_0^a (u(x,\sigma)-h(x,\sigma))w(x,\sigma)\,dx\,d\sigma.$$

Let

$$(u(x,\sigma)-h(x,\sigma))^+ = \max\{0, u(x,\sigma)-h(x,\sigma)\}.$$

From, $0\le w\le k$, we obtain

$$\int_0^a (u(x,\tau)-h(x,\tau))g(x)dx \le k\int_0^\tau \int_0^a (u(x,\sigma)-h(x,\sigma))^+\,dx\,d\sigma.$$

Since $g\in C^2(\overline{D})$ has compact support in D and $0\le g\le 1$, we have

$$\int_0^a (u(x,\sigma)-h(x,\sigma))^+dx \le k\int_0^\tau \int_0^a (u(x,\sigma)-h(x,\sigma))^+\,dx\,d\sigma.$$

By the Gronwall inequality,

$$\int_0^a (u(x,\sigma)-h(x,\sigma))^+dx \le 0,$$

which gives $u(x,\tau)\le h(x,\tau)$ for any $\tau>0$. Thus, the theorem is proved. □

Lemma 2.3. (i) *If $u_{xx}(x,0)\ge 0$ in $(0,a)$, then we obtain $u_t>0$ in $(0,a)\times(0,T)$.*

(ii) *If $u_{xx}(x,0)\le 0$ in $(0,a)$, then we obtain $u_t<0$ in$(0,a)\times(0,T)$.*

Proof. (i) Since $u_{xx}(x,0)\ge 0$ in $(0,a)$, $u_0'(0)=u_0^{-p}(0)$, $u_0'(a)=(1-u_0(a))^{-q}$, it follows that $u_0(x)$ is a lower solution of the problem (1.1) from Definition 2.1. The strong maximum principle implies that

$$u(x,t)\ge u_0(x) \quad \text{in } (0,a)\times(0,T).$$

Let h be a positive number less than T, and

$$z(x,t)=u(x,t+h)-u(x,t).$$

Then

$$\begin{gathered} z_t = z_{xx} \quad \text{in } (0,a)\times(0,T-h),\\ z(x,0)\ge 0 \quad \text{on } [0,a],\\ z_x(0,t)=-p\xi^{-p-1}(t)z(0,t), z_x(a,t)=q(1-\eta(t))^{-q-1}z(a,t), \quad 0<t<T-h, \end{gathered}$$

where $\xi(t)$ between $u(0,t+h)$ and $u(0,t)$, and $\eta(t)$ lies between $u(a,t+h)$ and $u(a,t)$. A proof similar to that of Theorem 2.2 shows that $z(x,t)\ge 0$. As $h\to 0$, we have $u_t\ge 0$ on $[0,a]\times(0,T)$.

Let $H=u_t$ in $[0,a]\times(0,T)$. Since

$$H_t - H_{xx} = 0 \text{ in } (0,a)\times(0,T),$$

it follows from the strong maximum principle that $H=u_t>0$ in $(0,a)\times(0,T)$.

(ii) If $u_{xx}(x,0)\le 0$ in $(0,a)$, then from the above proof we have $u_t\le 0$ on $[0,a]\times(0,T)$ and $u_t<0$ in $(0,a)\times(0,T)$. The proof is complete. □

Now we show that, if $u_{xx}(x,0)\le 0$ in $(0,a)$, namely, if u_0 is an upper solution, then we have quenching point at $x=0$.

Theorem 2.4. *If u_0 is an upper solution, then there exist a finite time T, such that the solution u of the problem* (1.2) *quenches at time T.*

Proof. Assume that u_0 is an upper solution. Then

$$\omega = -(1 - u(a, 0))^{-q} + u^{-p}(0, 0) > 0.$$

Introduce a mass function; $m(t) = \int_0^a u(x, t)dx$, $0 < t < T$. Then

$$m'(t) = (1 - u(a, t))^{-q} - u^{-p}(0, t) \leq -\omega,$$

by Lemma 2.3 (ii). Thus, $m(t) \leq m(0) - \omega t$; and so $m(T_0) = 0$ for some T_0, $(0 < T \leq T_0)$ which means u quenches in a finite time. □

Theorem 2.5. *If u_0 is an upper solution, then $x = 0$ is the only quenching point.*

Proof. Since $u_x(a, t) = (1 - u(a, t))^{-q} > 1$ and $u_{xx} = u_t < 0$ in $(0, a) \times (0, T)$, then u_x is a decreasing function and so, $u_x(x, t) > 1$ in $(0, a) \times (0, T)$. Let $\eta \in (0, a)$. Integrating this with respect to x from 0 to η, we have

$$u(\eta, t) > u(0, t) + \eta > 0.$$

So u does not quench in $(0, a]$. The proof is complete. □

Theorem 2.6. *If u_0 is an upper solution, then u_t blows up at quenching time.*

Proof. Suppose that u_t is bounded on $[0, a] \times [0, T)$. Then, there exists a positive constant M such that $u_t > -M$. We have $u_{xx} > -M$. Integrating this twice with respect to x from 0 to x, and then from 0 to a, we have

$$\frac{-a}{u^p(0, t)} > -\frac{Ma^2}{2} - u(a, t) + u(0, t).$$

As $t \to T^-$, the left-hand side tends to negative infinity, while the right-hand side is finite. This contradiction shows that u_t blows up somewhere. □

Now, we show that, if $u_{xx}(x, 0) \geq 0$ in $(0, a)$, namely u_0 is a lower solution then we have quenching point at $x = a$.

Theorem 2.7. *If u_0 is a lower solution, then there exist a finite time T, such that the solution u of the problem* (1.2) *quenches at time T.*

Proof. Assume that u_0 is a lower solution. Then, we obtain

$$\omega = (1 - u(a, 0))^{-q} - u^{-p}(0, 0) > 0.$$

Introduce a mass function $m(t) = \int_0^a (1 - u(x, t))dx$, $0 < t < T$. Then

$$m'(t) = -(1 - u(a, t))^{-q} + u^{-p}(0, t) \leq -\omega,$$

by Lemma 2.3 (i). Thus, $m(t) \leq m(0) - \omega t$; and so $m(T_0) = 0$ for some T_0, $(0 < T \leq T_0)$ which means u quenches in a finite time. □

Theorem 2.8. *If u_0 is a lower solution, then $x = a$ is the only quenching point.*

Proof. Since $u_x(0, t) = u^{-p}(0, t) > 1$ and $u_{xx} = u_t > 0$ in $(0, a) \times (0, T)$. Then, u_x is an increasing function and so, $u_x(x, t) > 1$ in $(0, a) \times (0, T)$. Let $\varepsilon \in (0, a)$. Integrating this with respect to x from $a - \varepsilon$ to a, we have

$$u(a - \varepsilon, t) < u(a, t) - \varepsilon < 1 - \varepsilon.$$

So u does not quench in $[0, a)$. □

Theorem 2.9. *If u_0 is a lower solution, then u_t blows up at quenching time.*

Proof. Suppose that u_t is bounded on $[0,1]\times[0,T)$. Then, there exists a positive constant M such that $u_t < M$. We have $u_{xx} < M$. Integrating this twice with respect to x from x to a, and then from 0 to a, we have

$$\frac{a}{(1-u(a,t))^q} < \frac{Ma^2}{2} + u(a,t) - u(0,t).$$

As $t \to T^-$, the left-hand side tends to infinity, while the right-hand side is finite. This contradiction shows that u_t blows up somewhere. □

Corollary 2.10. *We have the following results via Theorems 2.4–2.9:*

(i) If u_0 is an upper solution for the problem (1.2)*, then the solution u of the problem* (1.2) *quenches in a finite time, $x = 0$ is the only quenching point, and u_t blows up at quenching time.*

(ii) If u_0 is a lower solution for the problem (1.2)*, then the solution u of the problem* (1.2) *quenches in a finite time, $x = a$ is the only quenching point, and u_t blows up at quenching time.*

2.2. **Steady state.** The proof of the following lemma and theorem is analogous to that by Chan and Yuen [3, Section 3]. Let us consider the positive steady states of Problem (1.2),

$$U_{xx} = 0,\ U_x(0) = U^{-p}(0), \quad U_x(a) = (1-U(a))^{-q}. \tag{2.1}$$

We have $U = I + nx$, where

$$n = I^{-p}, \quad n = (1 - I - na)^{-q}.$$

From these, we have

$$U = I + I^{-p}x, \tag{2.2}$$

where

$$I^{-p} = (1 - I - I^{-p}a)^{-q},$$

which gives

$$a(I) = I^p(1-(I+I^{p/q})).$$

If we let $p = q$, then we obtain

$$a(I) = I^p(1-2I) = I^p - 2I^{p+1}. \tag{2.3}$$

Now, $a'(I) = 0$ implies

$$I = \frac{p}{2(p+1)}. \tag{2.4}$$

We note that $a(I) > 0$ for $0 < l < 1/2$. Since $a(0) = 0$ and $a(1/2) = 0$ and $a(I) > 0$, it follows from (2.4) that $\max_{0<l<1/2} a(I)$. We denote this value by A. From (2.3),

$$A = \frac{p^p}{2^p(p+1)^{p+1}}.$$

Lemma 2.11. *If $p = q$, then there is a solution u if and only if $0 < a \le A$. Furthermore, if $0 < a < A$, then there exist two positive solutions; if $a = A$, then there exists exactly one positive solution.*

Proof. Since $a(0) = 0 = a(1/2)$ and $a(I) > 0$ for $0 < l < 1/2$, the graph of $a(I)$ is concave downwards with maximum attained at A. Thus for $p = q$, the problem (3) has a solution if and only if $0 < a \le A$. To each $a \in (0, A)$, there are exactly two values of I. If $a = A$, then I is given by (2.4). □

Theorem 2.12. *If $p = q$ and $a \in (0, A)$, then u exists globally, provided $u_0 \le U(0)$.*

Proof. By Theorem 2.2, $u \le U$. Hence u exists globally. $\square$

3. Problem (1.1)

3.1. **Local solution.** It is well known that one of the most effective methods for obtaining existence and uniqueness of the solution of parabolic equations with initial conditions is monotone iterative techniques (for details see [4, 9]). For applications of monotone iterative techniques in quenching problem for a parabolic equation (see [2]).

Let $C^m(Q)$, $C^\alpha(Q)$ be the respective spaces of m-times differentiable and Hölder continuous functions in Q with exponent $\alpha \in (0,1)$, where Q is any domain. Denote by $C^{2,1}([0,a]\times[0,T))$ the set of functions that are twice continuously differentiable in x and continuously differentiable in t for $(x,t) \in [0,a]\times[0,T)$. It assumed that initial function $u_0(x)$ is in $C^{2+\alpha}$.

Definition 3.1. *A function $\widetilde{u}$ is called an upper solution of (1.1), if $\widetilde{u} \in C([0,a]\times[0,T)) \cap C^{2,1}((0,a)\times(0,T))$ and $\widetilde{u}$ satisfies the following conditions:*

$$\widetilde{u}_t - \widetilde{u}_{xx} \ge f(\widetilde{u}), \quad 0 < x < a,\ 0 < t < T,$$
$$\widetilde{u}_x(0,t) \le \widetilde{u}^{-p}(0,t), \quad \widetilde{u}_x(a,t) \ge (1-\widetilde{u}(a,t))^{-q}, \quad 0 < t < T,$$
$$\widetilde{u}(x,0) \ge u_0(x), \quad 0 \le x \le a\,.$$

A function $\widehat{u}$ is a lower solution of (1.1), if $\widehat{u} \in C([0,a]\times[0,T)) \cap C^{2,1}((0,a)\times(0,T))$, satisfies the reversing inequalities.

Lemma 3.2. *Let $\widetilde{u}$ and $\widehat{u}$ be a positive upper solution and a nonnegative lower solution of (1.1) in $[0,a]\times[0,T)$, respectively. Then, we obtain the following results:*

(a) *$\widetilde{u} \ge \widehat{u}$ in $[0,a]\times[0,T)$,*
(b) *if u^* is a solution, then $\widetilde{u} \ge u^* \ge \widehat{u}$ in $[0,a]\times[0,T)$.*

Proof. Let us prove it by utilizing [5, Lemma 3.1]. We select $f(v) = (1-v)^{-r}$ and we define $s(x,t) = \widetilde{u}(x,t) - \widehat{u}(x,t)$ in $[0,a]\times[0,T)$. Then $s(x,t)$ satisfies

$$s_t \ge s_{xx} + r(1-\eta)^{-r-1}s, \quad 0 < x < a,\ 0 < t < T,$$
$$s_x(0,t) \le -p\varphi^{-p-1}s(0,t), \quad s_x(a,t) \ge q(1-\xi(a,t))^{-q-1}s(a,t), \quad 0 < t < T,$$
$$s(x,0) \ge 0, \quad 0 \le x \le a,$$

where $\varphi(0,t)$ lies between $\widetilde{u}(0,t)$ and $\widehat{u}(0,t)$, $\eta(x,t)$ lies between $\widetilde{u}(x,t)$ and $\widehat{u}(x,t)$, and $\xi(a,t)$ lies between $\widetilde{u}(a,t)$ and $\widehat{u}(a,t)$.

For any fixed $\tau \in (0,T)$, let

$$L = \max_{0\le x\le a,\,0\le t\le\tau}\big(\frac{q}{2a}(1-\xi(x,t))^{-q-1}\big),$$
$$R = \max_{0\le x\le a,\,0\le t\le\tau}\big(\frac{p}{2a}\varphi^{-p-1}(x,t)\big),$$
$$M = 2L + 2R + \max_{0\le x\le a}(2Lx - 2R(a-x))^2 + \max_{0\le x\le a,\,0\le t\le\tau}(r(1-\eta(x,t))^{-r-1}).$$

Set $w(x,t) = e^{-Mt-Lx^2-R(a-x)^2}s(x,t)$. Then w satisfies

$$w_t \ge w_{xx} + (4Lx - 4R(a-x))w_x + cw, \quad 0 < x < a,\ 0 < t \le \tau,$$
$$w_x(0,t) \le kw(0,t), \quad w_x(a,t) \ge dw(a,t), \quad 0 < t \le \tau,$$

$$w(x,0) \geq 0, \quad 0 \leq x \leq a,$$

where $c = c(x,t) \leq 0$, $k = k(t) \geq 0$ and $d = d(t) \leq 0$. By the maximum principle and Hopf's lemma for parabolic equations, we obtain that $w \geq 0$ in $[0,a] \times [0,\tau]$. Thus, $\widetilde{u} \geq \widehat{u}$ in $[0,a] \times [0,T)$. For $f(v) = -v^{-r}$, a similar process follows.

(b) It is clear from Definition 3.1 that every solution of the problem (1.1) is an upper solution as well as a lower solution of the corresponding problem. If u^* is a solution, then we obtain

$$\widetilde{u} \geq u^*, \quad u^* \geq \widehat{u}, \quad \widetilde{u} \geq u^* \geq \widehat{u}$$

in $[0,a] \times [0,T)$ from Lemma 3.2 (a). □

For a given pair of ordered upper and lower solutions $\widetilde{u}$ and $\widehat{u}$ we set

$$S = \{u \in C([0,a] \times [0,T)) : \widehat{u} \leq u \leq \widetilde{u}\}.$$

Let

$$f(x,t,u(x,t)) = (1-u(x,t))^{-r} \quad \text{or} \quad f(x,t,u(x,t)) = -u^{-r}(x,t),$$
$$g(x,t,u(x,t)) = u^{-p}(x,t),\ h(x,t,u(x,t)) = (1-u(x,t))^{-q}$$

Throughout this section we assume the following hypothesis on the functions in Problem (1.1):

(H1) (i) The functions $f(x,t,\cdot)$ is in $C^{\alpha,\alpha/2}([0,a] \times [0,T))$, $g(x,t,.)$ is in $C^{1+\alpha,(1+\alpha)/2}(\{0\} \times (0,T))$ and $h(x,t,.)$ is in $C^{1+\alpha,(1+\alpha)/2}(\{a\} \times (0,T))$, respectively.

(ii) Let $f(.,u)$, $g(.,u)$ and $h(.,u)$ are C^1-functions of $u \in S$. Also,

$$\begin{aligned} f_u(x,t,u) \geq 0 \quad &\text{for } u \in S, (x,t) \in [0,a] \times [0,T), \\ g_u(x,t,u) \leq 0 \quad &\text{for } u \in S, (x,t) \in \{0\} \times (0,T), \\ h_u(x,t,u) \geq 0 \quad &\text{for } u \in S, (x,t) \in \{a\} \times (0,T). \end{aligned} \tag{3.1}$$

Condition (3.1) implies that $f(.,u)$ and $h(.,u)$ are non-decreasing in u, $g(.,u)$ is non-increasing in u, respectively, which is crucial for the construction of monotone sequences.

Next, we construct monotone sequences of functions which give the estimation of the solution u of problem (1.1). Specifically, by starting from any initial iteration u^0, we can construct a sequence $\{u^{(k)}\}$ from the linear iteration process

$$\begin{gathered} u_t^{(k)} - u_{xx}^{(k)} = f(x,t,u^{(k-1)}), \quad 0 < x < a,\ 0 < t < T, \\ u_x^{(k)}(0,t) = g(0,t,u^{(k-1)}), \quad u_x^{(k)}(a,t) = h(a,t,u^{(k-1)}), \quad 0 < t < T, \\ u^{(k)}(x,0) = u_0(x), \quad 0 \leq x \leq a. \end{gathered} \tag{3.2}$$

It is clear that the sequence governed by (3.2) is well defined and can be obtained by solving a linear initial boundary value problem. Starting from initial iteration $u^0 = \widetilde{u}$ and $u^0 = \widehat{u}$, we define two sequences of the functions $\{\overline{u}^{(k)}\}$ and $\{\underline{u}^{(k)}\}$ for $k = 1,2,\dots$ respectively, and refer to them as maximal and minimal sequences, respectively, where those elements satisfy the above linear problem.

Lemma 3.3. *The sequences $\{\overline{u}^{(k)}\}$, $\{\underline{u}^{(k)}\}$ possess the monotone property*

$$\widehat{u} \leq \underline{u}^{(k)} \leq \underline{u}^{(k+1)} \leq \overline{u}^{(k+1)} \leq \overline{u}^{(k)} \leq \widetilde{u}$$

for $(x,t) \in [0,a] \times [0,T)$ and every $k = 1,2,\dots$.

Proof. Let $\mu = \widetilde{u} - \overline{u}^{(1)}$. From (3.2) and from Definition 3.1, we obtain

$$\begin{gathered}
\mu_t - \mu_{xx} = \widetilde{u}_t - \widetilde{u}_{xx} - f(x,t,\widetilde{u}) \geq 0, \quad 0 < x < a,\ 0 < t < T, \\
\mu_x(0,t) = \widetilde{u}_x(0,t) - g(0,t,\widetilde{u}) \leq 0, \quad 0 < t < T, \\
\mu_x(a,t) = \widetilde{u}_x(a,t) - h(a,t,\widetilde{u}) \geq 0, \quad 0 < t < T, \\
\mu(x,0) = \widetilde{u}(x,0) - u_0(x) \geq 0, \quad 0 \leq x \leq a.
\end{gathered}$$

From the Maximum principle and Hopf's Lemma for parabolic equations, we obtain $\mu \geq 0$ for $(x,t) \in [0,a] \times [0,T)$, i.e. $\overline{u}^{(1)} \leq \widetilde{u}$. Similarly, using the property of a lower solution, we obtain $\underline{u}^{(1)} \geq \widehat{u}$.

Let $\mu^{(1)} = \overline{u}^{(1)} - \underline{u}^{(1)}$. From (3.1) and (3.2), we obtain

$$\begin{gathered}
\mu_t^{(1)} - \mu_{xx}^{(1)} = f(x,t,\widetilde{u}) - f(x,t,\widehat{u}) \geq 0, \quad 0 < x < a,\ 0 < t < T, \\
\mu_x^{(1)}(0,t) = g(0,t,\widetilde{u}) - g(0,t,\widehat{u}) \leq 0, \quad 0 < t < T, \\
\mu_x^{(1)}(a,t) = h(a,t,\widetilde{u}) - h(a,t,\widehat{u}) \geq 0, \quad 0 < t < T, \\
\mu^{(1)}(x,0) = u_0(x) - u_0(x) = 0, \quad 0 \leq x \leq a.
\end{gathered}$$

From the Maximum principle and Hopf's Lemma for parabolic equations, we obtain $\mu^{(1)} \geq 0$ for $(x,t) \in [0,a] \times [0,T)$, i.e. $\underline{u}^{(1)} \leq \overline{u}^{(1)}$. Therefore,

$$\widehat{u} \leq \underline{u}^{(1)} \leq \overline{u}^{(1)} \leq \widetilde{u}$$

for $(x,t) \in [0,a] \times [0,T)$.

Assume that

$$\underline{u}^{(k-1)} \leq \underline{u}^{(k)} \leq \overline{u}^{(k)} \leq \overline{u}^{(k-1)}$$

for $(x,t) \in [0,a] \times [0,T)$ and for some integer $k > 1$. Let $\mu^{(k)} = \overline{u}^{(k)} - \overline{u}^{(k+1)}$. From (3.1) and (3.2), we obtain

$$\begin{gathered}
\mu_t^{(k)} - \mu_{xx}^{(k)} = f(x,t,\overline{u}^{(k-1)}) - f(x,t,\overline{u}^{(k)}) \geq 0, \quad 0 < x < a,\ 0 < t < T, \\
\mu_x^{(k)}(0,t) = g(0,t,\overline{u}^{(k-1)}) - g(0,t,\overline{u}^{(k)}) \leq 0, \quad 0 < t < T, \\
\mu_x^{(k)}(a,t) = h(a,t,\overline{u}^{(k-1)}) - h(a,t,\overline{u}^{(k)}) \geq 0, \quad 0 < t < T, \\
\mu^{(k)}(x,0) = 0, \quad 0 \leq x \leq a.
\end{gathered}$$

From the Maximum principle and Hopf's Lemma for parabolic equations, we obtain $\mu^{(k)} \geq 0$ for $(x,t) \in [0,a] \times [0,T)$, i.e. $\overline{u}^{(k+1)} \leq \overline{u}^{(k)}$. A similar argument gives $\underline{u}^{(k+1)} \geq \underline{u}^{(k)}$ and $\overline{u}^{(k+1)} \geq \underline{u}^{(k+1)}$. Therefore, the result follows from the mathematical induction. □

Lemma 3.4. *For each positive integer k, $\overline{u}^{(k)}$ is an upper solution, $\underline{u}^{(k)}$ is a lower solution, $\underline{u}^{(k)} \leq \overline{u}^{(k)}$ for $(x,t) \in [0,1] \times [0,T)$.*

Proof. From (3.1), (3.2) and Lemma 3.2, $\overline{u}^{(k)}$ satisfies

$$\begin{aligned}
\overline{u}_t^{(k)} - \overline{u}_{xx}^{(k)} &= f(x,t,\overline{u}^{(k-1)}) \\
&= f(x,t,\overline{u}^{(k-1)}) - f(x,t,\overline{u}^{(k)}) + f(x,t,\overline{u}^{(k)}) \geq f(x,t,\overline{u}^{(k)}), \\
\overline{u}_x^{(k)}(0,t) &= g(0,t,\overline{u}^{(k-1)}) \\
&= g(0,t,\overline{u}^{(k-1)}) - g(0,t,\overline{u}^{(k)}) + g(0,t,\overline{u}^{(k)}) \leq g(0,t,\overline{u}^{(k)}),
\end{aligned}$$

$$\begin{aligned}
\overline{u}_x^{(k)}(a,t) &= h(a,t,\overline{u}^{(k-1)}) \\
&= h(a,t,\overline{u}^{(k-1)}) - h(a,t,\overline{u}^{(k)}) + h(a,t,\overline{u}^{(k)}) \geq h(a,t,\overline{u}^{(k)}), \\
\overline{u}^{(k)}(x,0) &= u_0(x), \quad 0 \leq x \leq a,
\end{aligned}$$

and $\underline{u}^{(k)}$ satisfies

$$\begin{aligned}
\underline{u}_t^{(k)} - \underline{u}_{xx}^{(k)} &= f(x,t,\underline{u}^{(k-1)}) \\
&= f(x,t,\underline{u}^{(k-1)}) - f(x,t,\underline{u}^{(k)}) + f(x,t,\underline{u}^{(k)}) \leq f(x,t,\underline{u}^{(k)}), \\
\underline{u}_x^{(k)}(0,t) &= g(0,t,\underline{u}^{(k-1)}) \\
&= g(0,t,\underline{u}^{(k-1)}) - g(0,t,\underline{u}^{(k)}) + g(0,t,\underline{u}^{(k)}) \geq g(0,t,\underline{u}^{(k)}), \\
\underline{u}_x^{(k)}(a,t) &= h(a,t,\underline{u}^{(k-1)}) \\
&= h(a,t,\underline{u}^{(k-1)}) - h(a,t,\underline{u}^{(k)}) + h(a,t,\underline{u}^{(k)}) \leq h(a,t,\underline{u}^{(k)}), \\
\underline{u}^{(k)}(x,0) &= u_0(x), 0 \leq x \leq a.
\end{aligned}$$

From Lemma 3.3 and from the above inequalities, the functions $\overline{u}^{(k)}$ and $\underline{u}^{(k)}$ are ordered upper and lower solutions of problem (3.2). $\square$

We have the following existence theorem for problem (1.1) via Lemmas 3.3 and 3.4.

Theorem 3.5. *Let $\widetilde{u}, \widehat{u}$ be a pair of ordered upper and lower solutions of the problem* (1.1)*, and let Hypothesis (H1) hold. Then the sequences $\{\overline{u}^{(k)}\}$, $\{\underline{u}^{(k)}\}$ given by Problem* (3.2) *with $u^0 = \widetilde{u}$ and $u^0 = \widehat{u}$ converge monotonically to a maximal solution $\overline{u}$ and minimal solution $\underline{u}$ of the problem* (1.1)*, respectively. Further,*

$$\widehat{u} \leq \underline{u}^{(k)} \leq \underline{u}^{(k+1)} \leq \underline{u} \leq \overline{u} \leq \overline{u}^{(k+1)} \leq \overline{u}^{(k)} \leq \widetilde{u} \tag{3.3}$$

for $(x,t) \in [0,a] \times [0,T)$ and each positive integer k. Furthermore if $\underline{u} = \overline{u}$ $(\equiv u^*)$, then u^* is the unique solution of the problem (1.1) *in S.*

Proof. The pointwise limits

$$\lim_{k\to\infty} \overline{u}^{(k)}(x,t) = \overline{u}(x,t), \quad \lim_{k\to\infty} \underline{u}^{(k)}(x,t) = \underline{u}(x,t)$$

exist and satisfy the relation (3.3). Indeed, the sequence $\{\overline{u}^{(k)}\}$ is monotone nonincreasing which is bounded from below, while the sequence $\{\underline{u}^{(k)}\}$ is monotone nondecreasing and is bounded from above as in Lemma 3.3.

Let $\Theta = \underline{u}(x,t) - \overline{u}(x,t)$. From (3.3), we have $\underline{u}(x,t) \leq \overline{u}(x,t)$ for $(x,t) \in [0,a] \times [0,T)$. Also, $\Theta(x,t)$ satisfies

$$\begin{gathered}
\Theta_t - \Theta_{xx} = f(x,t,\underline{u}) - f(x,t,\overline{u}), \quad 0 < x < a,\ 0 < t < T, \\
\Theta_x(0,t) = g(0,t,\underline{u}) - g(0,t,\overline{u}), \quad 0 < t < T, \\
\Theta_x(1,t) = h(a,t,\underline{u}) - h(a,t,\overline{u}), \quad 0 < t < T, \\
\Theta(x,0) = 0, \quad 0 \leq x \leq a.
\end{gathered}$$

By using the process of Lemma 3.2 (a) and Lemma 3.6, we obtain $\Theta \geq 0$ for $(x,t) \in [0,a] \times [0,T)$, i.e. $\underline{u}(x,t) \geq \overline{u}(x,t)$, and so, we obtain $\underline{u}(x,t) = \overline{u}(x,t)$.

If u^* is any other solution in S, then from Lemma 3.4 we obtain

$$\begin{gathered}
\overline{u} \geq u^*, \quad u^* \geq \underline{u}, \\
\overline{u} \geq u^* \geq \underline{u}
\end{gathered}$$

in $[0, a] \times [0, T)$. This implies that

$$\overline{u} = u^* = \underline{u}$$

and hence u^* is the unique solution of the problem (1.1). $\square$

3.2. **Quenching on the boundary.** In this subsection, we study quenching properties of the problem (1.1) via Section 2.1.

Lemma 3.6. *(i) $(f(v) = (1-v)^{-r})$ If $v_{xx}(x,0) + (1-v(x,0))^{-r} \geq 0$ in $(0,a)$ (i.e., if v_0 is a lower solution), then $v_t(x,t) \geq 0$ in $[0,a] \times [0,T)$. Also, we obtain $v_t(x,t) > 0$ in$(0,a) \times [0,T)$ by strong maximum principle.*

(ii) $(f(v) = -v^{-r})$ If $v_{xx}(x,0) - v^{-r}(x,0) \leq 0$ in $(0,a)$ (i.e., if v_0 is an upper solution), then $v_t(x,t) \leq 0$ in $[0,a] \times [0,T)$. Also, we obtain $v_t(x,t) < 0$ in$(0,a) \times [0,T)$ by the strong maximum principle.

Proof. (i) Let us prove it by using [5, Lemma 3.1]. We let $f(v) = (1-v)^{-r}$ and we define $s(x,t) = v_t(x,t)$ in $[0,a] \times [0,T)$. Then $s(x,t)$ satisfies

$$\begin{gathered}
s_t = s_{xx} + r(1-v)^{-r-1}s, \quad 0 < x < a,\ 0 < t < T, \\
s_x(0,t) = -pv^{-p-1}s(0,t), \quad s_x(a,t) = q(1-v(a,t))^{-q-1}s(a,t), \quad 0 < t < T, \\
s(x,0) = v_{xx}(x,0) + (1-v(x,0))^{-r} \geq 0, \quad 0 \leq x \leq a.
\end{gathered}$$

For a fixed $\tau \in (0,T)$, let

$$\begin{gathered}
L = \max_{0 \leq x \leq a,\, 0 \leq t \leq \tau} \Big(\frac{q}{2a}(1-v(x,t))^{-q-1}\Big), \\
R = \max_{0 \leq x \leq a,\, 0 \leq t \leq \tau} \Big(\frac{p}{2a}v^{-p-1}(x,t)\Big), \\
M = 2L + 2R + \max_{0 \leq x \leq a} (2Lx - 2R(a-x))^2 + \max_{0 \leq x \leq a,\, 0 \leq t \leq \tau} (r(1-\eta(x,t))^{-r-1}).
\end{gathered}$$

Set $w(x,t) = e^{-Mt - Lx^2 - R(a-x)^2} s(x,t)$. Then w satisfies

$$\begin{gathered}
w_t = w_{xx} + (4Lx - 4R(a-x))w_x + cw, \quad 0 < x < a,\ 0 < t \leq \tau, \\
w_x(0,t) = kw(0,t), \quad w_x(a,t) = dw(a,t),\ 0 < t \leq \tau, \\
w(x,0) = 0, \quad 0 \leq x \leq a,
\end{gathered}$$

where $c = c(x,t) \leq 0$, $k = k(t) \geq 0$ and $d = d(t) \leq 0$. By the maximum principle and Hopf's lemma for parabolic equations, we obtain that $w \geq 0$ in $[0,a] \times [0,\tau]$. Thus, $v_t \geq 0$ in $[0,a] \times [0,T)$. Also, we obtain $v_t(x,t) > 0$ in $(0,a) \times [0,T)$ by the strong maximum principle.

(ii) Now, if we let $f(v) = -v^{-r}$, and $v_{xx}(x,0) - v^{-r}(x,0) \leq 0$ in $(0,a)$, then using the same process above, we obtain $v_t(x,t) \leq 0$ in $[0,a] \times [0,T)$. Also, we obtain $v_t(x,t) < 0$ in $(0,a) \times [0,T)$ by the strong maximum principle. $\square$

Lemma 3.7. *If $v_x(x,0) \geq 0$, then $v_x \geq 0$ in $[0,a] \times (0,T)$.*

Proof. Let $H = v_x(x,t)$. Then

$$\begin{gathered}
H_t = H_{xx} + f'(v)H, \quad 0 < x < a,\ 0 < t < T, \\
H(0,t) = v^{-p}(0,t) > 0, \quad H(a,t) = (1-v(a,t))^{-q} > 0,\ 0 < t < T, \\
H(x,0) = v_x(x,0) \geq 0, \quad 0 \leq x \leq a.
\end{gathered}$$

From the maximum principle, it follows that $H \geq 0$ and hence $v_x \geq 0$, in $[0,a] \times (0,T)$. $\square$

Theorem 3.8. *(i)* $(f(v) = -v^{-r})$ *If* $u_0(x) \geq v_0(x)$ *and* u_0 *is an upper solution for problem* (1.2)*, then the solution* v *of problem* (1.1) *quenches in a finite time and* $x = 0$ *is the only quenching point.*

(ii) $(f(v) = (1-v)^{-r})$ *If* $u_0(x) \leq v_0(x)$ *and* u_0 *is a lower solution for the problem* (1.2)*, then the solution* v *of problem* (1.1) *quenches in a finite time and* $x = a$ *is the only quenching point.*

Proof. (i) First, let $f(v) = -v^{-r}$. If $u_0(x) \geq v_0(x)$, then the solution u of the problem (1.2) is an upper solution of (1.1) from Definition 3.1. Further, if $u_0(x)$ is an upper solution for the problem (1.2), then u quenches in a finite time, $\lim_{t\to T^-} u(0,t) \to 0$ from Corollary 3.10 (i). So, we obtain

$$u_0 \geq u \geq v$$

from Lemma 3.2. Thus, v quenches in a finite time, $\lim_{t\to T^-} v(0,t) \to 0$.

(ii) Now, let $f(v) = (1-v)^{-r}$. If $u_0(x) \leq v_0(x)$, then the solution u of (1.2) is a lower solution of (1.1) from Definition 3.1. Further, if $u_0(x)$ is a lower solution of (1.2), then u quenches in a finite time, $\lim_{t\to T^-} u(a,t) \to 1$ from Corollary 3.10 (ii). So, we obtain

$$u_0 \leq u \leq v$$

from Lemma 3.2. As a result, v quenches in a finite time, $\lim_{t\to T^-} v(a,t) \to 1$. $\square$

Theorem 3.9. *(i)* $(f(v) = -v^{-r})$ v_t *blows up at the quenching time at the boundary* $x = 0$.

(ii) $(f(v) = (1-v)^{-r})v_t$ *blows up at the quenching time at the boundary* $x = a$.

Proof. (i) $(f(v) = -v^{-r})$ Suppose that v_t is bounded on $[0,a] \times [0,T)$. Then, there exists a positive constant M such that $v_t > -M$. That is

$$v_{xx} - v^{-r} > -M.$$

Multiplying this inequality by v_x, and integrating with respect to x from 0 to x, we have

$$-\frac{1}{2}v^{-2p}(0,t) - \ln\big[\frac{1}{v(0,t)}\big] > \frac{1}{2}v_x^2 - \ln\big[\frac{1}{v(x,t)}\big] - M[v(a,t) - v(x,t)]$$

for $r = 1$, and

$$-\frac{1}{2}v^{-2p}(0,t) + \frac{v^{-r+1}(0,t)}{-r+1} > -\frac{1}{2}v_x^2 + \frac{v^{-r+1}(x,t)}{-r+1} - M[v(x,t) - v(0,t)]$$

for $r \neq 1$. We have, as $t \to T^-$, the left-hand side tends to negative infinity, while the right-hand side is finite. This contradiction shows that v_t blows up at the quenching point $x = 0$.

(ii) $(f(v) = (1-v)^{-r})$ Suppose that v_t is bounded on $[0,a] \times [0,T)$. Then, there exists a positive constant M such that $v_t < M$. That is,

$$v_{xx} + (1-v)^{-r} < M.$$

Multiplying this inequality by v_x, and integrating with respect to x from x to a, we have

$$\frac{1}{2}(1 - v(a,t))^{-2q} + \ln[\frac{1}{1 - v(a,t)}] < \frac{1}{2}v_x^2 + \ln[\frac{1}{1 - v(x,t)}] + M[v(a,t) - v(x,t)]$$

for $r = 1$, and

$$\frac{1}{2}(1-v(a,t))^{-2q} + \frac{(1-v(a,t))^{-r+1}}{r-1} < \frac{1}{2}v_x^2 + \frac{(1-v(x,t))^{-r+1}}{r-1} + M[v(a,t)-v(x,t)]$$

for $r \neq 1$. As $t \to T^-$, the left-hand side tends to infinity, while the right-hand side is finite. Hence, v_t blows up at the quenching point $x = a$. $\square$

Corollary 3.10. *We have the following results via Theorems 3.8 and 3.9:*

(i) $(f(v) = -v^{-r})$ *If* $u_0(x) \geq v_0(x)$ *and* u_0 *is an upper solution for the problem* (1.2)*, then the solution* v *of the problem* (1.1) *quenches in a finite time,* $x = 0$ *is the only quenching point, and* v_t *blows up at the quenching time.*

(ii) $(f(v) = (1-v)^{-r})$ *If* $u_0(x) \leq v_0(x)$ *and* u_0 *is a lower solution for the problem* (1.2)*, then the solution* v *of the problem* (1.1) *quenches in a finite time,* $x = a$ *is the only quenching point, and* v_t *blows up at the quenching time.*

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