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MATHEMATICAL MODEL FOR A MEMBRANE BIOREACTOR PROCESS

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Abstract. In this article, we consider a simple mathematical model involving biomass growth on organic materials in a membrane bioreactor for a waste water treatment. Details of qualitative analysis are provided. We proposed a high gain observer that permits the reconstruction of the biomass concentration and the endegenous decay based on on-line measurements of the chemical oxygen demand (CDO).

1. Introduction

Membrane bioreactors (MBRs) is a combination of a membrane process as microfiltration or ultrafiltration with a suspended growth bioreactor to treat waste water where the main focus is to reduce the chemical oxygen demand (COD) in the effluent discharged to natural waters. Membrane bioreactor process for the sludge retention and separation from the liquid has been one of the alternatives to the conventional activated sludge process (Figure 1). Two MBR configuration are possible, the submerged MBR, where the membrane is placed in the reactionnal medium, and the side stream MBR, where the membrane is out of the reactionnal medium. The amount of organic pollutants found in surface water, determined by COD measurements, gives us an idea on the water quality. The excess bacteria grown in the system are removed as sludge and this causes high costs.

In this study, we used a submerged BRM configuration (Figure 1, center). Assume that soluble COD in mixed liquor is equal to the effluent COD because the submerged membranes used in MBR don't remove dissolved materials (used membranes are mostly micro- or ultrafilters). Additionally, we assume that all organic material in feed solution are soluble. We neglect fouling phenomenon during membrane separation process.

The features of this article are the following:

• A simple mathematical model involving biomass growth on organic materials in a membrane bioreactor for a waste water treatment is proposed. Details of qualitative analysis are provided.

• A high gain observer is proposed that permits the reconstruction of the biomass concentration and the endegenous decay based on on-line measurements of the chemical oxygen demand (CDO).

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• Simulations are used to validate theoretical results provided above, and finally, concluding remarks are given.

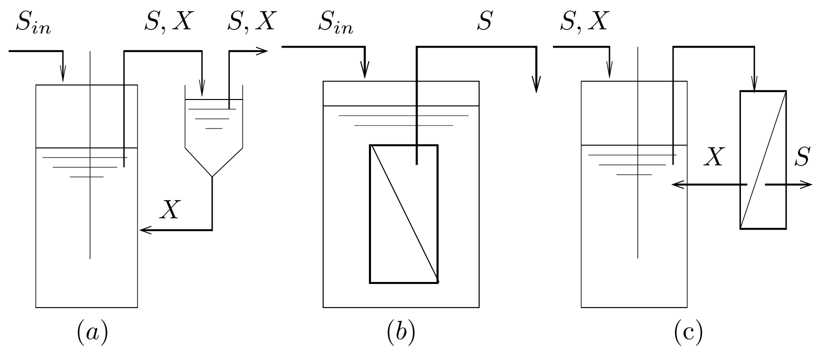

FIGURE 1. Membrane Bio-Reactor : (a) Conventional activated suldge process, (b) Submerged MBR configuration with integrated membrane unit, (c) Side stream MBR configuration with external membrane unit.

2. Mathematical model and results

Let S denote the soluble CDO and let X denote the total microorganisms present in the bioreactor at time t. The following ordinary system of differential equations describe the growth of total microorganisms on soluble CDO:

$$\begin{gathered}\dot{S} = D(S_{in} - S) - \frac{\mu(S)}{Y}X\,, \\ \dot{X} = (\mu(S) - m)X\,.\end{gathered} \tag{2.1}$$

S_{in} denotes the effluent CDO in the feed, Y is the conversion factor of COD converted to biomass, D denotes the dilution rate through the membrane and m is the endogenous decay constant.

Note that equation which describes the biomass growth is similar to batch culture [1] however the substrate equation is similar to classic continuous reactor [3]. It appears reasonable to assume that the endogenous decay constant m is smaller that the dilution rate D and that a priory bounds on parameter m are known. We use the following assumptions:

(A1) There exists constants m^- and m^+ such that $0 < m^- \le m \le m^+ < D$.

(A2) The growth function $\mu(\cdot)$ is a smooth increasing function such that $\mu(0) = 0$.

Let $x = \frac{X}{Y}$, $s = S$ and $s_{in} = S_{in}$. Then one obtains

$$\begin{gathered}\dot{s} = D(s_{in} - s) - \mu(s)x, \\ \dot{x} = \mu(s)x - mx,\end{gathered} \tag{2.2}$$

with positive initial condition $\big(s(0), x(0)\big) \in \mathbb{R}_+ \times \mathbb{R}_+$.

General properties.

Proposition 2.1. *(1) For any initial condition in $\mathbb{R}_+ \times \mathbb{R}_+$, the solution of* (2.2) *is bounded and has positive components and thus is defined for all $t > 0$.*

(2) System (2.2) *admits a positive invariant attractor set of solution given by* $\Omega = \big\{(s, x) \in \mathbb{R}_+ \times \mathbb{R}_+ : s + x \le \frac{D}{m}s_{in}\big\}$.

Proof. (1) The positivity of the solution is guaranteed by the fact that If $s = 0$ then $\dot{s} = Ds_{in} > 0$ and if $x = 0$ then $\dot{x} = 0$. Next we have to prove the boundedness of solutions of (2.2). By adding the two equations of system (2.2), one obtains, for $z = s + x - \frac{D}{m}s_{in}$, a single equation

$$\dot{z} \leq -m\big(s + x - \frac{D}{m}s_{in}\big) = -mz$$

then

$$0 \leq s + x \leq \frac{D}{m}s_{in} + Ke^{-mt} \quad \text{where} \quad K = z(0) = s(0) + x(0) - \frac{D}{m}\, s_{in}.$$

Since all terms of the sum are positive, then the solution is bounded

Part (2) is a direct consequence of the previous inequality. □

Stability. Let s^* be a solution of $\mu(s) = m$ and $x^* = \frac{D}{m}(s_{in} - s^*)$. The equilibrium points of system (2.2) are

$$F_0 = (s_{in}, 0) \quad \text{and} \quad F^* = (s^*, x^*).$$

Note that the trivial equilibrium point F_0 always exists and that F^* exists if and only if $\mu(s_{in}) > m$.

Proposition 2.2. (1) *There are no periodic orbits nor polycycles inside* Ω.

(2) *If* $\mu(s_{in}) > m$, F_0 *is a saddle point and* F^* *is globally asymptotically stable.*

(3) *If* $\mu(s_{in}) < m$, F_0 *is globally asymptotically stable.*

Proof. (1) Consider a trajectory of system (2.2) belonging to Ω. Let us transform the system (2.2) through the change of variables $\xi_1 = s$ and $\xi_2 = \ln(x)$. Then one obtains the system

$$\begin{gathered}\dot{\xi}_1 = h_1(\xi_1, \xi_2) := D(s_{in} - \xi_1) - \mu(\xi_1)e^{\xi_2},\\ \dot{\xi}_2 = h_2(\xi_1, \xi_2) := \mu(\xi_1) - m\,.\end{gathered} \tag{2.3}$$

We have

$$\frac{\partial h_1}{\partial \xi_1} + \frac{\partial h_2}{\partial \xi_2} = -D - \mu'(\xi_1)\, e^{\xi_2} < 0.$$

From Dulac criterion [3], we deduce that system (2.3) has no periodic trajectory. Hence system (2.2) has no periodic orbit inside Ω.

(2) Assume that $\mu(s_{in}) > m$. The Jacobian matrix J^* of system (2.2) at (s^*, x^*) is

$$J^* = \begin{bmatrix} -D - \mu'(s^*)x^* & -m \\ \mu'(s^*)x^* & 0 \end{bmatrix}.$$

One can easily verify that

$$\operatorname{tr}(J^*) = -D - \mu'(s^*)x^* < 0, \quad \det(J^*) = m\mu'(s^*)x^* > 0,$$

from where F^* is a stable node. The Jacobian matrix J_0 of system (2.2) at $(s_{in}, 0)$ is

$$J_0 = \begin{bmatrix} -D & 0 \\ 0 & \mu(s_{in}) - m \end{bmatrix}$$

One can easily verify that F_0 is a saddle point since $-D < 0$ and $\mu(s_{in}) - m > 0$. In this case $\Gamma_0 =]0, +\infty[\times\{0\}$ is the stable manifold of the saddle point F_0.

Let $s(0) \geq 0$ and $x(0) > 0$. System (2.2) has no periodic orbit inside Ω. Using the Poincaré-Bendixon Theorem [3], F^* is a globally asymptotically stable equilibrium point [5].

(3) If $\mu(s_{in}) < m$, then (2.2) admits F_0 as the only equilibrium point which is locally stable. As the omega limit set of any trajectory have to be in the 2D compact and positively invariant set Ω, and since F_0 lies on the boundary of Ω, F_0 must be globally asymptotically stable by the Poincaré-Bendixson Theorem [4]. $\square$

Remark 2.3. s^* is independent on the dilution rate D, contrarily to the classical continuous reactor (chemostat) [3]. Strict regulations regarding the maximum chemical oxygen demand allowed in wastewater before they can be returned to the environment are imposed by many governments. As COD at steady state (s^*) depends on the endegenous decay (m) then it follows that the used bacteria must have minimal decay constant.

In the following we assume that $\mu(s_{in}) > m$.

Observability. For the rest of this article, we shall use assumptions on the growth function $\mu(\cdot)$ and the yield coefficient Y of the classical Monod's system.

(A3) $\mu(\cdot)$ and Y are known.

Our aim is to estimate on-line both parameter m and unmeasured variable x, based on the measurements of CDO (s). Our system is not observable if $s = 0$ and/or $x = 0$ that is why we proposed a set on which we are in the ideal situation where the system is observable. We considering the set

$$\Omega = \big\{(s,x) \in \mathbb{R}_+^2 : s > 0,\ x > 0,\ s + x < \frac{D}{m} s_{in}\big\}$$

and deduced the following result.

Proposition 2.4. *Dynamics* (2.2) *leaves the domain* Ω *positively invariant.*

Letting $(s(0), x(0)) \in \bar{\Omega}$ and considering the state vector

$$\xi = \begin{bmatrix} s & \dot{s} & \ddot{s} \end{bmatrix}^T = \begin{bmatrix} \xi_1 & \xi_2 & \xi_3 \end{bmatrix}^T ,$$

one obtains the dynamics

$$\begin{gathered} \dot{\xi} = A\xi + \begin{pmatrix} 0 \\ 0 \\ \varphi(y,\xi) \end{pmatrix} \\ y = C\xi \end{gathered}$$

with

$$\begin{aligned} \varphi(y,\xi) = \Big(\xi_2 - D(s_{in} - y)\Big)\Big[\Big(\mu''(y)\mu(y) - (\mu'(y))^2\Big)\frac{\xi_2^2}{\mu^2(y)} + \frac{\mu'(y)}{\mu(y)}\xi_3 + \mu'(y)\xi_2\Big] \\ - D\xi_3 + \frac{(\xi_3 + D\xi_2)^2}{\xi_2 - D(s_{in} - y)} \,. \end{aligned}$$

and the pair (A, C) in the Brunovsky's canonical form

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}.$$

The unknown parameter m and the unknown state variable x are then made explicit as functions of the state vector ξ:

$$m = l_m(y,\xi) = \mu(y) + \frac{\mu'(y)}{\mu(y)}\xi_2 + \frac{D\,\xi_2 + \xi_3}{-\xi_2 + D(s_{in} - y)} \,,$$

$$x = l_x(y,\xi) = \frac{-\xi_2 + D(s_{in} - y)}{\mu(y)}.$$

One can notice that functions $\varphi(y,\cdot)$ and $l_m(y,\cdot)$ are not well defined on $\mathbb{R}^3$, but using the fact that $m^- \le m \le m^+$, we can consider (globally) Lipschitz extension of function $l_m(y,\cdot)$ (and then $\varphi(y,\cdot)$) away from the trajectories of the system, as follows:

$$\begin{aligned}
\tilde{l}_m(y,\xi) &= \max\Big(m^-, \min\Big(m^+, \mu(y) + \frac{\mu'(y)}{\mu(y)}\xi_2 + \frac{D\xi_2 + \xi_3}{-\xi_2 + D(s_{in} - y)}\Big)\Big),\\
\tilde{\varphi}(y,\xi) &= \Big(\xi_2 - D(s_{in} - y)\Big)\Big[\Big(\mu''(y)\mu(y) - (\mu'(y))^2\Big)\frac{\xi_2^2}{\mu^2(y)} + \frac{\mu'(y)}{\mu(y)}\xi_3 + \mu'(y)\xi_2\Big]\\
&\quad - D\xi_3 + \Big(\xi_3 + D\xi_2\Big)\Big(\tilde{l}_m(y,\xi) - \mu(y) - \frac{\mu'(y)}{\mu(y)}\xi_2\Big).
\end{aligned}$$

Then one obtains a construction of a high gain observer.

Proposition 2.5. *There exist numbers $a > 0$ and $b > 0$ such that the observer*

$$\begin{gathered}
\dot{\hat{\xi}} = A\hat{\xi} + \begin{bmatrix} 0 \\ 0 \\ \tilde{\varphi}(y,\hat{\xi}) \end{bmatrix} - \begin{bmatrix} 3\theta \\ 3\theta^2 \\ \theta^3 \end{bmatrix}(\hat{\xi}_1 - y),\\
(\hat{m}, \hat{x}) = \big(\tilde{l}_m(y,\hat{\xi}) l_x(y,\hat{\xi})\big)
\end{gathered} \tag{2.4}$$

guarantees the convergence

$$\max\Big(|\hat{m}(t) - m|, |\hat{x}(t) - x(t)|\Big) \le a e^{-b\theta t}\|\hat{\xi}(0) - \xi(0)\| \tag{2.5}$$

for any θ large enough and $t \ge 0$.

Proof. Consider a trajectory of dynamics (2.2). Define $K_\theta = -\begin{bmatrix} 3\theta & 3\theta^2 & \theta^3 \end{bmatrix}^T$. One can check that $K_\theta = -P_\theta^{-1}C^T$, where P_θ is solution of the algebraic equation

$$\theta P_\theta + A^T P_\theta + P_\theta A = C^T C.$$

Let $e = \hat{\xi} - \xi$ be the error vector. One has

$$\dot{e} = (A + K_\theta C)e + \begin{bmatrix} 0 \\ 0 \\ \tilde{\varphi}(y,\hat{\xi}) - \tilde{\varphi}(y,\xi) \end{bmatrix}$$

where $\tilde{\varphi}(y,\cdot)$ is (globally) Lipschitz on $\mathbb{R}^3$. Using the result in [2], there exists two constants $\alpha > 0$ and $\beta > 0$ such that $\|e(t)\| \le \alpha e^{-\beta\theta t}\|e(0)\|$ for θ large enough. Finally, functions $\tilde{l}_m(y,\cdot)$, $l_x(y,\cdot)$ being also (globally) Lipschitz on $\mathbb{R}^3$, one obtains the inequality (2.5). □

3. Numerical examples

Consider a Monod's growth function where $\mu_{\max} = 5$ and $k_s = 1$ then system (2.2) becomes

$$\begin{gathered}
\dot{s} = -\frac{5s}{1+s}x + 2(s_{in} - s),\\
\dot{x} = (\frac{5s}{1+s} - m)\,x.
\end{gathered}$$

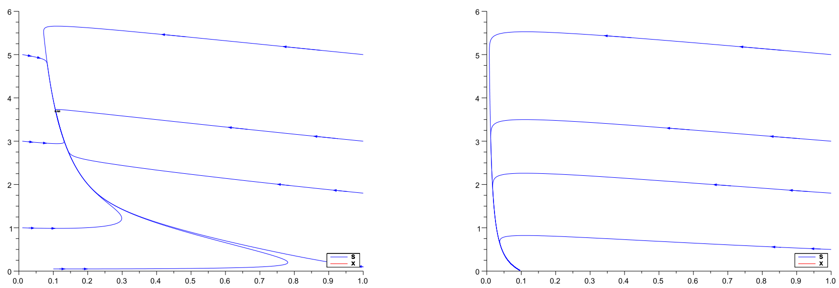

FIGURE 2. The $s - x$ behaviour for $s_{in} = 1$ (left) which satisfies $\mu(s_{in}) > m$ then E^* is GAS and for $s_{in} = 0.1$ (right) which satisfies $\mu(s_{in}) < m$ then E_0 is GAS.

In a first step we suppose that $m = 0.5$, $D = 2$ and we validate the stability results presented in Proposition 2.2 (see Figure 2).

In a second step, we suppose that parameter m is unknown and we used the observer proposed in Proposition 2.4 to reconstruct parameter m and state variable x. Considered initial conditions are $s(0) = 1$ and $x(0) = 1$ where $s_{in} = 5$ and $D = 2$. By assumption A1, parameter m is chosen, along with a priory bounds $m^- = 0.1 \le m = 0.5 \le m^+ = 1$. We have chosen a gain parameter $\theta = 4$ that provides a small error on the estimation of the parameter m and the state variable x (see Figure 3).

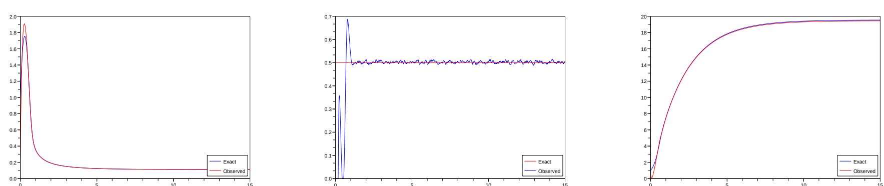

FIGURE 3. Graph of observation y, estimation of parameter m and state variable x in the case of noised measurements.

Conclusion. We considered a simple mathematical model involving biomass growth on organic materials in a waste water treatment plants (membrane bioreactor process). Details of qualitative analysis are given. A high gain observer is proposed that permits the reconstruction of the biomass concentration and the endegenous decay based on on-line measurements of the chemical oxygen demand (CDO).

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