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MULTIPLE POSITIVE SOLUTIONS FOR SUPERLINEAR KIRCHHOFF TYPE PROBLEMS ON $\mathbb{R}^N$

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Abstract. In this article, we study the multiplicity of positive solutions for a class of Kirchhoff type problems depending on two real functions and a non-negative parameter on an unbounded domain. Using the variational method and iterative techniques, we show that if the nonlinearity is subcritical and superlinear at zero and infinity, then the Kirchhoff type problems admits at least two positive solutions when the parameter is sufficiently small.

1. Introduction

The purpose of this article is to sutdy the multiplicity of positive solutions to the nonlinear Kirchhoff type problem

$$\Big(a+\lambda m\Big(\int_{\mathbb{R}^N}(|\nabla u|^2+bu^2)dx\Big)\Big)(-\Delta u+bu)=f(u)+h(x)|u|^{q-2}u, \quad \text{in } \mathbb{R}^N, \tag{1.1}$$

where $N\geq 3$, $1<q<2$, a, b are positive constants, $\lambda\geq 0$ is a parameter, and m,f,h are positive continuous functions.

Problem (1.1) is related to the stationary analogue of the Kirchhoff equation

$$u_{tt}-(a+b\int_\Omega|\nabla u|^2dx)\Delta u=f(x,u) \tag{1.2}$$

which was proposed by Kirchhoff in 1883 [14] as a generalization of the well-known d'Alembert's equation

$$\rho\frac{\partial^2 u}{\partial t^2}-\Big(\frac{P_0}{h}+\frac{E}{2L}\int_0^L|\frac{\partial u}{\partial x}|^2dx\Big)\frac{\partial^2 u}{\partial x^2}=f(x,u)$$

for free vibrations of elastic strings. Kirchhoff's model takes into account the changes in length of the string produced by transverse vibrations. Here, L is the length of the string, h is the area of the cross section, E is the Young modulus of the material, ρ is the mass density and P_0 is the initial tension. The readers can find some early classical research of Kirchhoff's equations in [4, 22]. However, (1.2) received great attention only after Lions [18] proposed an abstract framework to the problem. Some interesting results for problem (1.2) can be found in [1, 5, 10]

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and the references therein. More recently some mathematicians study the following Kirchhoff type problems on bounded domain

$$\begin{gathered} -\Big(a+b\int_\Omega |\nabla u|^2dx\Big)\Delta u = f(x,u), \quad \text{in } \Omega,\\ u=0, \quad \text{on } \partial\Omega. \end{gathered} \tag{1.3}$$

Some interesting studies for problem (1.3) by variational methods can be found in [2, 9, 17, 20, 21, 24, 28, 29] and the references therein. Especially, the authors [28] studied the existence of positive solution for Kirchhoff type problem on bounded domain using iterative techniques and variational methods.

Recently, authors have studied widely Kirchhoff type problems under various conditions on f and V on the whole space $\mathbb{R}^N$:

$$\Big(a+\lambda\Big(\int_{\mathbb{R}^N}(|\nabla u|^2+V(x)u^2)dx\Big)\Big)\big(-\Delta u+V(x)u\big)=f(x,u), \quad \text{in } \mathbb{R}^N. \tag{1.4}$$

When $f(x,u)=|u|^{p-2}u$, $p\in(2,2^*)$, Huang and Liu [12] considered (1.4) and studied existence and nonexistence of positive solution by variational methods; they also discussed the energy doubling property of nodal solutions by Nehari manifold. The results in [12] complement the corresponding results in [15, 16]. Li and Ye [15] showed that (1.4) has no nontrivial solution provided $f(x,u)=|u|^{p-2}u$, $p\in(2,3)$ when $\lambda>0$ is sufficiently large. If $V(x)=b$ and $f(x,u)=f(u)$ is superlinear at infinity, Li, Li and Shi [16] showed that (1.4) has at least one positive radial solution for $\lambda>0$ sufficiently small. Wu, Huang and Liu [26] gave a total description on the positive solutions to (1.4), and they made an observation on the sign-changing solutions. When $f(x,u)$ is asymptotically linear with respect to u at infinity, Ye and Yin [27] studied (1.4) and proved the existence of positive solution for λ sufficiently small and the nonexistence result for λ sufficiently large. Very recently, some authors extend the problem (1.4) to the p-Kirchhoff elliptic equations, see e.g. [6, 7, 8, 19] and the references therein.

In the spirit of [16, 28], for any continuous function m, we establish a multiplicity criterion of positive radial solutions to (1.1) using a variational method and an iterative technique. The main result of this article reads as follows.

Theorem 1.1. *Assume that $N\ge 3$, and a,b are positive constants, $\lambda\ge 0$ is a parameter and the following conditions hold:*

- (H1) *$f\in C(\mathbb{R}_+,\mathbb{R}_+)$ and there are positive constants c and $p\in(2,2^*)$ such that $f(t)\le c(1+t^{p-1})$ for $t\ge 0$, where $2^*=\frac{2N}{N-2}$ for $N\ge 3$;*
- (H2) *$\lim_{t\to 0}\frac{f(t)}{t}=0$;*
- (H3) *$\lim_{t\to\infty}\frac{f(t)}{t}=\infty$;*
- (H4) *$0\le h(x)=h(|x|)\in L^{q'}(\mathbb{R}^N)$, $\langle\nabla h(x),x\rangle\in L^{q'}(\mathbb{R}^N)$, where $q'=\frac{2^*}{2^*-q}$, $\langle\cdot,\cdot\rangle$ denotes the usual inner product in $\mathbb{R}^N$ and $1<q<2$.*

Then for any positive continuous function m, there exist two constants $\tilde\lambda>0$ and $m_0>0$ such that for any $\lambda\in[0,\tilde\lambda)$, problem (1.1) has at least two positive solutions if $\|h\|_{q'}<m_0$.

Since the result in Theorem 1.1 holds for $m(t)=t$, our result generalizes [16, Theorem 1.1]. In this paper, we give multiplicity results for the positive solutions of (1.1), while the authors [16] only studied the existence of positive solutions.

Furthermore, our method is different from that used in [16], we combine variational methods and iterative technique.

Our result can be regarded as an extension of the bounded case considered in [28] to the unbounded case. Also we give two positive solutions, while the authors [28] only studied the existence of positive solutions.

This article is organized as follows: In Section 2, we give some preliminaries. In Section 3 and 4 we present the proofs of the main results. Through out this paper, C, C_i are used in various places to denote distinct constants.

2. Preliminaries

Let $H^1(\mathbb{R}^N)$ be the usual Sobolev space equipped with the inner product and norm

$$\langle u,v\rangle = \int_{\mathbb{R}^N}(\nabla u\cdot\nabla v + buv)dx, \quad \|u\| = \langle u,u\rangle^{\frac{1}{2}}.$$

We denote by $\|\cdot\|_p$ the usual $L^p(\mathbb{R}^N)$ norm. We only consider positive solutions to (1.1), and we assume that $f(t)=0$ for $t<0$

To obtain our result, we have to overcome various difficulties. On one hand, it is well known that Sobolev embedding $H^1(\mathbb{R}^N)\hookrightarrow L^p(\mathbb{R}^N)$ is continuous but not compact for $p\in[2,2^*]$, and then it is usually difficult to prove that a minimizing sequence or a Palais-Smale sequence is strongly convergent if we seek solutions of (1.1) by variational methods. To overcome this difficulty, we usually restrict problem (1.1) in the radial function space. Let $H = H^1_r(\mathbb{R}^N)$ be the subspace of $H^1(\mathbb{R}^N)$ containing only the radial functions. We recall [25], $H\hookrightarrow L^p(R^N)$ compactly (continuously) for $p\in(2,2^*)(p\in[2,2^*])$. That is, there exists a $\gamma_p>0$ such that $\|u\|_p\le\gamma_p\|u\|$, $p\in[2,2^*]$. On the other hand, the nonlinearity f may not satisfy (AR) or 4-superlinearity, it is difficult to get the boundedness of any (PS) sequence even if a (PS) sequence has been obtained. To overcome this difficulty, we use a “freezing” technique whose formulation appears initially in [11]. This technique will help us to change problem (1.1) into semilinear equation. That is, for each fixed $\omega\in H$, we consider the “freezing” problem given by

$$\Big(a+\lambda m\Big(\int_{\mathbb{R}^N}(|\nabla\omega|^2+b\omega^2)dx\Big)\Big)\big(-\Delta u+bu\big) = f(u)+h(x)|u|^{q-2}u, \quad \text{in } \mathbb{R}^N,$$

and the associated function $J_\omega : H\to\mathbb{R}$ is defined by

$$J_\omega(u) = \frac{1}{2}\big(a+\lambda m(\|\omega\|^2)\big)\|u\|^2 - \int_{\mathbb{R}^N}F(u)dx - \frac{1}{q}\int_{\mathbb{R}^N}h(x)|u|^q dx, \quad u\in H,$$

where $F(t)=\int_0^t f(s)ds$. Clearly, by the assumptions imposed on f, h and m, we know that $J_\omega(u)$ is well defined on H, it is of class C^1 for all $\lambda\ge0$, and

$$\begin{aligned}\langle J'_\omega(u),v\rangle &= \big(a+\lambda m(\|\omega\|^2)\big)\int_{\mathbb{R}^N}(\nabla u\cdot\nabla v+buv)dx - \int_{\mathbb{R}^N}f(u)v\,dx\\ &\quad - \int_{\mathbb{R}^N}h(x)|u|^{q-2}uv\,dx, \quad u,v\in H.\end{aligned}$$

Next we recall a monotonicity method by Jeanjean [13] and Struwe [23], which will be used in our proof. The version here is from [13].

Theorem 2.1. *Let $(X, \|\cdot\|)$ be a Banach space and $I \subset \mathbb{R}_+$ an interval. Consider the family of C^1 functionals on X*

$$J_\mu(u) = A(u) - \mu B(u), \quad \mu \in I,$$

with B nonnegative and either $A(u) \to \infty$ or $B(u) \to \infty$ as $\|u\| \to \infty$ and such that $J_\mu(0) = 0$. For any $\mu \in I$, we set

$$\Gamma_\mu = \big\{\gamma \in C([0,1], X) : \gamma(0) = 0, J_\mu(\gamma(1)) < 0\big\}.$$

If for every $\mu \in I$, the set Γ_μ is nonempty and

$$c_\mu = \inf_{\gamma \in \Gamma_\mu} \max_{t \in [0,1]} J_\mu(\gamma(t)) > 0,$$

then for almost every $\mu \in I$, there exists a sequence $\{u_n\} \subset X$ such that

(i) *$\{u_n\}$ is bounded;*
(ii) *$J_\mu(u_n) \to c_\mu$ as $n \to \infty$;*
(iii) *$J'_\mu(u_n) \to 0$ as $n \to \infty$, in the dual space X^{-1} of X.*

3. First positive solution of (1.1)

In this section, we use Theorem 2.1 to obtain the first positive solution for (1.1). In the setting of Theorem 2.1, we have $X = H$, $I = [1/2, 1]$, and for each fixed $\omega \in H$,

$$A_\omega(u) = \frac{1}{2}\left(a + \lambda m(\|\omega\|^2)\right)\|u\|^2 - \frac{1}{q}\int_{\mathbb{R}^N} h(x)(u^+)^q dx, \quad B(u) = \int_{\mathbb{R}^N} F(u)dx,$$

where $u^+ = \max\{u, 0\}$. So the perturbed functional that we study is

$$I_{\omega,\tau}(u) = \frac{1}{2}\left(a + \lambda m(\|\omega\|^2)\right)\|u\|^2 - \frac{1}{q}\int_{\mathbb{R}^N} h(x)(u^+)^q dx - \tau\int_{\mathbb{R}^N} F(u)dx, \quad \tau \in I.$$

It follows from (H4) that

$$\begin{aligned}\frac{1}{2}\left(a + \lambda m(\|\omega\|^2)\right)\|u\|^2 - \frac{1}{q}\int_{\mathbb{R}^N} h(x)(u^+)^q dx &\geq \frac{a}{2}\|u\|^2 - \frac{1}{q}\|h\|_{q'}\|u\|_{2^*}^q \\ &\geq \frac{a}{2}\|u\|^2 - \frac{\gamma_{2^*}^q}{q}\|h\|_{q'}\|u\|^q,\end{aligned}$$

which implies that $A_\omega(u) \to \infty$ as $\|u\| \to \infty$ and obviously, $B(u) \geq 0$. Next, we give some lemmas that are important for proving our main result.

Lemma 3.1. *For each $\omega \in H$ and $\tau \in I$, each bounded (PS) sequence of the functional $I_{\omega,\tau}$ in H admits a convergent subsequence.*

Proof. For each given $\omega \in H$ and $\tau \in I$, let $\{u_n\}$ be a bounded (PS) sequence of the functional $I_{\omega,\tau}$, namely $\{u_n\}$ and $\{I_{\omega,\tau}(u_n)\}$ are bounded, and

$$I'_{\omega,\tau}(u_n) \to 0 \quad \text{in } H^{-1},$$

where H^{-1} is the dual space of H. Since $\{u_n\}$ is bounded, subject to a subsequence, we can assume that there exists $u \in H$ such that as $n \to \infty$,

$$\begin{gathered} u_n \rightharpoonup u, \quad \text{in } H; \\ u_n \to u, \quad \text{in } L^s(\mathbb{R}^N)\ (2 < s < \frac{2N}{N-2}); \\ u_n \to u, \quad \text{a.e } x \in \mathbb{R}^N. \end{gathered} \tag{3.1}$$

By (H1) and (H2), for any $\varepsilon > 0$, there exists $C_\epsilon > 0$ such that

$$|f(t)| \le b\varepsilon|t| + C_\varepsilon|t|^{p-1}, \quad t \in \mathbb{R}. \tag{3.2}$$

It follows from (3.2), the Hölder inequality, the Sobolev inequality and the boundedness of $\{u_n\}$ that

$$\begin{aligned}
\Big|\int_{\mathbb{R}^N} f(u_n)(u_n-u)dx\Big| &\le \int_{\mathbb{R}^N} |f(u_n)(u_n-u)|\,dx \\
&\le b\varepsilon\int_{\mathbb{R}^N}|u_n||u_n-u|dx + C_\varepsilon\int_{\mathbb{R}^N}|u_n|^{p-1}|u_n-u|dx \\
&\le b\varepsilon\|u_n\|_2\|u_n-u\|_2 + C_\varepsilon\|u_n\|_p^{p-1}\|u_n-u\|_p \\
&\le \varepsilon C\|u_n\|\|u_n-u\| + C_\varepsilon C\|u_n\|^{p-1}\|u_n-u\|_p \\
&\le \varepsilon C + C_\varepsilon C\|u_n-u\|_p.
\end{aligned}$$

Then, by (3.1) we can obtain

$$\limsup_{n\to\infty}\Big|\int_{\mathbb{R}^N} f(u_n)(u_n-u)dx\Big| \le \varepsilon C. \tag{3.3}$$

Therefore, using the arbitrariness of ε in (3.3), we have

$$\int_{\mathbb{R}^N} f(u_n)(u_n-u)dx \to 0, \quad \text{as } n\to\infty. \tag{3.4}$$

Using (3.1), we have

$$(u_n^+)^{q-1}(u_n-u) \to 0, \quad \text{a.e. } x\in\mathbb{R}^N.$$

Since

$$\begin{aligned}
\int_{\mathbb{R}^N}\big((u_n^+)^{q-1}(u_n-u)\big)^{2^*/q}dx &\le \Big(\int_{\mathbb{R}^N}(u_n^+)^{2^*}dx\Big)^{\frac{q-1}{q}}\Big(\int_{\mathbb{R}^N}(u_n-u)^{2^*}dx\Big)^{1/q} \\
&\le \|u_n\|_{2^*}^{\frac{(q-1)2^*}{q}}\|u_n-u\|_{2^*}^{2^*/q} \\
&\le C\|u_n\|^{\frac{(q-1)2^*}{q}}\|u_n-u\|^{2^*/q} < +\infty.
\end{aligned}$$

So, $(u_n^+)^{q-1}(u_n-u)$ is bounded in $L^{2^*/q}(\mathbb{R}^N)$. Hence, going if necessary to a subsequence, we can assume that $(u_n^+)^{q-1}(u_n-u) \rightharpoonup 0$ in $L^{2^*/q}(\mathbb{R}^N)$ and using (H4),

$$\int_{\mathbb{R}^N} h(x)(u_n^+)^{q-1}(u_n-u)dx \to 0, \quad \text{as } n\to\infty. \tag{3.5}$$

Thus, by using (3.4), (3.5) and $I'_{\omega,\tau}(u_n)\to 0$, we have

$$\begin{aligned}
\big(a+\lambda m(\|\omega\|^2)\big)\langle u_n, u_n-u\rangle &= \langle I'_{\omega,\tau}(u_n), u_n-u\rangle + \tau\int_{\mathbb{R}^N} f(u_n)(u_n-u)dx \\
&\quad + \int_{\mathbb{R}^N} h(x)(u_n^+)^{q-1}(u_n-u)dx \to 0;
\end{aligned}$$

that is, $\|u_n\|\to\|u\|$. This together with $u_n \rightharpoonup u$ shows that $u_n\to u$ in H. $\square$

Lemma 3.2. *For each $R>0$ and $\omega\in H$ with $\|\omega\|\le R$, there exists $\tilde\lambda = \tilde\lambda(R) > 0$, $m_0>0$ and $\tau_k\subset[1/2,1]$ satisfying that $\tau_k\to 1$ as $k\to\infty$, such that I_{ω,τ_k} has a nontrivial critical point u_{ω,τ_k} if $\lambda\in[0,\tilde\lambda)$, $\|h\|_{q'} < m_0$.*

Proof. We choose a function $\phi \in C_0^\infty(\mathbb{R}^N)$ with $\phi \geq 0$, $\|\phi\| = 1$ and $\operatorname{supp}(\phi) \subset B(0, R_0)$ for some $R_0 > 0$. For given constant $R > 0$, there exists $\tilde\lambda = \tilde\lambda(R) > 0$, such that if $\lambda \in [0, \tilde\lambda)$, we have $\lambda \max_{\xi\in[0,R^2]} m(\xi) \leq 1$. By (H3), for $\frac{2(a+1)}{\int_{B(0,R_0)} \phi^2 dx} > 0$, there exists $C_1 > 0$ such that

$$F(t) \geq \frac{2(a+1)}{\int_{B(0,R_0)} \phi^2 dx} t^2 - C_1, \quad t \geq 0.$$

So, for $t \geq 0$ we get

$$\begin{aligned} I_{\omega,\tau}(t\phi) &= \frac{t^2}{2}\left(a + \lambda m(\|\omega\|^2)\right)\|\phi\|^2 - \tau \int_{\mathbb{R}^N} F(t\phi)dx - \frac{t^q}{q}\int_{\mathbb{R}^N} h(x)\phi^q dx \\ &\leq \frac{t^2}{2}\left(a + \lambda m(\|\omega\|^2)\right) - \frac{t^2}{2}\frac{2(a+1)}{\int_{B(0,R_0)} \phi^2 dx}\int_{B(0,R_0)} \phi^2 dx - \frac{t^q}{q}\int_{\mathbb{R}^N} h(x)\phi^q dx \\ &\quad + \frac{C_1|B(0,R_0)|}{2} \\ &\leq -\frac{t^2}{2}(a+1) + \frac{C_1|B(0,R_0)|}{2} - \frac{t^q}{q}\int_{\mathbb{R}^N} h(x)\phi^q dx. \end{aligned} \tag{3.6}$$

On one hand, by (H4), we can obtain

$$I_{\omega,\tau}(t\phi) \leq -\frac{t^2}{2}(a+1) + \frac{C_1|B(0,R_0)|}{2} - \frac{t^q}{q}\int_{\mathbb{R}^N} h(x)\phi^q dx \to -\infty, \quad t \to +\infty;$$

on the other hand, by (3.6), we known that there exists a constant $C = C(R_0) > 0$ (depending on ω and τ) such that

$$\max_{t\geq 0} I_{\omega,\tau}(t\phi) \leq \frac{C_1|B(0,R_0)|}{2} := C. \tag{3.7}$$

Hence, we can choose $t > 0$ large enough such that $I_{\omega,\tau}(t\phi) < 0$; that is, $\Gamma_{\omega,\tau} \neq \emptyset$, where, $\Gamma_{\omega,\tau} = \{\gamma \in C([0,1], H) : \gamma(0) = 0, I_{\omega,\tau}(\gamma(1)) < 0\}$.

Using (H1) and (H2), for $\varepsilon \in (0, \frac{a}{2})$, there exists $C_2(\epsilon) > 0$ such that

$$F(t) \leq \frac{\varepsilon}{2} b t^2 + C_2(\varepsilon) t^p, \quad t \geq 0.$$

By Sobolev's embedding theorem, there exists $C_3(\varepsilon) > 0$ such that

$$\begin{aligned} I_{\omega,\tau}(u) &= \frac{1}{2}\left(a + \lambda m(\|\omega\|^2)\right)\|u\|^2 - \tau\int_{\mathbb{R}^N} F(u)dx - \frac{1}{q}\int_{\mathbb{R}^N} h(x)(u^+)^q dx \\ &\geq \frac{a}{2}\|u\|^2 - \frac{\varepsilon}{2} b\int_{\mathbb{R}^N} u^2 dx - C_2(\varepsilon)\int_{\mathbb{R}^N} |u|^p dx - \frac{1}{q}\|h\|_{q'}\|u\|_{2^*}^q \\ &\geq \frac{a}{4}\|u\|^2 - C_3(\varepsilon)\|u\|^p - \frac{\gamma_{2^*}^q}{q}\|h\|_{q'}\|u\|^q \\ &\geq \|u\|^q\Big(\frac{a}{4}\|u\|^{2-q} - C_3(\varepsilon)\|u\|^{p-q} - \frac{\gamma_{2^*}^q}{q}\|h\|_{q'}\Big). \end{aligned}$$

Setting

$$g(t) = \frac{a}{4} t^{2-q} - C_3(\varepsilon) t^{p-q}$$

for $t \geq 0$. Since $1 < q < 2 < p < 2^*$, we can choose a constant $\rho > 0$ sufficiently small such that $g(\rho) > 0$. Taking $m_0 := \frac{q}{2\gamma_{2^*}^q} g(\rho)$, it then follows that there exists a constant $c := \frac{1}{2}g(\rho)\rho^q > 0$ which is independent of τ, λ and ω such that

$$I_{\omega,\tau}(u)\big|_{\|u\|=\rho} \geq c > 0,$$

for any $\tau \in I$, $\omega \in H$ and all h satisfying $\|h\|_{q'} < m_0$. Fix $\tau \in I$ and for any $\gamma \in \Gamma_{\omega,\tau}$, by the definition of $\Gamma_{\omega,\tau}$, we have $\|\gamma(1)\| > \rho$. Since $\gamma(0) = 0$, then from intermediate value theorem, we deduce that there exists $t_\gamma \in (0,1)$ such that $\|\gamma(t_\gamma)\| = \rho$. Therefore, for any fixed $\tau \in I$,

$$c_{\omega,\tau} = \inf_{\gamma\in\Gamma_{\omega,\tau}} \max_{t\in[0,1]} I_{\omega,\tau}(\gamma(t)) \geq \inf_{\gamma\in\Gamma_{\omega,\tau}} I_{\omega,\tau}(\gamma(t_\gamma)) \geq c > 0.$$

Following Theorem 2.1, there are $\{\tau_k\} \subset [1/2, 1)$, with $\tau_k \to 1$ as $k \to \infty$, and for every k, there exists a sequence $\{u_{n,\omega,\tau_k}\} \subset H$, such that $\{u_{n,\omega,\tau_k}\}$ is bounded and $I_{\omega,\tau_k}(u_{n,\omega,\tau_k}) \to c_{\omega,\tau_k}, I'_{\omega,\tau_k}(u_{n,\omega,\tau_k}) \to 0$, where

$$c_{\omega,\tau_k} = \inf_{\gamma\in\Gamma_{\omega,\tau_k}} \sup_{u\in\gamma([0,1])} I_{\omega,\tau_k}(u),$$

$$\Gamma_{\omega,\tau_k} = \big\{\gamma \in C([0,1], H) | \gamma(0) = 0,\ I_{\omega,\tau_k}(\gamma(1)) < 0\big\}.$$

Furthermore, by Lemma 3.1, we can suppose that there exists $u_{\omega,\tau_k} \in H$ such that $u_{n,\omega,\tau_k} \to u_{\omega,\tau_k}$, and then

$$I_{\omega,\tau_k}(u_{\omega,\tau_k}) = c_{\omega,\tau_k}, \quad I'_{\omega,\tau_k}(u_{\omega,\tau_k}) = 0.$$

From the above discussion, we get that for given $R > 0$ and $\omega \in H$ with $\|\omega\| \leq R$, there exists $\tilde\lambda = \tilde\lambda(R) > 0$, $m_0 > 0$ and $\tau_k \subset [\frac{1}{2}, 1]$ satisfying that $\tau_k \to 1$ as $k \to \infty$, such that I_{ω,τ_k} has a nontrivial critical point u_{ω,τ_k} if $\lambda \in [0, \tilde\lambda)$, $\|h\|_{q'} < m_0$ and

$$c_{\omega,\tau_k} = I_{\omega,\tau_k}(u_{\omega,\tau_k}) \leq \max_{t\geq 0} I_{\omega,\tau_k}(t\phi) \leq C, \tag{3.8}$$

where C is given in (3.7). $\square$

Lemma 3.3. *Let u_{ω,τ_k} be a critical point of I_{ω,τ_k} at level c_{ω,τ_k}. Then $\{u_{\omega,\tau_k}\}$ are uniformly bounded.*

Proof. It follows from Lemma 3.2 that u_{ω,τ_k} is a weak solution of the problem

$$\big(a + \lambda m(\|\omega\|^2)\big)\,(-\Delta u + bu) = \tau_k f(u) + h(x)(u^+)^{q-1};$$

therefore,

$$\big(a + \lambda m(\|\omega\|^2)\big)\,(-\Delta u_{\omega,\tau_k} + bu_{\omega,\tau_k}) = \tau_k f(u_{\omega,\tau_k}) + h(x)(u^+_{\omega,\tau_k})^{q-1}. \tag{3.9}$$

Hence, we have the following Pohozaev identity

$$\begin{aligned}
&\Big(\frac{N-2}{2}\int_{\mathbb{R}^N} |\nabla u_{\omega,\tau_k}|^2 dx + \frac{Nb}{2}\int_{\mathbb{R}^N} u^2_{\omega,\tau_k} dx\Big)\,\big(a + \lambda m(\|\omega\|^2)\big) \\
&= N\tau_k \int_{\mathbb{R}^N} F(u_{\omega,\tau_k})dx + \frac{1}{q}\int_{\mathbb{R}^N}\Big(Nh + \langle\nabla h(x), x\rangle\Big)(u^+_{\omega,\tau_k})^q dx\,.
\end{aligned}\tag{3.10}$$

The proof is similar to that of [3, Proposition 1], we omit here.

By letting $c_{\omega,\tau_k} = I_{\omega,\tau_k}(u_{\omega,\tau_k})$, we have

$$\begin{aligned}
c_{\omega,\tau_k} &= \frac{1}{2}\,\big(a + \lambda m(\|\omega\|^2)\big)\,\|u_{\omega,\tau_k}\|^2 - \tau_k \int_{\mathbb{R}^N} F(u_{\omega,\tau_k})dx \\
&\quad - \frac{1}{q}\int_{\mathbb{R}^N} h(x)(u^+_{\omega,\tau_k})^q dx.
\end{aligned}\tag{3.11}$$

By (H4) and the Hölder inequality, we deduce that

$$\begin{aligned} \frac{1}{q}\Big|\int_{\mathbb{R}^N}\langle\nabla h(x),x\rangle(u^+_{\omega,\tau_k})^q dx\Big| &\le \frac{1}{q}\int_{\mathbb{R}^N}\big|\langle\nabla h(x),x\rangle(u^+_{\omega,\tau_k})^q\big|dx \\ &\le \frac{1}{q}\|\langle\nabla h(x),x\rangle\|_{q'}\|(u^+_{\omega,\tau_k})\|^q_{2^*} \\ &\le C_4\Big(\int_{\mathbb{R}^N}|\nabla u_{\omega,\tau_k}|^2dx\Big)^{q/2}. \end{aligned} \tag{3.12}$$

Therefore, by (3.8) and (3.10)-(3.12), we obtain

$$\begin{aligned} \int_{\mathbb{R}^N}|\nabla u_{\omega,\tau_k}|^2dx &= \frac{Nc_{\omega,\tau_k}-\frac{1}{q}\int_{\mathbb{R}^N}\langle\nabla h(x),x\rangle(u^+_{\omega,\tau_k})^q dx}{a+\lambda m(\|\omega\|^2)} \\ &\le \frac{Nc_{\omega,\tau_k}+C_4\left(\int_{\mathbb{R}^N}|\nabla u_{\omega,\tau_k}|^2dx\right)^{q/2}}{a+\lambda m(\|\omega\|^2)} \\ &\le \frac{NC+C_4\left(\int_{\mathbb{R}^N}|\nabla u_{\omega,\tau_k}|^2dx\right)^{q/2}}{a}. \end{aligned}$$

Because of $1<q<2$, $\int_{\mathbb{R}^N}|\nabla u_{\omega,\tau_k}|^2dx$ is uniformly bounded. That is, there exists a constant $C_5>0$, independent of τ, λ and ω, such that

$$\int_{\mathbb{R}^N}|\nabla u_{\omega,\tau_k}|^2dx\le C_5. \tag{3.13}$$

Furthermore, by (H1) and (H2), there exists a constant $C_6>0$ such that

$$|f(t)|\le\frac{ab}{2}|t|+C_6|t|^{2^*-1},\quad t\in\mathbb{R}. \tag{3.14}$$

Hence, by (3.9) and (3.14), we have

$$\begin{aligned} &\left(a+\lambda m(\|\omega\|^2)\right)\|u_{\omega,\tau_k}\|^2 \\ &= \tau_k\int_{\mathbb{R}^N}f(u_{\omega,\tau_k})u_{\omega,\tau_k}dx+\int_{\mathbb{R}^N}h(x)(u^+_{\omega,\tau_k})^q dx \\ &\le \frac{ab}{2}\int_{\mathbb{R}^N}|u_{\omega,\tau_k}|^2dx+C_6\int_{\mathbb{R}^N}|u_{\omega,\tau_k}|^{2^*}dx\ +\|h\|_{q'}\big(\int_{R^N}|u_{\omega,\tau_k}|^{2^*}dx\big)^{q/2^*} \\ &\le \frac{a}{2}\|u_{\omega,\tau_k}\|^2+C_6\int_{R^N}|u_{\omega,\tau_k}|^{2^*}dx+\|h\|_{q'}\big(\int_{R^N}|u_{\omega,\tau_k}|^{2^*}dx\big)^{q/2^*}. \end{aligned}$$

Using (3.13), we conclude that

$$\begin{aligned} \frac{a}{2}\|u_{\omega,\tau_k}\|^2 &\le C_6\int_{\mathbb{R}^N}|u_{\omega,\tau_k}|^{2^*}dx+\|h\|_{q'}\Big(\int_{R^N}|u_{\omega,\tau_k}|^{2^*}dx\Big)^{q/2^*} \\ &\le C_7\Big(\int_{\mathbb{R}^N}|\nabla u_{\omega,\tau_k}|^2dx\Big)^{2^*/2}+C_8\Big(\int_{\mathbb{R}^N}|\nabla u_{\omega,\tau_k}|^2dx\Big)^{q/2} \\ &\le C_7C_5^{2^*/2}+C_8C_5^{q/2}. \end{aligned}$$

Then $\|u_{\omega,\tau_k}\|^2\le C_9$, where $C_9=\frac{2}{a}C_7C_5^{2^*/2}+\frac{2}{a}C_8C_5^{q/2}$ which is independent of τ, λ and ω. If we set $R=\sqrt{C_9}$, then for any $\omega\in H$ with $\|\omega\|\le R$, there exist $\tilde\lambda>0$ $m_0>0$ which are independent of τ, λ and ω, such that I_{ω,τ_k} has a nontrivial critical point u_{ω,τ_k} with $\|u_{\omega,\tau_k}\|\le R$ when $\lambda\in[0,\tilde\lambda)$, $\|h\|_2<m_0$. And also, $\{u_{\omega,\tau_k}\}$ is uniformly bounded.

Now we choose $R = \sqrt{C_9}$ as above and construct a family of sequence by iterative techniques. For every k, if we let $\omega = \omega_0 \equiv 0$, by the previous arguments, we know I_{ω_0,τ_k} has a nontrivial critical point and denote it by $u_{1,k}$ with $\|u_{1,k}\| \le R$. Let $\omega = u_{1,k}$, then $I_{u_{1,k},\tau_k}$ has a nontrivial critical point and denote it by $u_{2,k}$ with $\|u_{2,k}\| \le R$. Hence, by induction, we can get a sequence $\{u_{n,k}\}$ with $\|u_{n,k}\| \le R$, $n = 1, 2, \dots$, such that $I'_{u_{n,k},\tau_k}(u_{n+1,k}) \cdot v = 0$ for all $v \in H$. $\square$

Existence of the first positive solution to (1.1)**.** To complete the proof, we proceed in two steps.

Step 1. For any fixed k, the iterative sequence $\{u_{n,k}\}$ constructed in Lemma 3.3 is convergent to a function u_k, which is a critical point of I_{u_k,τ_k}.

Since for fixed k and for all $n \in \mathbb{N}$, $\|u_{n,k}\| \le R$, if necessary going to a subsequence, we suppose that there exists $u_k \in H$ such that as $n \to \infty$,

$$\begin{gathered} u_{n,k} \rightharpoonup u_k, \quad \text{in } H; \\ u_{n,k} \to u_k, \quad \text{in } L^p(\mathbb{R}^N)(2 < p < 2^*); \\ u_{n,k} \to u_k, \quad \text{a.e. } x \in \mathbb{R}^N. \end{gathered} \tag{3.15}$$

Also we have $\|u_k\| \le R$, for all $k \in \mathbb{N}$. From the subcritical growth of f and (3.15), we see that

$$\int_{R^N} \big(f(u_{n,k}) - f(u_k)\big)(u_{n,k} - u_k)dx \to 0, \quad \text{as } n \to \infty; \tag{3.16}$$

$$\int_{R^N} \big(h(x)(u_{n,k}^+)^{q-1} - h(x)(u_k^+)^{q-1}\big)(u_{n,k} - u_k)dx \to 0, \quad \text{as } n \to \infty. \tag{3.17}$$

The proof is similar to that of (3.4) and (3.5), and we omit here. Then we have

$$\begin{aligned} &\big(a + \lambda m(\|u_{n-1,k}\|^2)\big)\, \|u_{n,k} - u_k\|^2 \\ &= \Big\langle I'_{u_{n-1,k},\tau_k}(u_{n,k}) - I'_{u_{n-1,k},\tau_k}(u_k), u_{n,k} - u_k \Big\rangle \\ &\quad + \tau_k \int_{R^N} \Big(f(u_{n,k}) - f(u_k)\Big)(u_{n,k} - u_k)dx \\ &\quad + \int_{R^N} \Big(h(x)(u_{n,k}^+)^{q-1} - h(x)(u_k^+)^{q-1}\Big)(u_{n,k} - u_k)dx \to 0 \quad \text{as } n \to \infty; \end{aligned}$$

that is, $u_{n,k} \to u_k$ in H as $n \to \infty$. Thus, for any $v \in H$, as $n \to \infty$, we have

$$\begin{gathered} a + \lambda m(\|u_{n-1,k}\|^2) \to a + \lambda m(\|u_k\|^2), \\ \int_{\mathbb{R}^N} (\nabla u_{n,k} \cdot \nabla v + b u_{n,k} v)\, dx \to \int_{\mathbb{R}^N} (\nabla u_k \cdot \nabla v + b u_k v)\, dx, \\ \tau_k \int_{\mathbb{R}^N} f(u_{n,k}) v\, dx \to \tau_k \int_{\mathbb{R}^N} f(u_k) v\, dx, \\ \tau_k \int_{\mathbb{R}^N} F(u_{n,k}) dx \to \tau_k \int_{\mathbb{R}^N} F(u_k) dx. \end{gathered}$$

Also, we have

$$\begin{gathered} \int_{\mathbb{R}^N} h(x)(u_{n,k}^+)^{q-1} v\, dx \to \int_{\mathbb{R}^N} h(x)(u_k^+)^{q-1} v\, dx, \\ \int_{\mathbb{R}^N} h(x)(u_{n,k}^+)^q dx \to \int_{\mathbb{R}^N} h(x)(u_k^+)^q dx, \quad \text{as } n \to \infty, \end{gathered}$$

the proof is similar to that of (3.5), and we omit here. So, we obtain

$$I'_{u_k,\tau_k}(u_k)\cdot v=\lim_{n\to\infty} I'_{u_{n-1,k},\tau_k}(u_{n,k})\cdot v=0,$$
$$I_{u_k,\tau_k}(u_k)=\lim_{n\to\infty} I_{u_{n-1,k},\tau_k}(u_{n,k})=\lim_{n\to\infty} c_{u_{n-1,k},\tau_k}\geq c>0;$$

that is, for any $v\in H$,

$$I'_{u_k,\tau_k}(u_k)\cdot v=0,\quad I_{u_k,\tau_k}(u_k)\geq c>0.$$

Step 2. The sequence $\{u_k\}$ obtained in step 1 is convergent to a nontrivial positive solution of (1.1).

Since $\|u_k\|\leq R$ for all $k\in\mathbb{N}$, without loss of generality, we can assume that there exists a function $u\in H$ such that

$$\begin{gathered} u_k\rightharpoonup u,\quad \text{in } H;\\ u_k\to u,\quad \text{in } L^p(\mathbb{R}^N)(2<p<2^*);\\ u_k\to u,\quad \text{a.e. } x\in\mathbb{R}^N. \end{gathered}\tag{3.18}$$

By the similar proof to that of (3.16) and (3.17), we have

$$\int_{\mathbb{R}^N}\big(f(u_k)-f(u)\big)(u_k-u)dx=o(1),$$
$$\int_{R^N}\big(h(x)(u_k^+)^{q-1}-h(x)(u^+)^{q-1}\big)(u_k-u)dx=o(1).$$

Now, taking into account that

$$\begin{aligned}
&\big(a+\lambda m(\|u_k\|^2)\big)\,\|u_k-u\|^2\\
&=\langle I'_{u_k,\tau_k}(u_k)-I'_{u_k,\tau_k}(u),u_k-u\rangle+\tau_k\int_{R^N}\Big(f(u_k)-f(u)\Big)(u_k-u)dx\\
&\quad+\int_{R^N}\Big(h(x)(u_k^+)^{q-1}-h(x)(u^+)^{q-1}\Big)(u_k-u)dx\to 0\quad\text{as } n\to\infty,
\end{aligned}$$

we deduce that $u_k\to u$ as $k\to\infty$. So for any $v\in H$, as $k\to\infty$, we have

$$a+\lambda m(\|u_k\|^2)\to a+\lambda m(\|u\|^2),$$
$$\int_{\mathbb{R}^N}(\nabla u_k\cdot\nabla v+bu_kv)dx\to\int_{\mathbb{R}^N}(\nabla u\cdot\nabla v+buv)dx,$$
$$\tau_k\int_{\mathbb{R}^N}f(u_k)v\,dx\to\int_{\mathbb{R}^N}f(u)v\,dx,$$
$$\tau_k\int_{\mathbb{R}^N}F(u_k)dx\to\int_{\mathbb{R}^N}F(u)dx,$$
$$\int_{\mathbb{R}^N}h(x)(u_k^+)^{q-1}v\,dx\to\int_{\mathbb{R}^N}h(x)(u^+)^{q-1}v\,dx,$$
$$\int_{\mathbb{R}^N}h(x)(u_k^+)^{q}dx\to\int_{\mathbb{R}^N}h(x)(u^+)^{q}dx.$$

So, for any $v\in H$, as $k\to\infty$, we can obtain

$$\begin{aligned}
&\big(a+\lambda m(\|u\|^2)\big)\int_{\mathbb{R}^N}\big(\nabla u\cdot\nabla v+buv\big)dx-\int_{\mathbb{R}^N}f(u)v\,dx-\int_{\mathbb{R}^N}h(x)(u^+)^{q-1}v\,dx\\
&=\lim_{k\to\infty}I'_{u_k,\tau_k}(u_k)\cdot v=0,
\end{aligned}$$

and

$$\begin{aligned}&\frac{1}{2}\big(a+\lambda m(\|u\|^2)\big)\int_{\mathbb{R}^N}\big(|\nabla u|^2+bu^2\big)dx-\int_{\mathbb{R}^N}F(u)dx-\int_{\mathbb{R}^N}h(x)(u^+)^q dx\\&=\lim_{n\to\infty}I_{u_k,\tau_k}(u_k)=c_{u_k,\tau_k}\geq c>0.\end{aligned}$$

Therefore, u is a nontrivial solution of (1.1). Setting $u^-=\max\{-u,0\}$, Since

$$(a+\lambda m(\|u\|^2))\langle u,u^-\rangle-\int_{\mathbb{R}^N}f(u)u^-dx-\int_{\mathbb{R}^N}h(x)(u^+)^{q-1}u^-dx=0,$$

by (H1) and (H4) we have $\|u^-\|=0$; this implies $u\geq 0$ a.e. in $\mathbb{R}^N$. So, by the strong maximum principle, we get that u is positive on H. Thus u is a positive solution of (1.1) if $\lambda\in[0,\tilde\lambda)$, $\|h\|_{q'}<m_0$.

4. Second positive solution of (1.1)

In this section, we prove the existence of local minimum solution for problem (1.1) by Ekeland's variational principle. Define the functional $I:H\to\mathbb{R}$ by

$$I_\lambda(u)=\frac{a}{2}\|u\|^2+\frac{\lambda}{2}M(\|u\|^2)-\int_{\mathbb{R}^N}F(u)dx-\frac{1}{q}\int_{\mathbb{R}^N}h(x)(u^+)^q dx,$$

where $M(t)=\int_0^t m(s)ds$. Then, it follows from (H1)–(H4) and the continuity of m that I_λ is well defined on H and is C^1 for all $\lambda\geq 0$, and

$$\begin{aligned}\langle I_\lambda'(u),v\rangle&=(a+\lambda m(\|u\|^2))\int_{\mathbb{R}^N}(\nabla u\cdot\nabla v+buv)dx\\&\quad-\int_{\mathbb{R}^N}f(u)v\,dx-\int_{\mathbb{R}^N}h(x)(u^+)^{q-1}v\,dx,\quad u,v\in H.\end{aligned}$$

Lemma 4.1. *Assume that* (H1), (H2), (H4) *are satisfied. Then there exist constants* $\rho,m_0,\alpha>0$ *such that* $I_\lambda(u)\big|_{\|u\|=\rho}\geq\alpha>0$ *with* $\|h\|_2<m_0$.

Proof. Using (H1) and (H2), for $\varepsilon\in(0,\frac{a}{2})$, there exists $C_{12}(\epsilon)>0$ such that

$$F(t)\leq\frac{\varepsilon}{2}bt^2+C_{12}(\varepsilon)t^p,\quad t\geq 0.$$

By Sobolev's embedding theorem, there exists $C_{13}(\varepsilon)>0$ such that

$$\begin{aligned}I_\lambda(u)&=\frac{a}{2}\|u\|^2+\frac{\lambda}{2}M(\|u\|^2)-\int_{\mathbb{R}^N}F(u)dx-\frac{1}{q}\int_{\mathbb{R}^N}h(x)(u^+)^q dx\\&\geq\frac{a}{2}\|u\|^2-\frac{\varepsilon}{12}b\int_{\mathbb{R}^N}u^2dx-C_{12}(\varepsilon)\int_{\mathbb{R}^N}|u|^p dx-\frac{1}{q}\|h\|_{q'}\|u\|_{2^*}^q\\&\geq\frac{a}{4}\|u\|^2-C_{13}(\varepsilon)\|u\|^p-\frac{\gamma_{2^*}}{q}\|h\|_{q'}\|u\|^q\\&\geq\|u\|^q\Big(\frac{a}{4}\|u\|^{2-q}-C_{13}(\varepsilon)\|u\|^{p-q}-\frac{\gamma_{2^*}}{q}\|h\|_{q'}\Big)\end{aligned}$$

So, setting

$$g(t)=\frac{a}{4}t^{2-q}-C_{13}(\varepsilon)t^{p-q}$$

for $t\geq 0$. Since $1<q<2<p<2^*$, we can choose a constant $\rho>0$ sufficiently small such that $g(\rho)>0$. Taking $m_0:=\frac{q}{2\gamma_{2^*}}g(\rho)$, it then follows that there exists a constant $\alpha:=\frac{1}{2}g(\rho)\rho^q>0$ which is independent of τ, λ and ω such that

$$I_\lambda(u)\big|_{\|u\|=\rho}\geq\alpha>0,$$

for any $\tau \in I$, $\omega \in H$ and all h satisfying $\|h\|_{q'} < m_0$. $\square$

Lemma 4.2. *Assume that* (H1)-(H4) *are satisfied, then there exist a function $e \in H$ with $\|e\| < \rho$ and a constant $0 < \lambda^* < \widetilde{\lambda}$ such that $I_\lambda(e) < 0$ for any $\lambda \in [0, \lambda^*)$, where ρ and $\widetilde{\lambda}$ are given by Lemma 4.1 and Lemma 3.2, respectively.*

Proof. We choose a function $0 \le \phi \in C_0^\infty(\mathbb{R}^N)$ with $\int_{B(0,R_0)} h(x)\phi^q dx \ge 0$ for some $R_0 > 0$. By (H1), for $t \ge 0$ we obtain

$$\begin{aligned} I_0(t\phi) &= \frac{at^2}{2}\|\phi\|^2 - \int_{\mathbb{R}^N} F(t\phi)dx - \frac{t^q}{q}\int_{\mathbb{R}^N} h(x)\phi^q dx \\ &\le \frac{at^2}{2}\|\phi\|^2 - \frac{t^q}{q}\int_{B(0,R_0)} h(x)\phi^q dx. \end{aligned} \tag{4.1}$$

Since $1 < q < 2$, it follows from (4.1) that $I_0(t\phi) < 0$ for $t > 0$ sufficiently small, which implies that there exist $e \in H$ with $\|e\| < \rho$ such that $I_0(e) < 0$, where ρ is given by Lemma 4.1. Since $I_\lambda(e) \to I_0(e)$ as $\lambda \to 0^+$, we see that there exists $\widetilde{\lambda} > \lambda^* > 0$ such that $I_\lambda(e) < 0$ for all $\lambda \in [0, \lambda^*)$, where $\widetilde{\lambda}$ is given by Lemma 3.2. $\square$

Second positive solution for (1.1)

Setting

$$c_1 := \inf\{I_\lambda(u) : u \in \overline{B}_\rho\},$$

where ρ is given by Lemma 4.1, $B_\rho = \{u \in H : \|u\| < \rho\}$. Using Lemma 4.1 and Lemma 4.2, we obtain

$$\inf_{\overline{B}_\rho} I_\lambda > -\infty, \quad \inf_{\partial B_\rho} I_\lambda > \alpha > 0, \quad c_1 < 0.$$

By Ekeland's variational principle, there exists a sequence $\{u_n\} \subset \overline{B}_\rho$ such that

$$\begin{gathered} c_1 \le I_\lambda(u_n) < c_1 + \frac{1}{n}, \\ I_\lambda(v) \ge I_\lambda(u_n) - \frac{1}{n}\|v - u_n\| \end{gathered}$$

for all $v \in \overline{B}_\rho$. Then by a standard procedure, we can show that $\{u_n\}$ is a bounded Palais-Smale sequence of I_λ. Using the similar proof to that of Lemma 3.1, we conclude that there exists a function $u_1 \in B_\rho$ such that $I_\lambda(u_1) = c_1 < 0$ and $I'_\lambda(u_1) = 0$.

Setting $u^- = \max\{0, -u\}$. Since

$$\big(a + \lambda m(\|u_1\|^2)\big)\langle u_1, u_1^-\rangle - \int_{\mathbb{R}^N} f(u_1)u_1^- dx - \int_{\mathbb{R}^N} h(x)(u_1^+)^{q-1}u_1^- dx = 0,$$

by (H1) and (H4) we have $\|u_1^-\| = 0$, which implies $u_1 \ge 0$ a.e. in $\mathbb{R}^N$. So, by the strong maximum principle, we obtain that u_1 is positive on H.

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