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# DYNAMICS OF THE $p$-LAPLACIAN EQUATIONS WITH NONLINEAR DYNAMIC BOUNDARY CONDITIONS

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Abstract. In this article, we study the long-time behavior of the $p$-Laplacian equation with nonlinear dynamic boundary conditions for both autonomous and non-autonomous cases. For the autonomous case, some asymptotic regularity of solutions is proved. For the non-autonomous case, we obtain the existence and structure of a compact uniform attractor in $L^{r_1}(\Omega) \times L^r(\Gamma)$ $(r = \min(r_1, r_2))$.

## 1. Introduction

In this article, we consider the asymptotic behavior of solutions of the following $p$-Laplacian equations with nonlinear dynamic boundary conditions:

$$\begin{aligned} &u_t - \Delta_p u + f(u) = h(x,t), \quad \text{in } \Omega, \\ &u_t + |\nabla u|^{p-2}\partial_n u + g(u) = 0, \quad \text{on } \Gamma, \end{aligned} \tag{1.1}$$

where $\Omega$ is a bounded domain in $\mathbb{R}^N$ $(N \geqslant 3)$ with a smooth boundary $\Gamma$, $\Delta_p$ denotes the $p$-Laplacian operator, which is defined as $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, $p \geqslant 2$, and about the external forcing $h(x,t)$, we consider two cases: the autonomous case $h(x,t) = h(x) \in L^{r_1'}(\Omega)$, where $r_1'$ is conjugate to $r_1$, and the non-autonomous case $h(x,t)$, which will be given later in Sections 3 and 4 respectively. The functions $f$ and $g \in C^1(\mathbb{R}, \mathbb{R})$, satisfy the following conditions:

$$C_1|s|^{r_1} - k_1 \le f(s)s \le C_2|s|^{r_1} + k_2, \quad r_1 \ge p, \tag{1.2}$$

$$C_3|s|^{r_2} - k_3 \le g(s)s \le C_4|s|^{r_2} + k_4, \quad r_2 \ge 2, \tag{1.3}$$

$$f'(s) \ge -l, \quad g'(s) \ge -m, \tag{1.4}$$

here $l, m > 0$, $C_i, k_i > 0$, $i = 1, 2, 3, 4$.

In the case $p = 2$, the problem (1.1) is a general reaction-diffusion equation, the dynamical behavior have been studied in [3, 4, 8, 22, 25, 26, 27, 31] for the Dirichlet boundary conditions and [10, 11, 14, 15, 28, 29] for the dynamic boundary conditions.

The long-time behavior of the solutions of (1.1) has been considered by many researchers, e.g., see [3, 4, 8, 27] and the references therein.

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For the autonomous systems; i.e., $h(x,t) = h(x)$, in the Dirichlet boundary case, the nonlinear eigenvalue problem for the $p$-Laplacian operator was considered in [18] by using the Ljusternik-Schnirelman principle. In [3], Babin & Vishik established the existence of a $(L^2(\Omega), (W_0^{1,p}(\Omega) \cap L^q(\Omega))_w)$-global attractor. In [27], a special case of $f = ku$ was discussed by Temam. In [5], Carvalho, Cholewa and Dlotko considered the existence of global attractors for problems with monotone operators, and as an application, they proved the existence of $(L^2(\Omega), L^2(\Omega))$-global attractor for $p$-Laplacian equation, see also Cholewa & Dlotko [8]. In [6], Carvalho & Gentile obtained that the corresponding semigroup has a $(L^2(\Omega), W_0^{1,p}(\Omega))$-global attractor under some additional conditions. In [30], Yang, Sun and Zhong obtained the existence of a $(L^2(\Omega), W_0^{1,p}(\Omega) \cap L^{r_1}(\Omega))$-global attractor, which holds only under the assumptions (1.2) and (1.4). Some asymptotic regularity of the solutions was proved by Liu, Yang and Zhong in [20]. In the dynamic boundary case, recently, Gal *et al* [16, 17] presented firstly the general result for the problem (1.1), the well-posedness and the asymptotic behavior of the solutions were studied.

Inspired by the ideas of [20, 26, 29], we obtain the asymptotic regularity of the solutions of equation (1.1), where we only assume the external forcing $h(x) \in L^{r_1'}(\Omega)$, $r_1'$ is conjugate to $r_1$. As a direct application of the asymptotic regularity results, we can obtain the existence of a global attractor in $(W^{1,p}(\Omega) \cap L^{r_1}(\Omega)) \times (W^{1-1/p,p}(\Gamma) \cap L^{r_2}(\Gamma))$ immediately. Moreover, we also can show further that the global attractor attracts every $L^2(\Omega) \times L^2(\Gamma)$-bounded subset with $(W^{1,p}(\Omega) \cap L^{r_1+\delta}(\Omega)) \times (W^{1-1/p,p}(\Gamma) \cap L^{r_2+\gamma}(\Gamma))$-norm for any $\delta, \gamma \in [0, \infty)$.

For the non-autonomous systems, in the Dirichlet boundary case, the existence of the $(L^2(\Omega), W_0^{1,p}(\Omega) \cap L^{r_1}(\Omega))$-uniform attractor was proved by Chen and Zhong in [7]. However, for the nonlinear dynamic boundary conditions, the non-autonomous $p$-Laplacian equation is less considered. In this article, we obtain the existence and structure of a compactly uniform attractor in $L^{r_1}(\Omega) \times L^r(\Gamma)$ $(r = \min(r_1, r_2))$, which holds only under the assumptions (1.2)–(1.4), and no any restrictions on $p, r_1, r_2$ and $N$.

The main results of this article are Theorem 3.4 (asymptotic regularity), Theorem 3.5 (global attractor) and Theorem 4.5 (uniform attractor and its structure).

Hereafter, we assume that

$$2 < p < N.$$

For the case $p = 2$, system (1.1) is a reaction-diffusion equation and we refer the reader to [15, 28]; and if $p \geqslant N$, then embeddings $W^{1,p}(\Omega) \hookrightarrow L^{s_1}(\Omega)$ and $W^{1,p}(\Omega) \hookrightarrow L^{s_2}(\Gamma)$ hold for any $s_1, s_2 \in [1, \infty)$, which make the nonlinear terms $f(\cdot)$ and $g(\cdot)$ to be trivial terms.

For convenience, hereafter $\|\cdot\|$ and $\|\cdot\|_\Gamma$ stand for the norm in $L^2(\Omega)$ and $L^2(\Gamma)$, $\langle \cdot, \cdot \rangle$ and $\langle \cdot, \cdot \rangle_\Gamma$ stand for the inner product in $L^2(\Omega)$ and $L^2(\Gamma)$, respectively. $C$, $C_i$ denote general positive constants, $i = 1, \dots$, which will be different in different estimates.

This article is organized as follows: in Section 2, we introduce some preliminary results; in Section 3, for the autonomous cases, i.e., $h(x,t) = h(x)$, we prove some asymptotic regularity of the solution; in Section 4, for the non-autonomous cases, the existence and structure of a uniform attractor in $L^{r_1}(\Omega) \times L^r(\Gamma)$ $(r = \min(r_1, r_2))$ is obtained.

## 2. Preliminaries

In this section, we give some auxiliary results which will be used later. We first introduce the spaces of time-dependent external forcing $h(x,t)$ to be considered in this article (see[4]).

**Definition 2.1** ([4]). A function $\varphi$ is said to be translation bounded in $L^2_{\rm loc}(\mathbb{R};X)$, if

$$\|\varphi\|_b^2 = \sup_{t\in\mathbb{R}} \int_t^{t+1} \|\varphi\|_X^2 ds < +\infty.$$

Denote by $L^2_b(\mathbb{R};X)$ the set of all translation bounded functions in $L^2_{\rm loc}(\mathbb{R};X)$.

We now introduce a class of functions that was defined first in [21].

**Definition 2.2** ([21]). A function $\varphi \in L^2_{\rm loc}(\mathbb{R};X)$ is said to be normal if for any $\varepsilon > 0$, there exists $\eta > 0$ such that

$$\sup_{t\in\mathbb{R}} \int_t^{t+\eta} \|\varphi\|_X^2 ds \le \varepsilon.$$

Denote by $L^2_n(\mathbb{R};X)$ the set of all normal functions in $L^2_{\rm loc}(\mathbb{R};X)$.

**Lemma 2.3** ([21]). *If $\varphi_0 \in L^2_n(\mathbb{R};X)$, then for any $\tau \in \mathbb{R}$,*

$$\lim_{\gamma\to\infty} \sup_{t\ge\tau} \int_\tau^t e^{-\gamma(t-s)} \|\varphi(s)\|_X^2 ds = 0,$$

*uniformly (with respect to $\varphi \in H(\varphi_0)$), where $H(\varphi_0) = \{\varphi_0(t+h) \mid h \in \mathbb{R}\}$ .*

The next result is an estimate of the $p$-Laplacian operator; see [9] for the proof.

**Lemma 2.4.** *Let $p \geqslant 2$. Then there exists constant $K > 0$ such that for any $a, b \in \mathbb{R}^N$,*

$$\langle |a|^{p-2}a - |b|^{p-2}b,\, a-b\rangle \geqslant K|a-b|^p, \tag{2.1}$$

*where $K$ depends only on $p$ and $N$; $\langle\cdot,\cdot\rangle$ denotes the inner product of $\mathbb{R}^N$.*

## 3. Autonomous cases: $h(x,t) = h(x)$

In this section, we consider the autonomous case of (1.1); that is,

$$\begin{gathered} u_t - \Delta_p u + f(u) = h(x), \quad \text{in } \Omega, \\ u_t + |\nabla u|^{p-2}\partial_n u + g(u) = 0, \quad \text{on } \Gamma, \\ u(x,0) = u_0(x), \end{gathered} \tag{3.1}$$

where $h(x) \in L^{r_1'}(\Omega)$, $r_1'$ is conjugate to $r_1$.

### 3.1. Mathematical setting.

At first, following [17], it is more convenient to introduce the unknown function $v(t) := u(t)_{|\Gamma}$, defined on the boundary $\Gamma$, so we think of our problem as a coupled system of two parabolic equations, one in the bulk $\Omega$ and the other on the boundary $\Gamma$. Thus, the function $u(t)$ tracks diffusion in the bulk, while $v(t)$ records the information on the boundary. Throughout the remainder of this section, we formulate the problem as following:

**Problem (P).** Let $\Omega \subset \mathbb{R}^N$ ($N \geq 3$) be a bounded domain with a smooth boundary $\Gamma := \partial\Omega$ (e.g., of class $C^2$). The nonlinearities $f$ and $g$ satisfy (1.2)–(1.4). For any given pair of initial data $(u_0, v_0) \in L^2(\Omega) \times L^2(\Gamma)$, find $(u(t), v(t))$ with

$$\begin{gathered}
(u, v) \in C([0, +\infty); L^2(\Omega) \times L^2(\Gamma)) \cap L^\infty((0, +\infty); W^{1,p}(\Omega) \times W^{1-1/p,p}(\Gamma)),\\
(u, v) \in W^{1,2}_{\rm loc}((0, \infty); L^2(\Omega) \times L^2(\Gamma)),\\
u \in L^p_{\rm loc}([0, +\infty); W^{1,p}(\Omega)),\\
v \in L^p_{\rm loc}([0, +\infty); W^{1-1/p,p}(\Gamma))
\end{gathered} \tag{3.2}$$

such that $(u(0), v(0)) = (u_0, v_0)$, and for almost all $t \geq 0$, $(u(t), v(t))$ satisfies $u(t)_{|\Gamma} = v(t)$ a.e. for $t \in (0, \infty)$, and the following partial differential equations:

$$\begin{gathered}
\partial_t u - \operatorname{div}(|\nabla u|^{p-2}\nabla u) + f(u) = h(x), \quad \text{in } \Omega \times (0, +\infty),\\
\partial_t v + |\nabla u|^{p-2}\partial_n u + g(v) = 0, \quad \text{on } \Gamma \times (0, +\infty).
\end{gathered} \tag{3.3}$$

Secondly, we give the following existence and uniqueness results, where we use the definition of weak solution as in [17, Definition 2.3]. For more details we refer the reader to [17].

**Theorem 3.1** ([17])**.** *Let $\Omega$ be a bounded smooth domain in $\mathbb{R}^N (N \geqslant 3)$, $f$ and $g$ satisfy* (1.2)–(1.4)*, $h(x) \in L^{r_1'}(\Omega)$. Then for any initial data $(u_0, v_0) \in L^2(\Omega) \times L^2(\Gamma)$ and any $T > 0$, the problem* (P) *has a unique weak solution $(u(t), v(t)) \in \mathcal{C}([0, T]; L^2(\Omega) \times L^2(\Gamma))$. In addition to the regularity stated in* (3.2)*, we also have that*

$$u(t) \in L^{r_1}(0, T; L^{r_1}(\Omega)), \quad v(t) \in L^{r_2}(0, T; L^{r_2}(\Gamma)).$$

*Furthermore, $(u_0, v_0) \mapsto (u(t), v(t))$ is continuous on $L^2(\Omega) \times L^2(\Gamma)$.*

By Theorem 2.3, we can define the operator semigroup $\{S(t)\}_{t\geqslant 0}$ on the phase space $L^2(\Omega) \times L^2(\Gamma)$ as follows:

$$S(t) : L^2(\Omega) \times L^2(\Gamma) \to L^2(\Omega) \times L^2(\Gamma), \quad S(t)(u_0, v_0) = (u(t), v(t)), \tag{3.4}$$

which is continuous in $L^2(\Omega) \times L^2(\Gamma)$.

Next, exactly as in [17], we have the following dissipative results.

**Lemma 3.2** ([17])**.** *Under the assumption of Theorem 2.3, $\{S(t)\}_{t\geqslant 0}$ has a positively invariant $(L^2(\Omega) \times L^2(\Gamma), W^{1,p}(\Omega) \cap L^{r_1}(\Omega) \times W^{1-1/p,p}(\Gamma) \cap L^{r_2}(\Gamma))$-bounded absorbing set; that is, there is a positive constant $M$, such that for any bounded subset $B \subset L^2(\Omega) \times L^2(\Gamma)$, there exists a positive constant $T$ which depends only on the $L^2(\Omega) \times L^2(\Gamma)$-norm of $B$ such that*

$$\int_\Omega |\nabla u(t)|^p\,dx + \int_\Omega |u(t)|^{r_1}\,dx + \int_\Gamma |v(t)|^{r_2}\,dS \leqslant M \quad \textit{for all } t \geqslant T \textit{ and } (u_0, v_0) \in B.$$

**Lemma 3.3** ([17])**.** *Under the assumption of Theorem 2.3, for any bounded subset $B \subset L^2(\Omega) \times L^2(\Gamma)$, there exists a positive constant $T_1$ which depends only on the $L^2(\Omega) \times L^2(\Gamma)$-norm of $B$ such that*

$$\int_\Omega |u_t(s)|^2\,dx + \int_\Gamma |v_t(s)|^2\,dS \leqslant M' \quad \textit{for all } s \geqslant T_1 \textit{ and } (u_0, v_0) \in B, \tag{3.5}$$

*where $M'$ is a positive constant which depends on $M$.*

Hereafter, from Lemma 3.2, we denote one of the positively invariant absorbing set by $B_0$ with

$$B_0 \subset \{(u(t), v(t)) : \|u(t)\|_{W^{1,p}(\Omega)\cap L^{r_1}(\Omega)} + \|v(t)\|_{W^{1-1/p,p}(\Gamma)\cap L^{r_2}(\Gamma)} \leqslant M\},$$

note that here the positive invariance means $S(t)B_0 \subset B_0$ for any $t \geqslant 0$.

3.2. **Asymptotic regularity.** In this subsection, we consider the asymptotic regularity of solutions of systems (3.1), which excel the regularity allowed by the corresponding elliptic equation.

At first, we consider the elliptic equation

$$\begin{gathered} -\operatorname{div}(|\nabla\phi|^{p-2}\nabla\phi) + f(\phi) = h(x) \quad \text{in } \Omega, \\ |\nabla\phi|^{p-2}\partial_n\phi + g(\phi) = 0 \quad \text{on } \Gamma. \end{gathered} \tag{3.6}$$

Due to the assumptions (1.2)–(1.4), from the classical results about elliptic equations, we know that (3.6) at least has one solution $\phi(x)$ with

$$\phi(x) \in W^{1,p}(\Omega) \cap L^{r_1}(\Omega). \tag{3.7}$$

For the rest of this article, we assume that $\phi(x)$ denotes a fixed solution of (3.6). Then, for the solution $(u(x,t), v(x,t))$ of (3.1), we decompose $(u(x,t), v(x,t))$ as follows

$$(u(x,t), v(x,t)) = (\phi(x) + w(x,t), \phi(x) + \widetilde{w}(x,t)) \tag{3.8}$$

with $u_0(x) = \phi(x) + w(x,0)$, $v_0(x) = \phi(x) + \widetilde{w}(x,0)$, where $(w(x,t), \widetilde{w}(x,t))$ solves the equation

$$\begin{gathered} w_t - \operatorname{div}(|\nabla u|^{p-2}\nabla u) + \operatorname{div}(|\nabla\phi|^{p-2}\nabla\phi) + f(u) - f(\phi) = 0 \quad \text{in } \Omega, \\ \widetilde{w}_t + |\nabla u|^{p-2}\partial_n u - |\nabla\phi|^{p-2}\partial_n\phi + g(v) - g(\phi) = 0, \quad \text{on } \Gamma, \\ \widetilde{w}(x,t) := w(x,t)_{|\Gamma}, \\ w(x,0) = u_0(x) - \phi(x), \\ \widetilde{w}(x,0) = v_0(x) - \phi(x). \end{gathered} \tag{3.9}$$

It is easy to see that this equation is also globally well posed. Moreover, thanks to Lemma 3.2, without loss of generality, hereafter we assume $(u_0, v_0) \in B_0$ and so $(w(x,0), \widetilde{w}(x,0)) \in (W^{1,p}(\Omega) \cap L^{r_1}(\Omega)) \times (W^{1-1/p,p}(\Gamma) \cap L^{r_2}(\Gamma))$.

At the same time, from the positive invariance of $B_0$ and (3.7) we have that

$$\|w(x,t)\|_{W^{1,p}(\Omega)\cap L^{r_1}(\Omega)} + \|\widetilde{w}(x,t)\|_{W^{1-1/p,p}(\Gamma)\cap L^{r_2}(\Gamma)} \leqslant M_1 \tag{3.10}$$

for all $t \geqslant 0$, with some positive constant $M_1$.

The main result of this section reads as follows.

**Theorem 3.4.** *Let $\Omega$ be a bounded smooth domain in $\mathbb{R}^N$ $(N \geqslant 3)$, $f$ and $g$ satisfy (1.2)–(1.4), $h(x) \in L^{r_1'}(\Omega)$, and suppose that $\{S(t)\}_{t\geq 0}$ is the semigroup generated by the solutions of equation (3.1) with initial data $(u_0, v_0) \in L^2(\Omega) \times L^2(\Gamma)$. Then, for any $\delta, \gamma \in [0, \infty)$, there exists a bounded subset $B_{\delta,\gamma}$ satisfying the following properties:*

$$\begin{aligned} B_{\delta,\gamma} = \Big\{(w, \widetilde{w}) : \|w\|_{W^{1,p}(\Omega)\cap L^{r_1+\delta}(\Omega)} \\ + \|\widetilde{w}\|_{W^{1-1/p,p}(\Gamma)\cap L^{r_2+\gamma}(\Gamma)} \leqslant \Lambda_{p,r_1,r_2,N,\delta,\gamma} < \infty\Big\}, \end{aligned}$$

*and for any bounded subset $B \subset L^2(\Omega) \times L^2(\Gamma)$, there exists a*

$$T = T(\|B\|_{L^2(\Omega)}, \|B\|_{L^2(\Gamma)}, \delta, \gamma)$$

*such that*

$$S(t)B \subset \phi(x) + B_{\delta,\gamma} \quad \text{for all } t \geqslant T, \tag{3.11}$$

*where* $\phi(x)$ *is a fixed solution of* (3.6), $(w(x,t), \widetilde{w}(x,t))$ *satisfies* (3.9)*; the constant* $\Lambda_{p,r_1,r_2,N,\delta,\gamma}$ *depends only on* $p, r_1, r_2, N, \delta, \gamma$.

*Proof.* We use the Moser-Alikakos iteration technique [2] to prove the following induction estimates about the solution of (3.9). For clarity, we separate our proof into two steps.

*Step 1*: We first claim that

*For each* $k = 0, 1, 2, \dots$, *there exist two positive constants* $T_k$ *and* $M_k$, *which depend only on* $k, p, r_1, r_2, N$ *and* $\|B_0\|_{W^{1,p}(\Omega)\cap L^{r_1}(\Omega)\times W^{1-1/p,p}(\Gamma)\cap L^{r_2}(\Gamma)}$, *such that for any* $(u_0, v_0) \in B_0$ *and* $t \geqslant T_k$, *we have*

$$\int_\Omega |w(t)|^{\sigma_k}\,dx + \int_\Gamma |\widetilde{w}(t)|^{\sigma_k}\,dS \leqslant M_k, \tag{$A_k$}$$

*and*

$$\int_t^{t+1}\Big(\int_\Omega |w(s)|^{\sigma_{k+1}}\,dx\Big)^{\frac{N-p}{N-1}} ds + \int_t^{t+1}\Big(\int_\Gamma |\widetilde{w}(s)|^{\sigma_{k+1}}\,dS\Big)^{\frac{N-p}{N-1}} ds \leqslant M_k. \tag{$B_k$}$$

*where* $(w(t), \widetilde{w}(t))$ *is the solution of equation* (3.9), *and*

$$\sigma_k = 2\Big(\frac{N-1}{N-p}\Big)^k + (p-2)\Big[\sum_{i=0}^k \Big(\frac{N-1}{N-p}\Big)^i - 1\Big], \quad k = 0, 1, 2, \dots. \tag{3.12}$$

(i) Initialization of the induction ($k = 0$). From (3.10), we can deduce ($A_0$) immediately. To prove ($B_0$), we multiply (3.9) by $w$ and $\widetilde{w}$, and integrate over $\Omega$, then we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_\Omega |w|^2\,dx + \frac{1}{2}\frac{d}{dt}\int_\Gamma |\widetilde{w}|^2\,dS + \int_\Omega \langle |\nabla u|^{p-2}\nabla u - |\nabla\phi|^{p-2}\nabla\phi, \nabla w\rangle\,dx \\ &+ \int_\Omega (f(u) - f(\phi))w\,dx + \int_\Gamma (g(v) - g(\phi))\widetilde{w}\,dS = 0. \end{aligned} \tag{3.13}$$

By (1.4), we have

$$\int_\Omega (f(u) - f(\phi))w\,dx \geqslant -l\int_\Omega |w|^2\,dx, \tag{3.14}$$

$$\int_\Gamma (g(v) - g(\phi))\widetilde{w}\,dS \geqslant -m\int_\Gamma |\widetilde{w}|^2\,dS. \tag{3.15}$$

Then applying Lemma 2.4, we have

$$\int_\Omega \langle |\nabla u|^{p-2}\nabla u - |\nabla\phi|^{p-2}\nabla\phi, \nabla w\rangle\,dx \geqslant K\int_\Omega |\nabla w|^p\,dx. \tag{3.16}$$

Inserting (3.14)–(3.16) into (3.13), we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_\Omega |w|^2\,dx + \frac{1}{2}\frac{d}{dt}\int_\Gamma |\widetilde{w}|^2\,dS + K\int_\Omega |\nabla w|^p\,dx \\ &\leqslant l\int_\Omega |w|^2\,dx + m\int_\Gamma |\widetilde{w}|^2\,dS \\ &\leqslant C\Big(\int_\Omega |w|^2\,dx + \int_\Gamma |\widetilde{w}|^2\,dS\Big). \end{aligned} \tag{3.17}$$

Then, for any $t \geqslant 0$, integrating the above inequality over $[t, t+1]$ and using (3.10), we deduce that

$$\int_t^{t+1} \int_\Omega |\nabla w(x,s)|^p \, dx \, ds \leqslant C_{K,M,M_1} \quad \text{for all } t \geqslant 0. \tag{3.18}$$

By the Sobolev embeddings (e.g., see Adams and Fourier [1])

$$W^{1,p}(\Omega) \hookrightarrow L^{\frac{p(N-1)}{N-p}}(\Omega), \quad W^{1,p}(\Omega) \hookrightarrow L^{\frac{p(N-1)}{N-p}}(\Gamma),$$

from (3.18), for all $t \geqslant 0$, we have

$$\begin{aligned} &\int_t^{t+1} \Big( \int_\Omega |w(x,s)|^{\frac{p(N-1)}{N-p}} \, dx \Big)^{\frac{N-p}{N-1}} ds \\ &\leqslant C_1 \int_t^{t+1} \int_\Omega |\nabla w(x,s)|^p \, dx \, ds \leqslant C_{K,M,M_1,N}, \end{aligned} \tag{3.19}$$

$$\begin{aligned} &\int_t^{t+1} \Big( \int_\Gamma |\widetilde{w}(x,s)|^{\frac{p(N-1)}{N-p}} \, dS \Big)^{\frac{N-p}{N-1}} ds \\ &\leqslant C_2 \int_t^{t+1} \int_\Omega |\nabla w(x,s)|^p \, dx \, ds \leqslant C_{K,M,M_1,N}, \end{aligned} \tag{3.20}$$

where $C_1, C_2$ are constants of embeddings $W^{1,p}(\Omega) \hookrightarrow L^{\frac{p(N-1)}{N-p}}(\Omega)$ and $W^{1,p}(\Omega) \hookrightarrow L^{\frac{p(N-1)}{N-p}}(\Gamma)$, note that here $C_1, C_2$ depend only on $N$. This implies $(B_0)$ holds.

(ii) The induction argument. We now assume that $(A_k)$ and $(B_k)$ hold for $k \geqslant 1$, and we need only to prove that $(A_{k+1})$ and $(B_{k+1})$ hold. Multiplying (3.9) by $|w|^{\sigma_{k+1}-2}w$ and $|\widetilde{w}|^{\sigma_{k+1}-2}\widetilde{w}$, and integrating over $\Omega$, we obtain

$$\begin{aligned} &\frac{1}{\sigma_{k+1}} \frac{d}{dt} \Big( \int_\Omega |w|^{\sigma_{k+1}} \, dx + \int_\Gamma |\widetilde{w}|^{\sigma_{k+1}} \, dS \Big) \\ &+ (\sigma_{k+1} - 1) \int_\Omega \langle |\nabla u|^{p-2} \nabla u - |\nabla \phi|^{p-2} \nabla \phi, \nabla w \rangle |w|^{\sigma_{k+1}-2} \, dx \\ &+ \int_\Omega \big( f(u) - f(\phi) \big) |w|^{\sigma_{k+1}-2} w \, dx + \int_\Gamma \big( g(v) - g(\phi) \big) |\widetilde{w}|^{\sigma_{k+1}-2} \widetilde{w} \, dS = 0. \end{aligned} \tag{3.21}$$

Similar to (3.14)–(3.16), we have

$$\int_\Omega \big( f(u) - f(\phi) \big) |w|^{\sigma_{k+1}-2} w \, dx \geqslant -l \int_\Omega |w|^{\sigma_{k+1}} \, dx, \tag{3.22}$$

$$\int_\Gamma (g(v) - g(\phi)) |\widetilde{w}|^{\sigma_{k+1}-2} \widetilde{w} \, dS \geqslant -m \int_\Gamma |\widetilde{w}|^{\sigma_{k+1}} \, dS, \tag{3.23}$$

$$\begin{aligned} &(\sigma_{k+1} - 1) \int_\Omega \langle |\nabla u|^{p-2} \nabla u - |\nabla \phi|^{p-2} \nabla \phi, \nabla w \rangle |w|^{\sigma_{k+1}-2} \, dx \\ &\geqslant K(\sigma_{k+1} - 1) \int_\Omega |\nabla w|^p |w|^{\sigma_{k+1}-2} \, dx, \end{aligned} \tag{3.24}$$

so we have

$$\begin{aligned} &\frac{1}{\sigma_{k+1}} \frac{d}{dt} \Big( \int_\Omega |w|^{\sigma_{k+1}} \, dx + \int_\Gamma |\widetilde{w}|^{\sigma_{k+1}} \, dS \Big) + K(\sigma_{k+1} - 1) \int_\Omega |\nabla w|^p |w|^{\sigma_{k+1}-2} \, dx \\ &\leqslant l \int_\Omega |w|^{\sigma_{k+1}} \, dx + m \int_\Gamma |\widetilde{w}|^{\sigma_{k+1}} \, dS \leqslant C \Big( \int_\Omega |w|^{\sigma_{k+1}} \, dx + \int_\Gamma |\widetilde{w}|^{\sigma_{k+1}} \, dS \Big). \end{aligned} \tag{3.25}$$

Then, combining with $(B_k)$ and application of the uniform Gronwall lemma to (3.25) we can get $(A_{k+1})$ immediately. For $(B_{k+1})$, we integrate the above inequality over $[t, t+1]$ and use $(A_{k+1})$, we have

$$\int_t^{t+1}\int_\Omega |\nabla w|^p |w|^{\sigma_{k+1}-2}\,dx\,ds \leqslant M_{k+1} \quad \text{for all } t \geqslant 0, \tag{3.26}$$

where $M_{k+1}$ depends on $k, p, r_1, r_2, N, M, M_1$. By the embeddings $W^{1,p}(\Omega) \hookrightarrow L^{\frac{p(N-1)}{N-p}}(\Omega)$ and $W^{1,p}(\Omega) \hookrightarrow L^{\frac{p(N-1)}{N-p}}(\Gamma)$ again, we have

$$\begin{aligned} &\Big(\int_\Omega |w|^{(\sigma_{k+1}-2+p)\frac{N-1}{N-p}}\,dx\Big)^{\frac{N-p}{N-1}} \\ &\leqslant C_1 \cdot \big(\frac{p}{\sigma_{k+1}-2+p}\big)^p \int_\Omega |w|^{\sigma_{k+1}-2}|\nabla w|^p\,dx, \end{aligned} \tag{3.27}$$

$$\begin{aligned} &\Big(\int_\Gamma |\widetilde{w}|^{(\sigma_{k+1}-2+p)\frac{N-1}{N-p}}\,dS\Big)^{\frac{N-p}{N-1}} \\ &\leqslant C_2 \cdot \big(\frac{p}{\sigma_{k+1}-2+p}\big)^p \int_\Omega |w|^{\sigma_{k+1}-2}|\nabla w|^p\,dx, \end{aligned} \tag{3.28}$$

and from the definition of $\sigma_k$, we have

$$(\sigma_{k+1}-2+p)\frac{N-1}{N-p} = \sigma_{k+2}. \tag{3.29}$$

Combining (3.26)–(3.29), we deduce $(B_{k+1})$ immediately.

*Step 2*: Based on Step 1, since $N \geqslant 3$, from the definition of $\sigma_k$ given in (3.12), it is easy to see that $\sigma_k \to \infty$ as $k \to \infty$.

Hence, for any $\delta, \gamma \in [0, \infty)$, we can take $k$ so large that $r_1 + \delta \leqslant \sigma_k$, $r_2 + \gamma \leqslant \sigma_k$. Consequently, we can define $\mathcal{B}_{\delta,\gamma}$ as

$$\begin{aligned} \mathcal{B}_{\delta,\gamma} := \Big\{(z, \tilde{z}) :\ &\|z+\phi\|^p_{W^{1,p}(\Omega)} + \|z\|^{r_1+\delta}_{L^{r_1+\delta}(\Omega)} \\ &+ \|\tilde{z}+\phi\|^p_{W^{1-1/p,p}(\Gamma)} + \|\tilde{z}\|^{r_2+\gamma}_{L^{r_2+\gamma}(\Gamma)} \leqslant M + M_k\Big\}, \end{aligned}$$

where $z(t)_{|\Gamma} = \tilde{z}(t)$, and recall that $\phi(x)$ is a fixed solution of (3.6). □

Hence, from Theorem 3.4, using the interpolation inequality, we can obtain immediately the following results.

**Theorem 3.5.** *Under the assumptions of Theorem 3.4, the semigroup $\{S(t)\}_{t\geqslant 0}$ has a $(L^2(\Omega) \times L^2(\Gamma), W^{1,p}(\Omega) \cap L^{r_1}(\Omega) \times W^{1-1/p,p}(\Gamma) \cap L^{r_2}(\Gamma))$-global attractor $\mathscr{A}$. Moreover, $\mathscr{A}$ attracts every $L^2(\Omega) \times L^2(\Gamma)$-bounded subset with $(W^{1,p}(\Omega) \cap L^{r_1+\delta}(\Omega)) \times (W^{1-1/p,p}(\Gamma) \cap L^{r_2+\gamma}(\Gamma))$-norm for any $\delta, \gamma \in [0, \infty)$; and $\mathscr{A}$ allows the decomposition $\mathscr{A} = \phi(x) + \mathscr{A}_0$ with $\mathscr{A}_0$ is bounded in $(W^{1,p}(\Omega) \cap L^{r_1+\delta}(\Omega)) \times (W^{1-1/p,p}(\Gamma) \cap L^{r_2+\gamma}(\Gamma))$ for any $\delta, \gamma \in [0, \infty)$, and $\phi(x)$ is a fixed solution of* (3.6).

*Proof.* From Theorem 3.4, combining with the $(L^2(\Omega) \times L^2(\Gamma), L^2(\Omega) \times L^2(\Gamma))$-asymptotic compactness (obtained in [17]) and the interpolation inequality, it is easily to verify that $\{S(t)\}_{t\geqslant 0}$ is asymptotically compact in $L^{r_1}(\Omega) \times L^{r_2}(\Gamma)$, then it is sufficient to verify that $\{S(t)\}_{t\geqslant 0}$ is asymptotically compact in $W^{1,p}(\Omega) \times W^{1-1/p,p}(\Gamma)$.

Let $B_0$ be a $(W^{1,p}(\Omega)\cap L^{r_1}(\Omega))\times(W^{1-1/p,p}(\Gamma)\cap L^{r_2}(\Gamma))$-bounded absorbing set obtained in Lemma 3.2, then we need only to show that

$$\text{for any } \{(u_{0n},v_{0n})\}\subset B_0 \text{ and } t_n\to\infty,\ \{(u_n(t_n),v_n(t_n))\}_{n=1}^{\infty} \text{ is precompact in } W^{1,p}(\Omega)\times W^{1-1/p,p}(\Gamma), \tag{3.30}$$

where $u_n(t_n)=S(t_n)u_{0n}, v_n(t_n)=S(t_n)v_{0n}$.

In fact, we know that $\{(u_n(t_n),v_n(t_n))\}_{n=1}^{\infty}$ is precompact in $L^2(\Omega)\times L^2(\Gamma)$ and in $L^{r_1}(\Omega)\times L^{r_2}(\Gamma)$.

Without loss of generality, we assume that $\{(u_{n_k}(t_{n_k}),v_{n_k}(t_{n_k}))\}_{n=1}^{\infty}$ is a Cauchy sequence in $L^2(\Omega)\times L^2(\Gamma)$ and $L^{r_1}(\Omega)\times L^{r_2}(\Gamma)$.

Now, we prove that $\{(u_{n_k}(t_{n_k}),v_{n_k}(t_{n_k}))\}_{n=1}^{\infty}$ is a Cauchy sequence in $W^{1,p}(\Omega)\times W^{1-1/p,p}(\Gamma)$. From Lemma 2.4, and after standard transformations, we know that there exists a constant $K>0$, such that

$$\begin{aligned}
&K\|\nabla(u_{n_k}(t_{n_k})-u_{n_j}(t_{n_j}))\|_{L^p(\Omega)}^p\\
&\le\Big\langle -\frac{d}{dt}u_{n_k}(t_{n_k})-f(u_{n_k}(t_{n_k}))+\frac{d}{dt}u_{n_j}(t_{n_j})+f(u_{n_j}(t_{n_j})),u_{n_k}(t_{n_k})-u_{n_j}(t_{n_j})\Big\rangle\\
&\quad+\Big\langle -\frac{d}{dt}v_{n_k}(t_{n_k})-g(v_{n_k}(t_{n_k}))+\frac{d}{dt}v_{n_j}(t_{n_j})+g(v_{n_j}(t_{n_j})),v_{n_k}(t_{n_k})-v_{n_j}(t_{n_j})\Big\rangle_\Gamma\\
&\le\int_\Omega\Big|\frac{d}{dt}u_{n_k}(t_{n_k})-\frac{d}{dt}u_{n_j}(t_{n_j})\Big||u_{n_k}(t_{n_k})-u_{n_j}(t_{n_j})|\\
&\quad+\int_\Omega|f(u_{n_k}(t_{n_k}))-f(u_{n_j}(t_{n_j}))||u_{n_k}(t_{n_k})-u_{n_j}(t_{n_j})|\\
&\quad+\int_\Gamma\Big|\frac{d}{dt}v_{n_k}(t_{n_k})-\frac{d}{dt}v_{n_j}(t_{n_j})\Big||v_{n_k}(t_{n_k})-v_{n_j}(t_{n_j})|\\
&\quad+\int_\Gamma|g(v_{n_k}(t_{n_k}))-g(v_{n_j}(t_{n_j}))||v_{n_k}(t_{n_k})-v_{n_j}(t_{n_j})|,
\end{aligned}$$

so we have

$$\begin{aligned}
&K\|\nabla(u_{n_k}(t_{n_k})-u_{n_j}(t_{n_j}))\|_{L^p(\Omega)}^p\\
&\le\Big\|\frac{d}{dt}u_{n_k}(t_{n_k})-\frac{d}{dt}u_{n_j}(t_{n_j})\Big\|\ \|u_{n_k}(t_{n_k})-u_{n_j}(t_{n_j})\|\\
&\quad+\Big\|\frac{d}{dt}v_{n_k}(t_{n_k})-\frac{d}{dt}v_{n_j}(t_{n_j})\Big\|_\Gamma\ \|v_{n_k}(t_{n_k})-v_{n_j}(t_{n_j})\|_\Gamma\\
&\quad+C\big(1+\|u_{n_k}(t_{n_k})\|_{L^{r_1}(\Omega)}^{r_1}+\|u_{n_j}(t_{n_j})\|_{L^{r_1}(\Omega)}^{r_1}\big)\ \big\|u_{n_k}(t_{n_k})-u_{n_j}(t_{n_j})\big\|_{L^{r_1}(\Omega)}\\
&\quad+\widetilde{C}\big(1+\|v_{n_k}(t_{n_k})\|_{L^{r_2}(\Gamma)}^{r_2}+\|v_{n_j}(t_{n_j})\|_{L^{r_2}(\Gamma)}^{r_2}\big)\ \big\|v_{n_k}(t_{n_k})-v_{n_j}(t_{n_j})\big\|_{L^{r_2}(\Gamma)}.
\end{aligned}\tag{3.31}$$

Combining Lemma 3.2, Lemma 3.3 and the compactness of $L^{r_1}(\Omega)\times L^{r_2}(\Gamma)$, and since $W^{1,p}(\Omega)\hookrightarrow W^{1-1/p,p}(\Gamma)$, we know that the norms on $W^{1,p}(\Omega)\times W^{1-1/p,p}(\Gamma)$ and $W^{1,p}(\Omega)$ are equivalent, (3.31) yields (3.30) immediately. □

## 4. Non-autonomous case

In this section, we discuss the non-autonomous case of (1.1); that is,

$$\begin{gathered}
u_t-\Delta_p u+f(u)=h(x,t),\quad \text{in } \Omega,\\
u_t+|\nabla u|^{p-2}\partial_n u+g(u)=0,\quad \text{on } \Gamma,\\
u(x,\tau)=u_\tau(x),\quad \text{in } \bar{\Omega},
\end{gathered}\tag{4.1}$$

where $h(x,t) \in L^2_b(\mathbb{R}; L^2(\Omega))$.

4.1. **Mathematical setting.** Similar to the autonomous cases (e.g., Problem (p) and Theorem 2.3), for each $h \in \Sigma$, we can also easily obtain the following well-posedness result and the time-dependent terms make no essential complications.

**Theorem 4.1** ([17]). *Let $\Omega$ be a bounded smooth domain in $\mathbb{R}^N$ $(N \geqslant 3)$, $f$ and $g$ satisfy (1.2)–(1.4), $h(x,t) \in L^2_b(\mathbb{R}; L^2(\Omega))$. Then for any initial data $(u_\tau, v_\tau) \in L^2(\Omega) \times L^2(\Gamma)$, and any $\tau, T \in \mathbb{R}$, $T > \tau$, the solution $(u(t), v(t))$ of problem (4.1) is globally defined and satisfies*

$$u(t) \in \mathcal{C}([\tau,T];\, L^2(\Omega)) \cap L^p_{\rm loc}(\tau,T; W^{1,p}(\Omega)) \cap L^{r_1}(\tau,T; L^{r_1}(\Omega)),$$
$$v(t) \in \mathcal{C}([\tau,T];\, L^2(\Gamma)) \cap L^p_{\rm loc}(\tau,T; W^{1-1/p,p}(\Gamma)) \cap L^{r_2}(\tau,T; L^{r_2}(\Gamma)),$$

*where $v(t) := u(t)_{|\Gamma}$. Furthermore, $(u_\tau, v_\tau) \mapsto (u(t), v(t))$ is continuous on $L^2(\Omega) \times L^2(\Gamma)$.*

We now define the symbol space $\Sigma$ for (4.1). Taking a fixed symbol $\sigma_0(s) = h_0(s)$, $h_0(s) \in L^2_b(\mathbb{R}; L^2(\Omega))$. We denote by $L^{2,w}_{\rm loc}(\mathbb{R}; L^2(\Omega))$ the space $L^2_{\rm loc}(\mathbb{R}; L^2(\Omega))$ endowed with local weak convergence topology. Set

$$\Sigma_0 = \{h_0(s+h) |\, h \in \mathbb{R}\}, \tag{4.2}$$

and let

$$\Sigma \text{ be the closure of } \Sigma_0 \text{ in } L^{2,w}_{\rm loc}(\mathbb{R}; L^2(\Omega)). \tag{4.3}$$

Systems (4.1) can be rewritten in the operator form

$$\partial_t y = A_{\sigma(t)}(y), \quad y|_{t=\tau} = y_\tau, \tag{4.4}$$

where $\sigma(t) = h(t)$ is the symbol of equation (4.4). Thus, from Theorem 4.1, we know that problem (4.1) is well posed for all $\sigma(s) \in \Sigma$ and generates a family of processes $\{U_\sigma(t,\tau)\}, \sigma \in \Sigma$ given by the formula $U_\sigma(t,\tau)y_\tau = y(t)$, and the $y(t)$ is the solution of (4.1).

4.2. **Existence of a bounded uniformly (w. r. t. $\sigma \in \Sigma$) absorbing set in $(W^{1,p}(\Omega) \cap L^{r_1}(\Omega)) \times (W^{1-1/p,p}(\Gamma) \cap L^{r_2}(\Gamma))$.** In this subsection, $(W^{1,p}(\Omega) \cap L^{r_1}(\Omega) \times W^{1-1/p,p}(\Gamma) \cap L^{r_2}(\Gamma))$-bounded uniformly (with respect to $\sigma \in \Sigma$) absorbing set is obtained. The proof is similar to [17] (autonomous case).

**Theorem 4.2.** *Let $\Omega$ be a bounded smooth domain in $\mathbb{R}^N$ $(N \geqslant 3)$, $f$ and $g$ satisfy (1.2)–(1.4), $h(x,t) \in L^2_b(\mathbb{R}; L^2(\Omega))$. Then the family of processes $\{U_\sigma(t,\tau)\}, \sigma \in \Sigma$ corresponding to (4.1) has a bounded uniformly (with respect to $\sigma \in \Sigma$) absorbing set $B_0$ in $(W^{1,p}(\Omega) \cap L^{r_1}(\Omega)) \times (W^{1-1/p,p}(\Gamma) \cap L^{r_2}(\Gamma))$, that is, there is a positive constant $M$, such that for any $\tau \in \mathbb{R}$ and any bounded subset $B$, there exists a positive constant $T = T(B,\tau) \geq \tau$ such that*

$$\int_\Omega |\nabla u(t)|^p\,dx + \int_\Omega |u(t)|^{r_1}\,dx + \int_\Gamma |v(t)|^{r_2}\,dS \leqslant M$$

*for all $t \geqslant T$, $(u_\tau, v_\tau) \in B$, $\sigma \in \Sigma$.*

*Proof.* Multiplying (4.1) by $u$ and $v$, and integrating by parts, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_\Omega |u|^2\,dx + \frac{1}{2}\frac{d}{dt}\int_\Gamma |v|^2\,dS + \int_\Omega |\nabla u|^p\,dx + \int_\Omega f(u)u\,dx + \int_\Gamma g(v)v\,dS \\ &= \int_\Omega h_0(t)u\,dx, \end{aligned} \tag{4.5}$$

combining with assumptions (1.2)–(1.4), Young's inequality and Poincaré inequality, we obtain

$$\begin{aligned} &\frac{d}{dt}\int_\Omega |u|^2\,dx + \frac{d}{dt}\int_\Gamma |v|^2\,dS + \mathcal{C}(\int_\Omega |u|^2\,dx + \int_\Gamma |v|^2\,dS) \\ &\le \mathcal{C}_{|\Omega|,S(\Gamma)} + \mathcal{C}\|h_0\|^2. \end{aligned} \tag{4.6}$$

Applying the suitable version of Gronwall's inequality to (4.6), we can find $T_0 > 0$ and $\rho_0 > 0$, such that

$$\|u(t)\|^2 + \|v(t)\|_\Gamma^2 \le \rho_0^2, \quad \text{for any } t \ge T_0. \tag{4.7}$$

Let $F(s) = \int_0^s f(\tau)d\tau$, $G(s) = \int_0^s g(\tau)d\tau$, by assumptions (1.2)–(1.3) again, from (4.5), we obtain

$$\begin{aligned} &\frac{d}{dt}\int_\Omega |u|^2\,dx + \frac{d}{dt}\int_\Gamma |v|^2\,dS + \int_\Omega |\nabla u|^p\,dx + \mathcal{C}_1\int_\Omega F(u)\,dx + \mathcal{C}_2\int_\Gamma G(v)\,dS \\ &\le \mathcal{C}_{|\Omega|,S(\Gamma)} + \mathcal{C}\|h_0\|^2. \end{aligned}$$

Integrating this inequality above from $t$ to $t+1$, and combining (4.7), it follows that for any $t \ge T_0$,

$$\begin{aligned} &\int_t^{t+1}(\int_\Omega |\nabla u|^p\,dx + \mathcal{C}_1\int_\Omega F(u)\,dx + \mathcal{C}_2\int_\Gamma G(v)\,dS)ds \\ &\le \mathcal{C}_{|\Omega|,S(\Gamma),\rho_0} + \mathcal{C}\int_t^{t+1}\|h_0\|^2 ds \\ &\le \mathcal{C}_{|\Omega|,S(\Gamma),\rho_0,\|h_0\|_b^2}. \end{aligned} \tag{4.8}$$

On the other hand, multiplying (1.1) by $u_t$ and $v_t$, we have

$$\begin{aligned} &\int_\Omega |u_t|^2\,dx + \int_\Gamma |v_t|^2\,dS + \frac{1}{p}\frac{d}{dt}\int_\Omega |\nabla u|^p\,dx + \frac{d}{dt}\big(\int_\Omega F(u)\,dx + \int_\Gamma G(v)\,dS\big) \\ &\le \frac{1}{2}\int_\Omega |h_0|^2\,dx + \frac{1}{2}\int_\Omega |u_t|^2\,dx, \end{aligned} \tag{4.9}$$

so we obtain

$$\frac{d}{dt}(\int_\Omega |\nabla u|^p\,dx + p\int_\Omega F(u)\,dx + p\int_\Gamma G(v)\,dS) \le \mathcal{C}\|h_0\|^2. \tag{4.10}$$

Combining (4.8) and (4.10), by the uniformly Gronwall lemma, we have that for any $t \ge T_0 + 1$, $\sigma \in \Sigma$,

$$\int_\Omega |\nabla u|^p\,dx + \int_\Omega F(u)\,dx + \int_\Gamma G(v)\,dS \le \mathcal{C}_{|\Omega|,S(\Gamma),\rho_0,\|h\|_b^2}, \tag{4.11}$$

which implies that for any $t \ge T_0 + 1$, $\sigma \in \Sigma$,

$$\int_\Omega |\nabla u|^p\,dx + \int_\Omega |u|^{r_1}\,dx + \int_\Gamma |v|^{r_2}\,dS \le M, \tag{4.12}$$

where $M$ depends on $|\Omega|, S(\Gamma), \rho_0, \|h\|_b^2$. □

As a direct result of Theorem 4.2, we have the existence of a uniform attractor in $L^2(\Omega) \times L^2(\Gamma)$:

**Corollary 4.3.** *Under the assumptions of Theorem 4.2, the family of processes* $\{U_\sigma(t,\tau)\}, \sigma \in \Sigma$ *corresponding to* (4.1) *has a uniform attractor* $\mathcal{A}_\Sigma$ *in* $L^2(\Omega) \times L^2(\Gamma)$ *, which is compact in* $L^2(\Omega)\times L^2(\Gamma)$ *and attracts every* $L^2(\Omega)\times L^2(\Gamma)$*-bounded subset with* $L^2(\Omega) \times L^2(\Gamma)$*-norm. Moreover,*

$$\mathcal{A}_\Sigma = \omega_{0,\Sigma}(B_0) = \cup_{\sigma\in\Sigma}\mathcal{K}_\sigma(s), \quad \forall\, s \in \mathbb{R},$$

*where* $\mathcal{K}_\sigma(s)$ *is the section at* $t = s$ *of the kernel* $\mathcal{K}_\sigma$ *of the process* $\{U_\sigma(t,\tau)\}$ *with symbol* $\sigma$*.*

*Proof.* Theorem 4.2 and the Sobolev compactness imbedding theorem imply the existence of a uniform attractor $\mathcal{A}_\Sigma$ in $L^2(\Omega) \times L^2(\Gamma)$ immediately. □

4.3. **Existence of a uniform attractor in** $L^{r_1}(\Omega) \times L^r(\Gamma)$ **(**$r = \min(r_1, r_2)$**).** First, we give some a priori estimates for the solution of (4.1) to verify the uniformly asymptotic compactness in $L^{r_1}(\Omega)\times L^{r_1}(\Gamma)$. The idea of the proof comes from [31].

**Theorem 4.4.** *Assume that* $h(t)$ *is normal in* $L^2_{\rm loc}(\mathbb{R}; L^2(\Omega))$*,* $f$ *and* $g$ *satisfy* (1.2)–(1.3)*. Then for any* $\varepsilon > 0$*,* $\tau \in \mathbb{R}$ *and any bounded subset* $B \subset L^2(\Omega) \times L^2(\Gamma)$*, there exist two positive constants* $T = T(B, \varepsilon, \tau)$ *and* $M = M(\varepsilon)$*, such that*

$$\int_{\Omega(|U_\sigma(t,\tau)u_\tau|\geq M)} |U_\sigma(t,\tau)u_\tau|^{r_1} + \int_{\Gamma(|U_\sigma(t,\tau)v_\tau|\geq M)} |U_\sigma(t,\tau)v_\tau|^{r_1} \leq \varepsilon,$$

*for all* $t \geq T$*,* $(u_\tau, v_\tau) \in B$*,* $\sigma \in \Sigma$*.*

*Proof.* We multiply (4.1) by $(u - M)_+^{r_1-1}$ and $(v - M)_+^{r_1-1}$, and integrate over $\Omega$, then we have

$$\begin{aligned}
&\frac{1}{r_1}\frac{d}{dt}\int_{\Omega(u\geq M)} |u - M|^{r_1}\, dx + \frac{1}{r_1}\frac{d}{dt}\int_{\Gamma(v\geq M)} |v - M|^{r_1}\, dS \\
&+ (r_1 - 1)\int_{\Omega(u\geq M)} (u - M)^{r_1-2}|\nabla u|^p\, dx + \int_{\Omega(u\geq M)} f(u)(u - M)^{r_1-1}\, dx \\
&+ \int_{\Gamma(v\geq M)} g(v)(v - M)^{r_1-1}\, dS \\
&= \int_{\Omega(u\geq M)} h_0(t)(u - M)^{r_1-1}\, dx,
\end{aligned} \tag{4.13}$$

where $(u - M)_+$ denotes the positive part of $(u - M)$; that is,

$$(u - M)_+ = \begin{cases} u - M, & u \geq M, \\ 0, & u \leq M. \end{cases}$$

From conditions (1.2)–(1.3), we can take $M$ large enough such that

$$\begin{aligned}
\mathcal{C}_3|v|^{r_2-1} &\leq g(v), \quad \text{in } \Gamma(v(t) \geq M), \\
\mathcal{C}_4|u|^{r_1-1} &\leq f(u), \quad \text{in } \Omega(u(t) \geq M).
\end{aligned}$$

Let $\Omega_1 = \Omega(u(t) \geq M)$, $\Gamma_1 = \Gamma(v(t) \geq M)$, using Young's inequality and the inequalities above, we obtain

$$
\begin{aligned}
&\frac{1}{r_1}\frac{d}{dt}\int_{\Omega_1} |u-M|^{r_1}\,dx + \frac{1}{r_1}\frac{d}{dt}\int_{\Gamma_1} |v-M|^{r_1}\,dS \\
&+ (r_1-1)\int_{\Omega_1} (u-M)^{r_1-2}|\nabla u|^p\,dx \\
&+ \mathcal{C}_4\int_{\Omega_1} |u|^{r_1-1}(u-M)^{r_1-1}\,dx + \mathcal{C}_3\int_{\Gamma_1} |v|^{r_2-1}(v-M)^{r_1-1}\,dS \\
&\leq \frac{\mathcal{C}_4}{2}\int_{\Omega_1} |u-M|^{2r_1-2}\,dx + \frac{1}{2\mathcal{C}_4}\int_{\Omega_1} |h_0(t)|^2\,dx,
\end{aligned}
\tag{4.14}
$$

so we have

$$
\begin{aligned}
&\frac{1}{r_1}\frac{d}{dt}\int_{\Omega_1} |u-M|^{r_1}\,dx + \frac{1}{r_1}\frac{d}{dt}\int_{\Gamma_1} |v-M|^{r_1}\,dS \\
&+ (r_1-1)\int_{\Omega_1} (u-M)^{r_1-2}|\nabla u|^p\,dx \\
&+ \frac{\mathcal{C}_4 M^{r_1-2}}{2}\int_{\Omega_1} |u-M|^{r_1}\,dx + \mathcal{C}_3 M^{r_2-2}\int_{\Gamma_1} |v-M|^{r_1}\,dS \\
&\leq \frac{1}{2\mathcal{C}_4}\int_{\Omega_1} |h_0(t)|^2\,dx.
\end{aligned}
$$

By using the Gronwall lemma and together with the Lemma 2.3, we can choose $M$ large enough, such that

$$
\int_{\Omega_1} |u-M|^{r_1}\,dx + \int_{\Gamma_1} |v-M|^{r_1}\,dS \leq \varepsilon. \tag{4.15}
$$

Noting that

$$
\frac{1}{2^{r_1}}\int_{\Omega(u\geq 2M)} |u|^{r_1}\,dx \leq \int_{\Omega(u\geq M)} |u-M|^{r_1}\,dx, \tag{4.16}
$$

$$
\frac{1}{2^{r_1}}\int_{\Gamma(v\geq 2M)} |v|^{r_1}\,dS \leq \int_{\Gamma(v\geq M)} |v-M|^{r_1}\,dS, \tag{4.17}
$$

combining (4.15)–(4.17), we obtain

$$
\int_{\Omega(u\geq 2M)} |u(t)|^{r_1}\,dx + \int_{\Gamma(v\geq 2M)} |v(t)|^{r_1}\,dS \leq 2^{r_1}\varepsilon. \tag{4.18}
$$

Repeating the same steps above, just taking $(u+M)_-^{r_1-1}$ instead of $(u-M)_+^{r_1-1}$, $(v+M)_-^{r_1-1}$ instead of $(v-M)_+^{r_1-1}$, we deduce that

$$
\int_{\Omega(u\leq -2M)} |u(t)|^{r_1}\,dx + \int_{\Gamma(v\leq -2M)} |v(t)|^{r_1}\,dS \leq 2^{r_1}\varepsilon. \tag{4.19}
$$

Combining (4.18)–(4.19), we obtain

$$
\int_{\Omega(|u(t)|\geq 2M)} |u(t)|^{r_1}\,dx + \int_{\Gamma(|v(t)|\geq 2M)} |v(t)|^{r_1}\,dS \leq 2^{r_1}\varepsilon. \tag{4.20}
$$

□

Now we state the existence and structure of a uniform attractor in $L^{r_1}(\Omega)\times L^r(\Gamma)$ ($r=\min(r_1,r_2)$).

**Theorem 4.5.** *Assume that $h(t)$ is normal in $L^2_{\rm loc}(\mathbb{R}; L^2(\Omega))$, $f$ and $g$ satisfy (1.2)–(1.4). Then the family of processes $\{U_\sigma(t,\tau)\}, \sigma \in \Sigma$ corresponding to (4.1) has a compact uniform (with respect to $\sigma \in \Sigma$) attractor $\mathscr{A}_\Sigma$ in $L^{r_1}(\Omega) \times L^r(\Gamma)$ ($r = \min(r_1, r_2)$) and $\mathscr{A}_\Sigma$ satisfies*

$$\mathscr{A}_\Sigma = \omega_{0,\Sigma}(B_0) = \cup_{\sigma\in\Sigma}\mathcal{K}_\sigma(s), \quad \forall s \in \mathbb{R},$$

*where $\mathcal{K}_\sigma(s)$ is the section at $t = s$ of the kernel $\mathcal{K}_\sigma$ of the process $\{U_\sigma(t,\tau)\}$ with symbol $\sigma$.*

*Proof.* From Corollary 4.3 and Theorem 4.4, it is easy to verify that $\{U_\sigma(t,\tau)\}, \sigma \in \Sigma$ has uniformly asymptotic compactness in $L^{r_1}(\Omega) \times L^{r_1}(\Gamma)$, which combining with Theorem 4.2, we can obtain the existence of a compactly uniform attractor in $L^{r_1}(\Omega) \times L^r(\Gamma)$ ($r = \min(r_1, r_2)$). Then, similar to [24, 28], we can obtain the structure of $\mathscr{A}_\Sigma$, see more details in [24, 28]. □

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