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NONUNIQUENESS OF SOLUTIONS OF INITIAL-VALUE PROBLEMS FOR PARABOLIC p-LAPLACIAN

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Abstract. We construct a positive solution to a quasilinear parabolic problem in a bounded spatial domain with the p-Laplacian and a nonsmooth reaction function. We obtain nonuniqueness for zero initial data. Our method is based on sub- and supersolutions and the weak comparison principle.

Using the method of sub- and supersolutions we construct a positive solution to a quasilinear parabolic problem with the p-Laplacian and a reaction function that is non-Lipschitz on a part of the spatial domain. Thereby we obtain nonuniqueness for zero initial data.

1. Introduction

The problem of *uniqueness* and *nonuniqueness* of solutions to various types of initial (and boundary) value problems for quasilinear parabolic equations has been an interesting research topic for several decades (see, e.g., Fujita and Watanabe [3] and the references therein, Guedda [4], Ladyzhenskaya and Ural'tseva [6], and Oleinik and Kruzhkov [10]).

In this work we focus on the following problem with the p-Laplacian and a (partly) nonsmooth reaction function:

$$\begin{gathered} \frac{\partial u}{\partial t} - \Delta_p u = q(x)|u|^{\alpha-1}u \quad \text{for } (x,t) \in \Omega \times (0,T)\,; \\ u(x,t) = 0 \quad \text{for } (x,t) \in \partial\Omega \times (0,T)\,, \\ u(x,0) = 0 \quad \text{for } x \in \Omega\,. \end{gathered} \tag{1.1}$$

Here, $\Delta_p u := \operatorname{div}\big(|\nabla u|^{p-2}\nabla u\big)$ denotes the p-Laplacian for $1 < p < \infty$, $\alpha \in (0,1)$ is a given number, $0 < T < \infty$, and the potential q satisfies

(Q) $q \in C(\overline{\Omega})$, $q \geq 0$, and $q(x_0) > 0$ for some $x_0 \in \Omega$.

We assume that $\Omega \subset \mathbb{R}^N$ is a bounded domain with a $C^{1+\mu}$-boundary $\partial\Omega$ where $\mu \in (0,1)$.

In particular, we deal with degenerate (singular) diffusion if $2 < p < \infty$ ($1 < p < 2$, respectively) and the reaction function $f(x,u) := q(x)|u|^{\alpha-1}u$. Notice that if $q(x_0) > 0$ then the function $u \mapsto f(x_0,u)$ satisfies neither a local Lipschitz nor

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an Osgood (see [11]) condition near $u = 0$ provided $\alpha \in (0,1)$. The case $p = 2$ (the Laplace operator) was treated in Fujita and Watanabe [3] by entirely different methods based on the Green's function for the heat equation. An important special case, $N = 1$, $1 < p < \infty$, and $q(x) \equiv \lambda > 0$ (a constant), was treated in Guedda [4] also by different methods. The main purpose of the present article is to fill in the gap left open for $1 < p < \infty$, $p \neq 2$, and $q \in C(\overline{\Omega})$, $q \geq 0$, where q is not necessarily positive everywhere in Ω. Because of this possibly nonuniform positivity of q over Ω, the method used in [4] cannot be applied here. We use a different approach based on sub- and supersolutions and the weak comparison principle. As a trivial consequence of the fact that problem (1.1) possesses a nontrivial nonnegative solution (see our main result, Theorem 1), we conlude that the weak comparison principle does not hold for problem (1.1) considered with nontrivial initial conditions, say, in $W_0^{1,p}(\Omega)$.

Observe that our assumption (Q) implies that there exists $R > 0$ such that $q(x) \geq q_0 \equiv \text{const} > 0$ for all $x \in B_R(x_0)$ where

$$B_R(x_0) := \{x \in \mathbb{R}^N : |x - x_0| < R\} \subset \Omega.$$

Let $(\lambda_1, \varphi_{1,R})$ denote the first eigenpair for the operator $-\Delta_p \colon W_0^{1,p}(B_R(x_0)) \to W^{-1,p'}(B_R(x_0))$; that is,

$$\begin{gathered} -\Delta_p \varphi_{1,R} = \lambda_{1,R}\, \varphi_{1,R}^{p-1} \quad \text{in } B_R(x_0)\,; \\ \varphi_{1,R} = 0 \quad \text{on } \partial B_R(x_0)\,, \end{gathered} \tag{1.2}$$

and $\varphi_{1,R} \in W_0^{1,p}(B_R(x_0))$ is normalized by $\varphi_{1,R}(x_0) = 1$. Note that this normalization yields $0 < \varphi_{1,R}(x) \leq 1$ for all $x \in B_R(x_0)$. Moreover, we denote by

$$\widetilde{\varphi}_{1,R}(x) := \begin{cases} \varphi_{1,R}(x) & \text{for } x \in B_R(x_0)\,; \\ 0 & \text{for } x \in \overline{\Omega} \setminus B_R(x_0)\,, \end{cases} \tag{1.3}$$

the natural zero extension of $\varphi_{1,R}$ from $B_R(x_0)$ to the whole of $\overline{\Omega}$. Our main theorem is the following nonuniqueness result.

Theorem 1.1. *Assume that $0 < \alpha < \min\{1, p-1\}$ and (Q) are satisfied. Then there exists $T > 0$ small enough, such that problem (1.1) possesses (besides the trivial solution $u \equiv 0$) a nontrivial, nonnegative weak solution*

$$u \in C\big([0,T] \to L^2(\Omega)\big) \cap L^p\big((0,T) \to W^{1,p}(\Omega)\big)$$

which is bounded below by a subsolution $\underline{u} : \Omega \times (0,T) \to \mathbb{R}_+$ of type

$$\underline{u}(x,t) = \theta(t)\widetilde{\varphi}_{1,R}(x)^\beta \geq 0 \quad \text{in } \Omega \times (0,T)\,,$$

where $\theta \colon [0,T] \to \mathbb{R}_+$ is a strictly increasing, continuously differentiable function with $\theta(0) = 0$, and $\beta \in (1,\infty)$ is a suitable number.

In contrast with this nonuniqueness result, several uniqueness results have been established in [2].

Remark 1.2. Assume that $q \in L^\infty(\Omega)$ satisfies $0 \leq q(x) \leq \lambda_1$ a.e. in Ω, where λ_1 stands for the principal eigenvalue of $-\Delta_p$ with zero Dirichlet boundary conditions on Ω. Then the condition $\alpha < p - 1$ is essential for obtaining our nonuniqueness

result. Namely, if $\alpha = p-1$ then $u \equiv 0$ is the unique weak solution of (1.1). The uniqueness follows directly from the following standard energy estimate:
$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |u(x,t)|^2\,\mathrm{d}x + \int_\Omega |\nabla u|^p\,\mathrm{d}x = \int_\Omega q(x)|u|^p\,\mathrm{d}x \le \lambda_1 \int_\Omega |u|^p\,\mathrm{d}x\,.$$
By the variational characterization of λ_1 (Poincaré's inequality in Lindqvist [8]), we get
$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |u(x,t)|^2\,\mathrm{d}x \le -\int_\Omega |\nabla u|^p\,\mathrm{d}x + \lambda_1\int_\Omega |u|^p\,\mathrm{d}x \le 0\,,$$
which implies $u(x,t) \equiv 0$ in $\Omega \times (0,T)$, thanks to $u(x,0) \equiv 0$ in Ω.

A weaker result than our Theorem 1.1 has recently been published in Merchán, Montoro, and Peral [9, Theorem 2.2, p. 248]. There, a very strong uniform positivity condition on the potential q is assumed, $q_0 = \inf_\Omega q > 0$. This means that it suffices to treat the constant case $q(x) \equiv q_0 = \text{const} > 0$ and then use the resulting solution as a subsolution for the general case $q(x) \ge q_0 = \text{const} > 0$. In contrast, our Theorem 1.1 above does not assume $q_0 > 0$; we assume only $q \ge 0$ and $q \not\equiv 0$ in Ω. Nevertheless, our proof of this result, especially our construction of a nonzero subsolution, is simpler than in [9].

2. Proof of Theorem 1.1

Note that $\widetilde{\varphi}_{1,R}$ defined in (1.3) is continuous on $\overline{\Omega}$ and $\widetilde{\varphi}_{1,R}^\beta$ is continuously differentiable for any constant $\beta > 1$. We need to establish a few additional properties of $\varphi_{1,R}(x) \equiv \varphi_{1,R}(|x-x_0|) = \varphi_{1,R}(r)$, with $r = |x-x_0|$ and the usual harmless abuse of notation.

Lemma 2.1. *If $\beta \in (0,\infty)$ then*
$$-\Delta_p\Big(\varphi_{1,R}^\beta\Big) = \beta^{p-1}\varphi_{1,R}^{(p-1)(\beta-1)-1}\big[\lambda_{1,R}\varphi_{1,R}^p - (p-1)(\beta-1)|\nabla\varphi_{1,R}|^p\big] \tag{2.1}$$
holds pointwise a.e. in $B_R(x_0)$. In particular, for $\beta \ge 1$ we have
$$\frac{-\Delta_p(\varphi_{1,R}^\beta)}{\varphi_{1,R}^\beta} \le C \equiv \text{const} < \infty \quad \textit{pointwise a.e. in } B_R(x_0)\,. \tag{2.2}$$

Proof. Any function $u\colon B_R(x_0) \to \mathbb{R}$ that is radially symmetric around x_0 can be written as $u(x) = u(r)$ where $r = |x-x_0|$. Using this notation we obtain, by formal differentiation,
$$\begin{aligned}\Delta_p u(|x-x_0|) &= \operatorname{div}\Big(|u'(r)|^{p-2}u'(r)\frac{x-x_0}{r}\Big)\\ &= \Big(|u'(r)|^{p-2}u'(r)\Big)' + \frac{N-1}{r}|u'(r)|^{p-2}u'(r)\,.\end{aligned} \tag{2.3}$$
It is well-known that the first eigenfunction $\varphi_{1,R}$ is radially symmetric around x_0, positive, and C^2 in $\overline{B_R}(x_0) \setminus \{x_0\}$, see e.g. [1]. Therefore, we get a.e. in $B_R(x_0)$,
$$\begin{aligned}&\Delta_p\Big(\varphi_{1,R}^\beta(r)\Big)\\ &= \Big(\beta^{p-1}\varphi_{1,R}^{(p-1)(\beta-1)}|\varphi_{1,R}'|^{p-2}\varphi_{1,R}'\Big)'\\ &\quad + \frac{N-1}{r}\beta^{p-1}\varphi_{1,R}^{(p-1)(\beta-1)}|\varphi_{1,R}'|^{p-2}\varphi_{1,R}'\end{aligned}$$

$$\begin{aligned}
&= \beta^{p-1}\Big\{(p-1)(\beta-1)\varphi_{1,R}^{(p-1)(\beta-1)-1}|\varphi_{1,R}'|^p \\
&\quad + \varphi_{1,R}^{(p-1)(\beta-1)}\Big(|\varphi_{1,R}'|^{p-2}\varphi_{1,R}'\Big)' + \frac{N-1}{r}\varphi_{1,R}^{(p-1)(\beta-1)}|\varphi_{1,R}'|^{p-2}\varphi_{1,R}'\Big\} \\
&= \beta^{p-1}\varphi_{1,R}^{(p-1)(\beta-1)-1}\Big\{(p-1)(\beta-1)|\varphi_{1,R}'|^p - \lambda_{1,R}\varphi_{1,R}^p\Big\} \\
&= \beta^{p-1}\varphi_{1,R}^{(p-1)\beta}\Big\{(p-1)(\beta-1)\,\frac{|\varphi_{1,R}'|^p}{\varphi_{1,R}^p} - \lambda_{1,R}\Big\}\,.
\end{aligned}$$

Hence,

$$-\Delta_p\big(\varphi_{1,R}^\beta\big) \le \beta^{p-1}\lambda_{1,R}\varphi_{1,R}^{(p-1)\beta}$$

for $\beta \ge 1$. For $p \ge 2$ this yields

$$\frac{-\Delta_p\big(\varphi_{1,R}^\beta\big)}{\varphi_{1,R}^\beta} \le \beta^{p-1}\lambda_{1,R}\varphi_{1,R}^{(p-2)\beta} \le \beta^{p-1}\lambda_{1,R}\,,$$

thanks to our normalization $0 < \varphi_{1,R} \le 1$. On the other hand, for $1 < p < 2$,

$$\frac{-\Delta_p\big(\varphi_{1,R}^\beta\big)}{\varphi_{1,R}^\beta} = \beta^{p-1}\varphi_{1,R}^{(p-2)\beta}\big\{\lambda_{1,R} - (p-1)(\beta-1)\varphi_{1,R}^{-p}|\varphi_{1,R}'|^p\big\}. \tag{2.4}$$

Since $\varphi_{1,R}$ is radially decreasing and satisfies the Hopf maximum principle on the boundary of $B_R(x_0)$, we can choose $\varepsilon > 0$ such that $\varphi_{1,R}'(r) < \varphi_{1,R}'(R)/2 < 0$ for all $r \in (R-\varepsilon, R)$.

Hence, (2.4) implies (2.2) for $R-\varepsilon \le r < R$ provided $\varepsilon > 0$ is small enough, such that

$$\lambda_{1,R} - (p-1)(\beta-1)\varphi_{1,R}^{-p}|\varphi_{1,R}'|^p \le 0 \quad \text{for } R-\varepsilon \le r < R\,.$$

At the same time, the ratio $-\Delta_p\big(\varphi_{1,R}^\beta\big)/\varphi_{1,R}^\beta$ is bounded for $0 < r \le R-\varepsilon$. Thus, estimate (2.2) holds a.e. in $B_R(x_0)$. □

Proposition 2.2. *Assume that $0 < \alpha < \min\{1, p-1\}$ and (Q) are satisfied. Given any fixed number $S \in (0,\infty)$, we define*

$$\underline{u}(x,t) := \theta(t)\widetilde{\varphi}_{1,R}(x)^\beta \quad \textit{for } (x,t) \in \Omega \times [0,S]\,,$$

where $\beta > 1$, $\widetilde{\varphi}_{1,R}$ is given by (1.3), and $\theta : [0,S] \to \mathbb{R}_+$ is the positive solution of the Cauchy problem

$$\frac{\mathrm{d}\theta}{\mathrm{d}t}(t) = \frac{q_0}{2}\theta^\alpha(t) \quad \textit{for } t \in (0,S)\,; \quad \theta(0) = 0\,, \tag{2.5}$$

such that $0 < \theta(t) < \infty$ for every $t \in (0,S)$. Then $\underline{u} : \Omega \times (0,S) \to \mathbb{R}_+$ is a subsolution of problem (1.1) in a smaller domain $\Omega \times (0,\underline{\sigma})$, i.e., for $t \in (0,\underline{\sigma})$ only, where $\underline{\sigma} \in (0,S)$ is small enough.

Proof. We will show that the following inequality holds

$$\frac{\partial \underline{u}}{\partial t} - \Delta_p \underline{u} \le q(x)|\underline{u}|^{\alpha-1}\underline{u}.$$

Using $0 < \alpha < \min\{1, p-1\}$, equation (2.5), and the continuity of $\theta\colon [0,S) \to \mathbb{R}_+$, we get

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} \le -C\theta(t)^{p-1} + q_0\theta(t)^\alpha \quad \text{for all } t \in [0,\underline{\sigma}]\,, \tag{2.6}$$

where $\underline{\sigma} \in (0, S)$ is small enough, such that $\theta(t)^{p-1-\alpha} \le q_0/(2C)$ holds for all $t \in [0, \underline{\sigma}]$.

Inserting the inequality

$$\varphi_{1,R}^{-\beta} \Delta_p(\varphi_{1,R}^{\beta}) \ge -C \equiv \text{const}$$

in Ω from Lemma 2.1, inequality (2.2), into (2.6), we obtain

$$\begin{aligned}\frac{\mathrm{d}\theta}{\mathrm{d}t} &\le \varphi_{1,R}^{-\beta} \Delta_p(\varphi_{1,R}^{\beta})\theta(t)^{p-1} + q_0\theta(t)^{\alpha} \\ &\le \varphi_{1,R}^{-\beta} \Delta_p(\varphi_{1,R}^{\beta})\theta(t)^{p-1} + q_0\varphi_{1,R}^{(\alpha-1)\beta}\theta(t)^{\alpha},\end{aligned}$$

thanks to the normalization $0 < \varphi_{1,R} \le 1$ in $B_R(x_0)$ combined with $(\alpha - 1)\beta < 0$. Finally, multiplying by $\varphi_{1,R}^{\beta}$, we arrive at

$$\begin{aligned}\frac{\mathrm{d}\theta}{\mathrm{d}t}\varphi_{1,R}^{\beta} &\le \Delta_p(\varphi_{1,R}^{\beta})\theta(t)^{p-1} + q_0\theta(t)^{\alpha}\varphi_{1,R}^{\alpha} \\ &\le \Delta_p(\varphi_{1,R}^{\beta})\theta(t)^{p-1} + q(x)\theta(t)^{\alpha}\varphi_{1,R}^{\alpha}\,.\end{aligned}$$

This inequality, combined with our definition of the function $\widetilde{\varphi}_{1,R}$, guarantees that $\underline{u}(x,t) = \theta(t)\widetilde{\varphi}_{1,R}(x)$ is a subsolution to problem (1.1). □

Proof of Theorem 1.1. First, let us observe that $\overline{u}(x,t) = \|q\|_\infty^{\frac{1}{1-\alpha}} t$ is a supersolution of (1.1) for $0 < t \le 1$. Indeed, a straightforward calculation shows that

$$\frac{\partial \overline{u}}{\partial t} - \Delta_p \overline{u} = \|q\|_\infty^{\frac{1}{1-\alpha}} \ge q(x)\Big(\|q\|_\infty^{\frac{1}{1-\alpha}} t\Big)^{\alpha} = q(x)|\overline{u}|^{\alpha-1}\overline{u}$$

holds for $0 < t \le 1$, since $q \in C(\overline{\Omega})$, $q \ge 0$, and $\|q\|_\infty = \sup_{x\in\Omega} q(x)$.

Second, we show now that $\underline{u} \le \overline{u}$ for all $x \in \Omega$ and all $t > 0$ sufficiently small, say, $0 < t \le \overline{\sigma}$. Evidently,

$$\underline{u}(x,t) = \theta(t)\widetilde{\varphi}_1(x)^{\beta} = c_1 t^{\frac{1}{1-\alpha}} \widetilde{\varphi}_1(x)^{\beta} \le c_1 t^{\frac{1}{1-\alpha}} \le \overline{u}(x,t) = \|q\|_\infty^{\frac{1}{1-\alpha}} t$$

for $0 < t \le \overline{\sigma}$, where $\overline{\sigma}$ satisfies

$$\overline{\sigma}^{\alpha} \le \|q\|_\infty / c_1^{1-\alpha}\,.$$

Now it remains to show the existence of weak solution u for (1.1), such that

$$\underline{u} \le u \le \overline{u} \quad \text{in } \Omega \times (0,T)\,, \quad \text{where } T := \min\{\underline{\sigma}, \overline{\sigma}\} > 0.$$

Let us define a sequence of functions $u_n \colon \Omega \times (0,T) \to \mathbb{R}$ recursively for $n = 1, 2, 3, \dots$, such that u_n is the unique weak solution of

$$\begin{gathered}\frac{\partial u_n}{\partial t} - \Delta_p u_n = q(x)|u_{n-1}|^{\alpha-1}u_{n-1}, \quad (x,t) \in \Omega \times (0,T), \\ u_n(x,0) = 0, \quad x \in \Omega, \\ u_n(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T),\end{gathered} \tag{2.7}$$

with $u_0 = \underline{u}$. By a weak solution of (2.7), we mean a Lebesgue-measurable function $u_n \colon \Omega \times (0,T) \to \mathbb{R}$ that satisfies

$$u_n \in C([0,T] \to L^2(\Omega)) \cap L^p\big((0,T) \to W_0^{1,p}(\Omega)\big)$$

and the equation

$$\begin{aligned}
&\int_\Omega u_n(x,t)\phi(x,t)\,\mathrm{d}x - \int_0^t \int_\Omega u_n(x,s)\frac{\partial \phi}{\partial t}(x,s)\,\mathrm{d}x\,\mathrm{d}s \\
&+ \int_0^t \int_\Omega |\nabla u_n(x,s)|^{p-2}\langle \nabla u_n(x,s), \nabla\phi(x,s)\rangle\,\mathrm{d}x\,\mathrm{d}s \qquad (2.8)\\
&= \int_0^t \int_\Omega q(x)|u_{n-1}(x,s)|^{\alpha-1}u_{n-1}(x,s)\phi(x,s)\,\mathrm{d}x\,\mathrm{d}s
\end{aligned}$$

for every $t \in (0,T)$ and every test function

$$\phi \in C\left([0,T] \to L^2(\Omega)\right) \cap L^p\left((0,T) \to W_0^{1,p}(\Omega)\right) \cap W^{1,p'}\left((0,T) \to W^{-1,p'}(\Omega)\right).$$

The questions of existence and uniqueness of weak solutions of problems of type (2.7) obtained by monotone iterations have been discussed in [12, Appendix A, §A.1]. Let us deduce from the fact that $u_0 = \underline{u}$ is a subsolution of (1.1) the inequalities $u_{n-1} \le u_n$ in $\Omega \times (0,T)$ for every $n = 1, 2, 3, \dots$. The proof is by induction on n. The first inequality, $u_0 \le u_1$ in $\Omega \times (0,T)$, holds by the Weak Comparison Principle (see [12, Lemma 4.9, p. 618]) and the fact that $u_0 = \underline{u}$ is a subsolution of (1.1). Now assume that $u_{n-1} \le u_n$ in $\Omega \times (0,T)$ for some $n \in \mathbb{N}$. Then we have

$$\frac{\partial u_n}{\partial t} - \Delta_p u_n = |u_{n-1}|^{\alpha-1}u_{n-1} \le |u_n|^{\alpha-1}u_n = \frac{\partial u_{n+1}}{\partial t} - \Delta_p u_{n+1}$$

in $\Omega \times (0,T)$ and consequently $u_n \le u_{n+1}$ in $\Omega \times (0,T)$ again, by [12, Lemma 4.9, p. 618]. Therefore, monotonicity holds: $\underline{u} = u_0 \le u_1 \le u_2 \le \dots \le \overline{u}$ in $\Omega \times (0,T)$. The comparison with the supersolution $\overline{u}$ is deduced again from the Weak Comparison Principle. Hence, u_n is uniformly bounded in $\Omega \times (0,T)$ by $\underline{u} \le u \le \overline{u}$. A global regularity result from [7, Theorem 0.1, p. 552] (cf. [12, Lemma 4.6, p. 617]) guarantees $u_n \in C^{1+\gamma,\frac{1+\gamma}{2}}(\overline{\Omega} \times [0,T])$ uniformly for $n \in \mathbb{N}$, where $\gamma \in (0,1)$ is independent of n. We follow the notations and definitions of Hölder spaces of functions on $\Omega \times [0,T]$ from [5, Chpt. 1, p. 7]. Thus, by the Arzelà-Ascoli theorem, $\{u_n\}$ is relatively compact in $C^{1,0}(\overline{\Omega} \times [0,T])$. Hence, the sequence $\{u_n\}$ possesses a subsequence which converges to $u \in C^{1,0}(\overline{\Omega} \times [0,T])$. Therefore, in the weak formulation of (2.8) we may pass to the limit as $n \to \infty$, thus verifying that the limit function u is a weak solution of (1.1) in $\Omega \times (0,T)$, such that $\underline{u} \le u \le \overline{u}$. □

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