EXISTENCE OF SOLUTIONS FOR KIRCHHOFF TYPE EQUATIONS

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ABSTRACT. In this article, we prove the existence of solutions for Kirchhoff type equations with Dirichlet boundary-value condition. We use the Mountain Pass Theorem in critical point theory, without the (PS) condition.

1. INTRODUCTION AND STATEMENT OF MAIN RESULTS

Consider the Kirchhoff type problem

$$-\left(a + b\int_\Omega |\nabla u|^2 dx\right)\Delta u = f(x,u), \quad \text{in } \Omega,$$
$$u = 0, \quad \text{on } \partial\Omega, \tag{1.1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$, $a, b > 0$, and $f(x,t) : \overline{\Omega} \times \mathbb{R}$ is a continuous real function and satisfies the subcritical condition

$$|f(x,t)| \le C(|t|^{p-1} + 1) \quad \text{for some } 2 < p < 2^* = \begin{cases} \frac{2N}{N-2}, & N \ge 3, \\ \infty, & N = 1,2, \end{cases} \tag{1.2}$$

where C denotes some positive constant.

It is pointed out in [7] that the similar nonlocal problems model several physical and biological systems where u describes a process which depends on the average of itself, for example that of the population density.

Problem (1.1) is related to the stationary analogue of the Kirchhoff equation

$$u_{tt} - \left(a + b\int_\Omega |\nabla u|^2 dx\right)\Delta u = g(x,u),$$

proposed by Kirchhoff in [13] as an extension of the classical d'Alembert wave equation for free vibrations of elastic strings. Kirchhoff's model takes into account the changes in length of the string produced by transverse vibrations. It received great attention only after Lions [16] proposed an abstract framework for the problem.

Positive solutions are considered by authors, such as Alves et al [1], Ma and Rivera [19], Cheng and Wu [6], Yang and Zhang [29]. Problems on the unbounded domain $\mathbb{R}^3$ have also been considered by authors, such as Jin and Wu [12], Nie and Wu [21], Wu [28], Liu and He [27], Li et al [15], Li and Ye [14]. And more recently

the concentration behavior of positive solutions has been studied by He and Zou [9], Wang et al [26], He et al [11]. A result with Hartree-type nonlinearities can be found in Lü [18]. Ground state nonlinear with critical growth is considered in He and Zou [10]. The readers may consult Bernstein [2] and Pohožaev [23], Sun and Tang [24], Chen et al [4], Cheng [5], Perera and Zhang [22, 31], Mao and Zhang [20], Sun and Liu [25] and the references therein, for more information on this problem.

There are many solvability conditions for problem (1.1) with f, like the asymptotical linear case (at infinity) in [29] and people are more interested in the superlinear case (at infinity):

(S1) there exists $\theta \geq 1$ such that $\theta G(t) \geq G(st)$ for all $t \in \mathbb{R}$ and $s \in [0,1]$, where $G(t) = f(t)t - 4F(t)$ (see [24]);

(S2) $\lim_{|t|\to\infty} G(t) = \infty$ and there exist $\sigma > \max\{1, N/2\}$ and $C > 0$ such that $|f(t)|^\sigma \leq CG(t)|t|^\sigma$ for $|t|$ large (see [20]); or some limitation forms,

(S3) $\lim_{|t|\to\infty}[f(t)t - 4F(t)] = \infty$ (see [30]);

(S4) $\liminf_{|t|\to\infty} \frac{f(x,t)t - 4F(x,t)}{|t|^\tau} > -\alpha$ uniformly in $x \in \Omega$, where $\tau \in [0,2]$ and $0 < \alpha < a\lambda_1$, λ_1 is the first eigenvalue of $\big(-\Delta, H_0^1(\Omega)\big)$ (see [5]).

All kinds of conditions are mainly making sure the boundness of the Cerami or Palais-Smale sequences. The following condition on f which is called Ambrosetti-Rabinowitz condition is often used:

(S5) there exists $\theta > 4$ such that $f(x,t)t \geq \theta F(x,t)$ for $|t|$ large, where $F(x,t) = \int_0^t f(x,s)ds$.

We consider the nonlinear eigenvalue problem

$$-\Big(\int_\Omega |\nabla u|^2 dx\Big)\Delta u = \mu u^3, \quad \text{in } \Omega,$$
$$u = 0, \quad \text{on } \partial\Omega, \tag{1.3}$$

whose the eigenvalues are the critical values of the functional

$$J(u) = \|u\|^4, \quad u \in S := \big\{u \in H_0^1(\Omega) : \int_\Omega |u|^4 dx = 1\big\}, \tag{1.4}$$

where $\|u\| = \big(\int_\Omega |\nabla u|^2 dx\big)^{1/2}$. We already know the first eigenvalue $\mu_1 > 0$ and the first eigenfunction $\psi_1 > 0$ (see [31]).

Now, we can state our main results.

Theorem 1.1. *Assume that* $f \in C(\Omega \times \mathbb{R}, \mathbb{R})$ *satisfies* (1.2) *and*

(F1) $\lim_{|t|\to\infty} \big(\frac{a\lambda_1}{2}t^2 + \frac{b\mu_1}{4}t^4 - F(x,t)\big) = +\infty$ *uniformly in* $x \in \Omega$;

(F2) *there exists* $\lambda > \lambda_1$ *such that* $F(x,t) \geq \frac{a\lambda}{2}t^2$ *for* $|t|$ *small.*

Then (1.1) *has at least one nontrivial solution.*

Remark 1.2. Theorem 1.1 is a new for the case $\liminf_{|t|\to\infty} \frac{F(x,t)}{t^4} \leq \frac{b\mu_1}{4}$. The condition (F1) is weaker than (F3) in [30]. So our theorem is different from their theorems and obtains one nontrivial solution by adding the condition (F2) near zero.

Theorem 1.3. *Assume that* $f \in C(\Omega \times \mathbb{R}, \mathbb{R})$ *satisfies* (1.2) *and*

(F3) $\liminf_{|t|\to\infty} \frac{F(x,t)}{t^4} > \frac{b\mu_1}{4}$ *uniformly in* $x \in \Omega$;

(F4) $\lim_{|t|\to\infty} \big(\frac{1}{4}f(x,t)t - F(x,t) + \frac{a\lambda_1}{4}t^2\big) = +\infty$ *uniformly in* $x \in \Omega$;

(F5) *there exists* $\mu < \mu_1$ *such that* $F(x,t) \leq \frac{a\lambda_1}{2}t^2 + \frac{b\mu}{4}t^4$ *for* $|t|$ *small.*

Then (1.1) *has at least one nontrivial solution.*

Remark 1.4. Condition (F4) is a new condition for a class of function $f(x,t)$ and is weaker than (S1)–(S5). For example, let

$$f(x,t) = \frac{a\lambda_1}{8}\Big(8t^3 \ln(1+t^2) + \frac{4t^5}{1+t^2} + 4t^3 \cos t^4\Big).$$

A simple computation shows that

$$\frac{1}{4}f(x,t)t - F(x,t) + \frac{a\lambda_1}{4}t^2 = \frac{a\lambda_1}{8}\Big(\frac{t^6(1+\cos t^4)}{1+t^2} + \frac{t^4(2+\cos t^4)+2t^2}{1+t^2} - \sin t^4\Big)$$

and

$$\lim_{|t|\to\infty}\Big(\frac{1}{4}f(x,t)t - F(x,t) + \frac{a\lambda_1}{4}t^2\Big) = +\infty.$$

Hence, $f(x,t)$ satisfies all the assumptions of Theorem 1.3, but it does not satisfy any conditions of (S1)–(S5).

2. Preliminaries

We consider $H := H_0^1(\Omega)$ endowed with the norm $\|u\| = \big(\int_\Omega |\nabla u|^2 dx\big)^{1/2}$. We denote the usual $L^p(\Omega)$-norm by $|\cdot|_p$. Since Ω is a bounded domain, it is well known that $H \hookrightarrow L^p(\Omega)$ continuously for $p \in [1, 2^*]$, and compactly for $p \in [1, 2^*)$. Moreover there exists $\gamma_p > 0$ such that

$$|u|_p \leq \gamma_p \|u\|, \quad u \in H. \tag{2.1}$$

Seeking a weak solution of problem (1.2) is equivalent to finding a critical point of the C^1 functional

$$I(u) := \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \int_\Omega F(x,u)dx, \quad u \in H, \tag{2.2}$$

which implies that

$$\langle I'(u), v\rangle = (a + b\|u\|^2)\int_\Omega \nabla u \cdot \nabla v dx - \int_\Omega f(x,u)v dx, \quad u, v \in H. \tag{2.3}$$

Let

$$E_j := \oplus_{i \leq j} \ker(-\Delta - \lambda_i),$$

where $0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_i \leq \ldots$ are the eigenvalues of $(-\Delta, H)$. We denote a subsequence of a sequence $\{u_n\}$ as $\{u_n\}$ to simplify the notation unless specified. We need the following concept, which was introduced by Cerami [3] and is a weak version of the (PS) condition.

Definition 2.1 ([3]). Let $J \in C^1(X, \mathbb{R})$, we say that J satisfies the Cerami condition at the level $c \in \mathbb{R}$ ($(Ce)_c$ for short), if any sequence $\{u_n\} \subset X$ with

$$J(u_n) \to c, \quad (1 + \|u_n\|)J'(u_n) \to 0 \quad \text{as } n \to \infty,$$

possesses a convergence subsequence in X; J satisfies the (Ce) condition if J satisfies the $(Ce)_c$ for all $c \in \mathbb{R}$.

The following lemma, which can be found in [8], is our main tool in this article.

Lemma 2.2 (Mountain Pass Theorem)**.** *Let H be a real Banach space and $I \in C^1(H, \mathbb{R})$ satisfying the (Ce) condition. Suppose $I(0) = 0$,*

(i) *there are constants $\rho, \beta > 0$ such that $I|_{\partial B_\rho} \geq \beta$ where*
$$B_\rho = \{u \in H : \|u\| \leq \rho\};$$

(ii) *there is $u_1 \in H$ and $\|u_1\| > \rho$ such that $I(u_1) < 0$.*

Then I possesses a critical value $c \geq \beta$. Moreover c can be characterized as
$$c = \inf_{g \in \Gamma} \max_{u \in g([0,1])} I(u), \quad \Gamma = \{g \in C([0,1], H) : g(0) = 0, g(1) = u_1\}.$$

We give a lemma about the (Ce) condition which will play an important role in the proof of our theorems.

Lemma 2.3. *Assume that $f(x,t)$ satisfies (1.2) and (F4), then I satisfies the (Ce) condition.*

Proof. Suppose that $\{u_n\}$ is a $(Ce)_c$ sequence for $c \in \mathbb{R}$
$$I(u_n) \to c, \quad (1 + \|u_n\|)I'(u_n) \to 0 \quad \text{as } n \to \infty. \tag{2.4}$$

Now firstly, we prove that $\{u_n\}$ is a bounded sequence. From (2.2), (2.3) and (2.4), we obtain
$$1 + c \geq I(u_n) - \frac{1}{4}I'(u_n)u_n = \frac{a}{4}\|u_n\|^2 + \int_\Omega \Big(\frac{1}{4}f(x, u_n)u_n - F(x, u_n)\Big)dx. \tag{2.5}$$

By (F4), there exists $M > 0$ such that
$$\frac{1}{4}f(x,t)t - F(x,t) + \frac{a\lambda_1}{4}|t|^2 \geq -M \tag{2.6}$$

for all $x \in \Omega$ and $t \in \mathbb{R}$. And let $u_n = \phi_n + w_n$, where $\phi_n \in E_1$ and $w_n \in E_1^\perp$. From (2.5) and (2.6), one obtains

$$\begin{aligned} 1 + c &\geq I(u_n) - \frac{1}{4}I'(u_n)u_n \\ &= \frac{a}{4}\|u_n\|^2 - \frac{a\lambda_1}{4}|u_n|_2^2 + \int_\Omega \Big(\frac{1}{4}f(x, u_n)u_n - F(x, u_n) + \frac{a\lambda_1}{4}|u_n|^2\Big)dx \\ &\geq \frac{a}{4}\Big(1 - \frac{\lambda_1}{\lambda_2}\Big)\|w_n\|^2 - M|\Omega| \end{aligned} \tag{2.7}$$

which implies that $\|w_n\|$ is bounded. We claim that $\{u_n\}$ is a bounded sequence. Otherwise, there is a subsequence of $\{u_n\}$ satisfying $\|u_n\| \to +\infty$ as $n \to +\infty$. Then we obtain
$$\frac{w_n}{\|u_n\|} \to 0 \in H.$$

Since $\phi_n/\|u_n\|$ is bounded in E_1 (E_1 has finite dimension), we have $\phi_n/\|u_n\| \to v$ in E_1. By
$$v_n := \frac{u_n}{\|u_n\|} = \frac{\phi_n + w_n}{\|u_n\|} = \frac{\phi_n}{\|u_n\|} + \frac{w_n}{\|u_n\|} \to v \in E_1,$$

one has
$$\frac{u_n(x)}{\|u_n\|} \to v(x) \quad \text{a.e. in } \Omega. \tag{2.8}$$

From $\|v_n\| = 1$, we obtain that $\|v\| = 1$. And by $v \in E_1$, one has that $v(x) > 0$ or $v(x) < 0$, which implies that
$$|u_n(x)| \to +\infty \quad \text{as } n \to +\infty \tag{2.9}$$

for all $x \in \Omega$ by (2.8). It follows from (2.7), (2.9) and Fatou's lemma that

$$
\begin{aligned}
1 + c \geq I(u_n) - \frac{1}{4}I'(u_n)u_n \\
= \frac{a}{4}\|u_n\|^2 + \int_\Omega \Big(\frac{1}{4}f(x, u_n)u_n - F(x, u_n)\Big)dx \\
\geq \int_\Omega \Big(\frac{1}{4}f(x, u_n)u_n - F(x, u_n) + \frac{a\lambda_1}{4}|u_n|^2\Big)dx \\
\to +\infty \quad \text{as } n \to +\infty,
\end{aligned}
$$

which is a contradiction. Then we get that $\{u_n\}$ is bounded in H. Since $f(x, t)$ is subcritical growth, we can easily obtain that $\{u_n\}$ has a convergence subsequence. Hence, I satisfies the (Ce) condition. $\qquad\square$

3. Proof of main results

Proof of Theorem 1.1. Let

$$
\bar{u} = \Big(\int_\Omega \nabla u \cdot \nabla \phi_1 dx\Big)\phi_1, \quad \widetilde{u} = u - \bar{u},
$$

where the ϕ_1 is the first eigenfunction corresponding to λ_1.

The following statements come from [17]. First, there exist a real function $g \in L^1(\Omega)$, and $G \in C(\mathbb{R}, \mathbb{R})$ which is subadditive; that is,

$$
G(s + t) \leq G(s) + G(t)
$$

for all $s, t \in \mathbb{R}$, and coercive; that is, $G(t) \to +\infty$ as $|t| \to \infty$, and satisfies

$$
G(t) \leq |t| + 4
$$

for all $t \in \mathbb{R}$, such that

$$
F(x, t) - \frac{a\lambda_1}{2}t^2 - \frac{b\mu_1}{4}t^4 \leq -G(t) + g(x)
$$

for all $t \in \mathbb{R}$ and $x \in \Omega$.

Second, the functional $\int_\Omega G(v)dx$ is coercive on E_1 (this result also can be seen in [17]). We claim that $I(u)$ is coercive.

$$
\begin{aligned}
\int_\Omega \Big(F(x, u) - \frac{a\lambda_1}{2}u^2 - \frac{b\mu_1}{4}u^4\Big)dx \\
\leq -\int_\Omega G(u)dx + \int_\Omega g(x)dx \\
\leq -\int_\Omega \big(G(\bar{u}) - G(-\widetilde{u})\big)dx + \int_\Omega g(x)dx \\
\leq -\int_\Omega G(\bar{u})dx + |\widetilde{u}|_1 + 4|\Omega| + \int_\Omega g(x)dx \\
\leq -\int_\Omega G(\bar{u})dx + C_1(\|\widetilde{u}\| + 1)
\end{aligned}
$$

for all $u \in H$ and some

$$
C_1 = C + 4|\Omega| + \int_\Omega g(x)dx,
$$

where C is a positive constant in Sobolev's inequality,

$$
|u|_1 \leq C\|u\|, \quad |u|_2 \leq C\|u\|
$$

for all $u \in H$. Hence we have

$$
\begin{aligned}
I(u_n) &= \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \int_\Omega F(x,u)dx \\
&= \frac{a}{2}\|u_n\|^2 - \frac{a\lambda_1}{2}|u_n|_2^2 + \frac{b}{4}\|u_n\|^4 - \frac{b\mu_1}{4}|u_n|_4^4 \\
&\quad + \int_\Omega \Big(\frac{a\lambda_1}{2}|u_n|^2 + \frac{b\mu_1}{4}|u_n|^4 - F(x,u_n)\Big)dx \\
&\geq \frac{a}{2}\|u_n\|^2 - \frac{a\lambda_1}{2}|u_n|_2^2 + \int_\Omega \Big(\frac{a\lambda_1}{2}|u_n|^2 + \frac{b\mu_1}{4}|u_n|^4 - F(x,u_n)\Big)dx \\
&\geq \frac{a}{2}\Big(1 - \frac{\lambda_1}{\lambda_2}\Big)\|\widetilde{u}\|^2 + \int_\Omega G(\overline{u})dx - C_1(\|\widetilde{u}\| + 1)
\end{aligned}
$$

for all $u \in H$. By the coercivity of the functional $\int_\Omega G(v)dx$ on E_1 and that fact

$$\|u\|^2 = \|\overline{u}\|^2 + \|\widetilde{u}\|^2,$$

which implies that the functional $I(u)$ is coercive. I satisfies the (Ce) condition and is bounded from below. By (F2), we have

$$F(x,t) \geq \frac{a\lambda}{2}t^2 - C|t|^p$$

for all $x \in \Omega$ and $t \in \mathbb{R}$, which implies that

$$
\begin{aligned}
I(u) &\leq \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{a\lambda}{2}|u|_2^2 + C|u|_p^p \\
&= \frac{a}{2}\Big(1 - \frac{\lambda}{\lambda_1}\Big)\|u\|^2 + \frac{b}{4}\|u\|^4 + C\|u\|^p < 0
\end{aligned}
$$

for $u \in E_1 \cap B_\delta$, $\lambda > \lambda_1$, where $\delta > 0$ small enough and E_1 is the subspace of H spanned by ϕ_1 the eigenfunctions of λ_1. Then $I(u)$ achieves the negative infimum. This completes the proof ☐

Proof of Theorem 1.3. By Lemmas 2.2 and 2.3, it is sufficient to show that I satisfies (i) and (ii).

Step 1. There are constants ρ, $\beta > 0$ such that $I(u) \geq \beta$ for all $\|u\| = \rho$. In fact, by (F5), it is easy to see that

$$F(x,t) \leq \frac{a\lambda_1}{2}t^2 + \frac{b(\mu_1 - \varepsilon)}{4}t^4 + C|t|^p$$

for all $t \in \mathbb{R}$ and $x \in \Omega$,

$$
\begin{aligned}
I(u) &\geq \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{a\lambda_1}{2}|u|_2^2 - \frac{b(\mu_1 - \varepsilon)}{4}|u|_4^4 - C\int_\Omega |u|^p dx \\
&\geq \frac{b}{4}\Big(1 - \frac{\mu_1 - \varepsilon}{\mu_1}\Big)\|u\|^4 - C\gamma_p\|u\|^p.
\end{aligned}
$$

Note that $4 < p < 2^*$, then for ε small enough. So there exists $\beta > 0$ such that $I(u) \geq \beta$ for all $\|u\| = \rho$, where $\rho > 0$ small enough.

Step 2. There exists $u_1 \in H$ and $\|u_1\| > \rho$ such that $I(u_1) < 0$. Indeed, for small $\varepsilon > 0$, by the definition of μ_1, we can choose $u \in S$ satisfying

$$\mu_1 + \frac{\varepsilon}{2} \geq \|u\|^4. \tag{3.1}$$

It follows from (F3) that

$$F(x,t) \geq \frac{b(\mu_1 + \varepsilon)}{4}t^4 - C. \tag{3.2}$$

Hence, by (3.1) and (3.2), we have

$$\begin{aligned}
I(tu) &\leq \frac{a}{2}t^2\|u\|^2 + \frac{b}{4}t^4\|u\|^4 - \frac{b}{4}t^4(\mu_1 + \varepsilon) + C|\Omega| \\
&\leq \frac{a}{2}t^2\|u\|^2 + \frac{b}{4}t^4\mu_1 + \frac{b\varepsilon}{8}t^4 - \frac{b}{4}t^4(\mu_1 + \varepsilon) + C|\Omega| \\
&= -\frac{b\varepsilon}{8}t^4 + \frac{a}{2}t^2\|u\|^2 + C|\Omega|.
\end{aligned} \tag{3.3}$$

Thus, $I(tu) \to -\infty$ as $t \to \infty$. Therefore, there is $u_1 \in H$ with $\|u_1\| > \rho$ such that $I(u_1) < 0$. This completes the proof. $\square$

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