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COLLAGE-TYPE APPROACH TO INVERSE PROBLEMS FOR ELLIPTIC PDES ON PERFORATED DOMAINS

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ABSTRACT. We present a collage-based method for solving inverse problems for elliptic partial differential equations on a perforated domain. The main results of this paper establish a link between the solution of an inverse problem on a perforated domain and the solution of the same model on a domain with no holes. The numerical examples at the end of the paper show the goodness of this approach.

1. INTRODUCTION

In recent years a great deal of attention has been paid to the problem of parameter estimation in distributed systems, that is the determination of unknown parameters in the functional form of the governing model of the phenomenon under study [8, 17, 18, 20]. In the mathematical literature this kind of problem is called an *inverse problem*. According to Keller [7], "we call two problems *inverse* of one another if the formulation of each involves all or part of the solution of the other. Often, for historical reasons, one of the two problems has been studied extensively for some time, while the other one is newer and not so well understood. In such cases, the former is called the *direct problem*, while the latter is the *inverse problem*". There is a fundamental difference between the direct and the inverse problem; often the direct problem is *well-posed* while the corresponding inverse problem is *ill-posed*. Hadamard [6] introduced the concept of *well-posed problem* to describe a mathematical model that has the properties of existence, uniqueness and stability of the solution. When one of these properties fails to hold, the mathematical model is said to be an *ill-posed problem*. There are many inverse problems in the literature that are *ill-posed* whereas the corresponding direct problems are *well-posed*. The literature is rich in papers studying ad hoc methods to address *ill-posed* inverse problems by minimizing a suitable approximation error along with utilizing some regularization techniques [19].

Many inverse problems may be recast as the approximation of a target element x in a complete metric space (X, d) by the fixed point $\bar{x}$ of a contraction mapping $T : X \to X$. Thanks to a simple consequence of Banach's Fixed Point Theorem known as the *Collage Theorem*, most practical methods of solving the inverse problem for

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fixed point equations seek an operator T for which the *collage distance* $d(x, Tx)$ is as small as possible.

Theorem 1.1 (Collage Theorem [1])**.** *Let (X, d) be a complete metric space and $T : X \to X$ a contraction mapping with contraction factor $c \in [0, 1)$. Then for any $x \in X$,*

$$d(x, \bar{x}) \leq \frac{1}{1-c} d(x, Tx), \tag{1.1}$$

where $\bar{x}$ is the fixed point of T.

This theorem vastly simplifies this type of inverse problem as it is much easier to estimate $d(x, Tx)$ than it is to find the fixed point $\bar{x}$ and then compute $d(x, \bar{x})$. One now seeks a contraction mapping T that minimizes the so-called *collage error* $d(x, Tx)$ – in other words, a mapping that sends the target x as close as possible to itself. This is the essence of the method of *collage coding* which has been the basis of most, if not all, fractal image coding and compression methods. Barnsley [1] was the first to see the potential of using the Collage Theorem above for the purpose of *fractal image approximation* and *fractal image coding* [5]. However, this method of *collage coding* may be applied in other situations where contractive mappings are encountered. We have shown this to be the case for inverse problems involving several families of differential equations: ordinary differential equations [9, 14], random differential equations [10, 12], boundary value problems [2, 11, 13], parabolic partial differential equations [15], stochastic differential equations [3], and others.

In practical applications, from a family of contraction mappings T_λ, $\lambda \in \Lambda \subset \mathbb{R}^n$, one wishes to find the parameter $\bar{\lambda}$ for which the approximation error $d(x, \bar{x}_\lambda)$ is as small as possible. In practice the feasible set is often taken to be $\Lambda_c = \{\lambda \in \mathbb{R}^n : 0 \leq c_\lambda \leq c < 1\}$ which guarantees the contractivity of T_λ for any $\lambda \in \Lambda_c$. A difference between this "collage" approach and the one based on Tikhonov regularization is the following: in the collage approach, the constraint $\lambda \in \Lambda_c$ guarantees that T_λ is a contraction and, therefore, replaces the effect of the regularization term in the Tikhonov approach (see [19] and [20]). The collage approach works well for low-dimensional parametrization in particular, while Tikhonov regularization is a fundamentally non-parametric methodology. The collage-based inverse problem can be formulated as an optimization problem as follows:

$$\min_{\lambda \in \Lambda_c} d(x, T_\lambda x). \tag{1.2}$$

This is typically a nonlinear and nonsmooth optimization model. Several algorithms can be used to solve it including, for instance, penalization methods, particle swarm ant colony techniques, and so on.

The article is organized as follows: Section 2 recalls the extended approach based on the Generalized Collage Theorem to solving inverse problems for elliptic partial differential equations. Section 3 presents a brief introduction of porous media and perforated domains and the formulation of the inverse problem. Section 4 illustrates the main results and, finally, Section 5 lists some numerical examples.

2. Inverse problems for elliptic PDEs by the generalized collage theorem

Many physical phenomena in science and engineering can be described through partial differential equations which include the parameters of the process in the operators of the model. The direct problem typically requires finding the unique solution of such a well-posed problem. The inverse problem seeks to estimate the parameter values given information about the solution.

Let us consider the following variational equation associated with an elliptic equation:

$$a(u,v) = \phi(v), \quad v \in H, \tag{2.1}$$

where $\phi(v)$ and $a(u,v)$ are linear and bilinear maps, respectively, both defined on a Hilbert space H. Let us denote by $\langle \cdot, \cdot \rangle$ the inner product in H, $\|u\|^2 = \langle u, u \rangle$ and $d(u,v) = \|u - v\|$, for all $u, v \in H$. The inverse problem may now be viewed as follows: Suppose that we have an observed solution u and a given (restricted) family of bounded, coercive bilinear functionals $a^\lambda(u,v)$, $\lambda \in \mathbb{R}^n$. We now seek "optimal" values of λ. The existence and uniqueness of solutions to this kind of equation are provided by the classical Lax-Milgram representation theorem. Suppose that we have a "target" element $u \in H$, a family of bilinear functionals a^λ, and a family of linear functionals ϕ^λ. Then, by the Lax-Milgram theorem, there exists a unique vector $u^\lambda \in H$ such that $\phi^\lambda(v) = a^\lambda(u^\lambda, v)$ for all $v \in H$. We would like to determine if there exists a value of the parameter λ such that $u^\lambda = u$ or, more realistically, such that $\|u^\lambda - u\|$ is small enough. The following theorem will be useful for the solution of this problem.

Theorem 2.1 (Generalized Collage Theorem)**.** [11] *For all $\lambda \in \Lambda$, suppose that $a^\lambda(u,v) : \Lambda \times H \times H \to \mathbb{R}$ is a family of bilinear forms and $\phi^\lambda : \Lambda \times H \to \mathbb{R}$ is a family of linear functionals. Let u^λ denote the solution of the equation $a^\lambda(u,v) = \phi^\lambda(v)$ for all $v \in H$ as guaranteed by the Lax-Milgram theorem. Then, given a target element $u \in H$,*

$$\|u - u^\lambda\| \leq \frac{1}{m^\lambda} F^\lambda(u), \tag{2.2}$$

where

$$F^\lambda(u) = \sup_{v \in H,\ \|v\|=1} \left| a^\lambda(u,v) - \phi^\lambda(v) \right| \tag{2.3}$$

and $m^\lambda > 0$ is the coercivity constant of a^λ.

To ensure that the approximation u^λ is close to a target element $u \in H$, we can, by the Generalized Collage Theorem, try to make the term $F^\lambda(u)/m_\lambda$ as close to zero as possible. The appearance of the m^λ factor complicates the procedure as does the factor $1/(1-c)$ in standard collage coding, i.e., (1.1). If $\inf_{\lambda \in \Lambda} m^\lambda \geq m > 0$ then the inverse problem can be reduced to the minimization of the function $F^\lambda(u)$ on the space Λ; that is,

$$\min_{\lambda \in \Lambda} F^\lambda(u). \tag{2.4}$$

The choice of λ according to (2.4) for minimizing the residual is, in general, not stabilizing (see [4]). However, as the next sections show, under the condition $\inf_{\lambda \in \Lambda} m^\lambda \geq m > 0$ our approach is stable. Following our earlier studies of inverse problems using fixed points of contraction mappings, we shall refer to the minimization of the functional $F^\lambda(u)$ as a "generalized collage method." Such an optimization problem has a solution that can be approximated with a suitable discrete and quadratic program, derived from the application of the Generalized Collage Theorem and an adequate use of an orthonormal basis in the Hilbert space H, as seen in [11].

Example 2.2. As an illustrative example, we choose $K(x, y) = K_{true}(x, y) = 8 + x^2 + 2y^2$ and $f(x, y) = x^2 + 4y^2$ and consider the steady-state diffusion problem

$$\begin{gathered} \nabla \cdot (K(x, y)\nabla u(x, y)) = f(x, y), \quad \Omega = [0, 1]^2, \\ u(x, y) = 0, \quad \partial\Omega. \end{gathered} \tag{2.5}$$

We solve the diffusion problem numerically and sample the solution u at 36 uniformly distributed points strictly inside Ω, $(x_i, y_j) = (\frac{i}{7}, \frac{j}{7})$, $i, j = 1, \dots, 6$. The level curves of the solution are illustrated in Figure 1, which also presents the mesh used by the numerical solver.

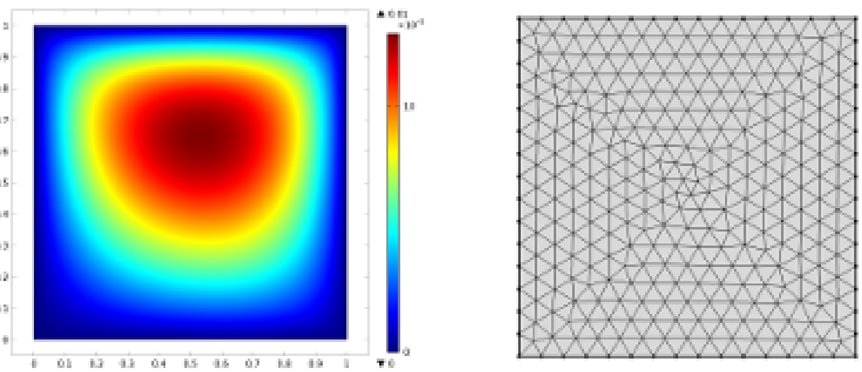

FIGURE 1. Level curves of solutions and the numerical solver mesh for Example 2.2.

Next, we define $K^\lambda(x, y) = \lambda_0 + \lambda_1 x^2 + \lambda_2 y^2$ and $f^\lambda(x, y) = \lambda_3 x^2 + \lambda_4 y^2$. Note that if we leave all of the parameters in K^λ variable, then, due to linearity, any nonzero multiple of the resulting parameter vector will correspond to the same solution, so we fix $\lambda_0 = 1$. Using the 36 data points, we seek to estimate the values of λ_i in $K_\lambda(x, y)$ and/or $f_\lambda(x, y)$ by applying the generalized collage theorem. To four decimal places, we obtain $(\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4) = (1, 0.1291, 0.0988, 0.1330, 0.4574)$, corresponding to $(8, 1.0327, 0.7906, 1.0641, 3.6590)$. If we increase the number of points, the results improve. The results are also robust with respect to the introduction of low-amplitude additive noise [11, 13].

3. Inverse problems on perforated domains

A porous medium (or perforated domain) is a material characterized by a partitioning of the total volume into a solid portion often called the "matrix" and a pore space usually referred to as "holes" that can be either materials different from that of the matrix or real physical holes. When formulating differential equations over porous media, the term "porous" implies that the state equation is written in the matrix only, while boundary conditions should be imposed on the whole boundary of the matrix, including the boundary of the holes. Porous media can be found in many areas of applied sciences and engineering including petroleum engineering, chemical engineering, civil engineering, aerospace engineering, soil science, geology, material science, and many more areas. Figure 2 presents an example of a two-dimensional perforated domain.

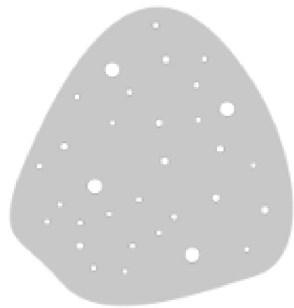

Figure 2. A two-dimensional perforated domain.

Since porosity in materials can take different forms and appear in varying degrees, solving differential equations over porous media is often a complicated task and the holes' size and their distribution play an important role in its characterization. Furthermore numerical simulations over perforated domains need a very fine discretization mesh which often requires a significant computational time. The mathematical theory of differential equations on perforated domains is usually based on the theory of "homogenization" in which heterogeneous material is replaced by a fictitious homogeneous one. Of course this implies the need of convergence results linking together the model on a perforated domain and on the associated homogeneous one. In the case of porous media, or heterogeneous media in general, characterizing the properties of the material is a tricky process and can be done on different levels, mainly the microscopic and macroscopic scales, where the microscopic scale describes the heterogeneities and the macroscopic scale describes the global behavior of the composite.

In this article we focus on the analysis of inverse problems for elliptic partial differential equations on perforated domains. Thus far, we have illustrated the importance of inverse problems for practical applications and some results for the case of homogeneous media. Now, starting from a target function, which is supposed to be the solution to a partial differential equation on a perforated domain for certain values of unknown parameters, we aim to estimate these parameters by solving an inverse problem on a homogenized domain with no holes. The next section establishes some results relating the solution to an inverse problem on a porous medium and the corresponding problem on a homogenized domain.

4. Main results

Given a compact and convex set Ω, in the following let us denote by Ω_B the collection of circular holes $\cup_{j=1}^m B(x_j,\varepsilon)$ where $x_j \in \Omega$, ε is a strictly positive number, and the holes $B(x_j,\varepsilon)$ are nonoverlapping and lie strictly inside Ω. We denote by Ω_ε the closure of the set $\Omega\backslash\Omega_B$. In the remaining part of this section we consider the problem

$$\begin{gathered} \nabla\cdot(K^\lambda(x,y)\nabla u(x,y)) = f^\lambda(x,y), \quad \text{in } \Omega_\varepsilon, \\ u(x,y)=0, \quad \text{on } \partial\Omega_\varepsilon, \end{gathered} \tag{4.1}$$

and the problem

$$\begin{aligned} \nabla \cdot (K^\lambda(x,y)\nabla u(x,y)) = f^\lambda(x,y), \quad \text{in } \Omega, \\ u(x,y) = 0, \quad \text{on } \partial\Omega. \end{aligned} \tag{4.2}$$

where λ is a parameter belonging to the compact set $\Lambda \subset \mathbb{R}^n$. The results provided in this section are related to the Dirichlet problem but they can be easily extended to the case of Neumann boundary conditions ($\frac{\partial u}{\partial n} = 0$ on $\partial\Omega_B$).

Let us introduce, using classical notation, the Sobolev spaces $H = H_0^1(\Omega)$ and $H_\varepsilon = H_0^1(\Omega_\varepsilon)$ and the variational formulation of the above equations (4.1) and (4.2) as follows:

- (P_ε) Find $u \in H_\varepsilon$ such that
$$a_\varepsilon^\lambda(u,v) = \phi_\varepsilon^\lambda(v), \quad \forall v \in H_\varepsilon \tag{4.3}$$
- (P) Find $u \in H$ such that
$$a^\lambda(u,v) = \phi^\lambda(v), \quad \forall v \in H \tag{4.4}$$

As any function in H_ε can be extended to be zero over the holes, it is trivial to prove that H_ε can be embedded in H. In the sequel, let $\Pi_\varepsilon u$ be the projection of $u \in H$ onto H_ε. It is easy to prove that $\|u - \Pi_\varepsilon u\|_H \to 0$ whenever $\varepsilon \to 0$. When Neumann boundary conditions are considered, it is still possible to extend a function in H_ε to a function of H: these extension conditions are well studied (see [16]) and they typically hold when the domain Ω has a particular structure. In any case, it holds for a wide class of disperse media, that is media consisting of two media that do not mix.

Let us also assume the following hypotheses:

- the continuous and bilinear forms a_ε^λ and a^λ are uniformly coercive and bounded with respect to λ and ε, namely there exists two positive constants m and M such that
$$\begin{aligned} a_\varepsilon^\lambda(u,u) \geq m\|u\|^2 \quad \forall u \in H_\varepsilon \\ a_\varepsilon^\lambda(u,v) \leq M\|u\|\|v\| \quad \forall u,v \in H_\varepsilon \\ a^\lambda(u,u) \geq m\|u\|^2 \quad \forall u \in H \\ a^\lambda(u,v) \leq M\|u\|\|v\| \quad \forall u \in H \end{aligned} \tag{4.5}$$
- the linear functionals ϕ_ε^λ and ϕ^λ are uniformly bounded with respect to λ and ε, namely there exists a positive constant μ such that
$$\begin{aligned} \phi_\varepsilon^\lambda(u) \leq \mu\|u\| \quad \forall u \in H_\varepsilon \\ \phi^\lambda(u) \leq \mu\|u\| \quad \forall u \in H \end{aligned} \tag{4.6}$$

Using classical results from the theory of PDEs we know that, under the hypotheses (4.5) and (4.6) above, (4.3) and (4.4) have unique solutions u_ε^λ and u^λ for each $\lambda \in \Lambda$ and for each positive ε.

The inverse problem of interest can now be stated as follows:
Given a target u, which is a solution of (4.1) for certain unknown values λ and ε, determine an estimation of λ using (4.2) instead. In other words, we want to estimate the unknown parameter λ by solving an inverse problem on a domain with no holes.

From a practical perspective, starting from a set of data u_i, $i = 1, \dots, s$, sampled on the porous domain Ω_ε, u is obtained from u_i by applying some interpolation technique.

The following results demonstrate some relationships between (4.1) and (4.2). For this purpose and for each $u \in H_\varepsilon$, let us introduce the function

$$F_\varepsilon^\lambda(u) = \sup_{v \in H_\varepsilon,\ \|v\|_{H_\varepsilon}=1} |a_\varepsilon^\lambda(u,v) - \phi_\varepsilon^\lambda(v)|. \tag{4.7}$$

associated with problem (4.1).

Proposition 4.1. *The following estimate holds:*

$$\|\Pi_\varepsilon u - u_\varepsilon^\lambda\|_{H_\varepsilon} \le \frac{F^\lambda(u)}{m} + \frac{M}{m}\|u - \Pi_\varepsilon u\|_H \tag{4.8}$$

Proof. Let us first notice that the function $\Pi_\varepsilon u$ is an element of H_ε. The thesis follows from the following chain of inequalities and the observation

$$\|\Pi_\varepsilon u - u_\varepsilon^\lambda\|_{H_\varepsilon} \le \frac{1}{m} F_\varepsilon^\lambda(\Pi_\varepsilon u) \le \frac{1}{m} F^\lambda(\Pi_\varepsilon u) \le \frac{F^\lambda(u)}{m} + \frac{M}{m}\|u - \Pi_\varepsilon u\|_H$$

for all $\lambda \in \Lambda$, $\varepsilon > 0$. □

Proposition 4.2. *There exists a constant C, that does not depend on ε, such that the following estimate holds:*

$$F^\lambda(\Pi_\varepsilon u) \le F_\varepsilon^\lambda(\Pi_\varepsilon u) + C\varepsilon \tag{4.9}$$

for all $\lambda \in \Lambda$, $\varepsilon > 0$.

Proof. The following calculations hold:

$$\begin{aligned}
F^\lambda(\Pi_\varepsilon u) &= \sup_{v \in H,\ \|v\|_H=1} |a^\lambda(\Pi_\varepsilon u, v) - \phi^\lambda(v)| \\
&\le \sup_{v \in H,\ \|v\|_H=1} |a^\lambda(\Pi_\varepsilon u, v) - a^\lambda(\Pi_\varepsilon u, \Pi_\varepsilon v)| \\
&\quad + \sup_{v \in H,\ \|v\|_H=1} |a^\lambda(\Pi_\varepsilon u, \Pi_\varepsilon v) - \phi^\lambda(\Pi_\varepsilon v)| \\
&\quad + \sup_{v \in H,\ \|v\|_H=1} |\phi^\lambda(v) - \phi^\lambda(\Pi_\varepsilon v)| \\
&= F_\varepsilon^\lambda(\Pi_\varepsilon u) + (M\|\Pi_\varepsilon u\|_H + \mu) \sup_{v \in H,\ \|v\|_H=1} \|v - \Pi_\varepsilon v\|_H \\
&\le F_\varepsilon^\lambda(\Pi_\varepsilon u) + C\varepsilon
\end{aligned}$$

□

Proposition 4.3. *Suppose that $F^\lambda(u), F_\varepsilon^\lambda(v) : \Lambda \to \mathbb{R}_+$ are continuous for all $u \in H$, $v \in H_\varepsilon$, and $\varepsilon > 0$. Let λ_ε be a sequence of minimizers of $F_\varepsilon^\lambda(u)$ over Λ. Then there exists $\varepsilon_n \to 0$ and $\lambda^* \in \Lambda$ such that $\lambda_{\varepsilon_n} \to \lambda^*$, with λ^* a minimizer of $F^\lambda(u)$ over Λ.*

Proof. As λ_ε is a sequence of vectors in the compact space Λ, there exists a convergent subsequence $\lambda_{\varepsilon_n} \to \lambda^* \in \Lambda$ when $\varepsilon_n \to 0$. Computing we have:

$$\begin{aligned}
F^{\lambda^*}(u) &= \lim_{\varepsilon_n \to 0} F^{\lambda_{\varepsilon_n}}(\Pi_{\varepsilon_n} u) \le \lim_{\varepsilon_n \to 0} F_{\varepsilon_n}^{\lambda_{\varepsilon_n}}(\Pi_{\varepsilon_n} u) + C\varepsilon_n \\
&\le \lim_{\varepsilon_n \to 0} F_{\varepsilon_n}^\lambda(\Pi_{\varepsilon_n} u) + C\varepsilon_n
\end{aligned}$$

$$\leq \lim_{\varepsilon_n \to 0} F^\lambda(u) + M\|u - \Pi_{\varepsilon_n} u\|_H + C\varepsilon_n = F^\lambda(u)$$

□

In closing this section, we note that all of the above results can be extended to the case where the radii of the holes are different, that is $B(x_j, \varepsilon_j)$, in which case we define $\varepsilon = \max_j \varepsilon_j$.

5. Numerical examples

We provide two numerical examples of an inverse problem on a perforated domain. In both cases, we set $\Omega = [0,1]^2$.

Example 5.1. We extend Example 2.2, placing nine holes of assorted sizes inside Ω, as in Figure 3

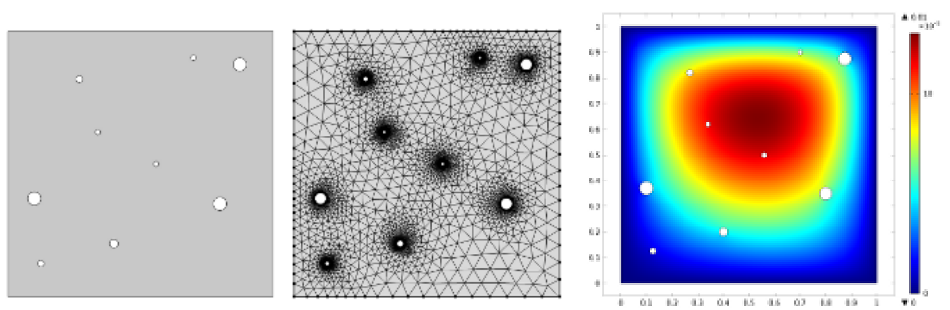

Figure 3. The domain, mesh, and level curves of solutions for Example 5.1.

As in Example 2.2, we choose $K(x,y) = K_{true}(x,y) = 8 + x^2 + 2y^2$ and $f(x,y) = x^2 + 4y^2$ and consider

$$\begin{gathered} \nabla \cdot (K(x,y)\nabla u(x,y)) = f(x,y), \quad \text{in } \Omega_\varepsilon, \\ u(x,y) = 0, \quad \text{on } \partial\Omega, \\ \frac{\partial u}{\partial n}(x,y) = 0, \quad \text{on } \partial\Omega_B, \end{gathered} \tag{5.1}$$

where Ω_B is the union of the nine holes. We solve the diffusion problem numerically and sample the solution u_ε at $M \times M$ uniformly-distributed points strictly inside Ω. The level curves of the solution are illustrated in Figure 3. If a sample point lies inside a hole, we obtain no information at the point. We define $K^\lambda(x,y) = \lambda_0 + \lambda_1 x^2 + \lambda_2 y^2$ and $f^\lambda(x,y) = \lambda_3 x^2 + \lambda_4 y^2$. Using the M^2 (or fewer) data points, we seek to estimate the values of λ_i in $K^\lambda(x,y)$ and/or $f^\lambda(x,y)$ by applying the generalized collage theorem to solve the related inverse problem on Ω with no holes.

The results for various cases are presented in Table 1. In the case that we seek to recover all five of the parameters, we choose to normalize $\lambda_0 = 1$, so the desired values of the other parameters are scaled by 1/8. We mention that if we instead set $\lambda_0 = 0$, the solution we obtain to the inverse problem is very poor, as we would expect. We see that the estimates obtained are quite good.

Estimating $K_\lambda(x,y) = \lambda_0 + \lambda_1 x^2 + \lambda_2 y^2$ given $f_\lambda(x,y) = x^2 + 4y^2$					
M	λ_0	λ_1	λ_2		
4	7.9606	0.9177	1.8065		
5	8.1955	0.8381	1.3709		
6	8.0474	0.9170	1.6289		
Estimating $f_\lambda(x,y) = \lambda_3 x^2 + \lambda_4 y^2$ given $K_\lambda(x,y) = 8 + x^2 + 2y^2$					
M				λ_3	λ_4
4				0.9817	4.1071
5				0.9766	4.1385
6				0.9827	4.1106
Estimating both $K_\lambda(x,y) = 1 + \lambda_1 x^2 + \lambda_2 y^2$ and $f_\lambda(x,y) = \lambda_3 x^2 + \lambda_4 y^2$					
M		λ_1	λ_2	λ_3	λ_4
4		0.0178	0.3233	0.0977	0.5323
5		0.0879	0.1429	0.1196	0.4784
6		0.0832	0.1707	0.1178	0.4837
9		0.1264	0.2111	0.1015	0.4728

TABLE 1. Results for the inverse problem in Example 5.1. For the top problem, the true values are $(\lambda_0, \lambda_1, \lambda_2) = (8, 1, 2)$; for the middle problem, the true values are $(\lambda_3, \lambda_4) = (1, 4)$; and for the bottom problem, the true values are $(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = (0.0125, 0.2500, 0.0125, 0.500)$.

Example 5.2. For $\varepsilon \in \{0.1, 0.025, 0.01\}$, define $N_\varepsilon = \frac{1}{10\varepsilon}$ and

$$\Omega_B = \cup_{i,j=1}^{N_\varepsilon} B_\varepsilon\Big(\big(i - \frac{1}{2}\big)\varepsilon, \big(j - \frac{1}{2}\big)\varepsilon\Big),$$

a domain with N_ε^2 uniformly-distributed holes of radius ε. Choosing $K(x,y) = K_{true}(x,y) = 10 + 2x + 3y$, we consider the steady-state diffusion problem

$$\begin{gathered} \nabla \cdot (K(x,y)\nabla u(x,y)) = x^2 + y^2, \quad \text{in } \Omega_\varepsilon, \\ u(x,y) = 0, \quad \text{on } \partial\Omega, \\ \frac{\partial u}{\partial n}(x,y) = 0, \quad \text{on } \partial\Omega_B. \end{gathered} \tag{5.2}$$

For a fixed value of ε, we solve the diffusion problem numerically and sample the solution at $M \times M$ uniformly-distributed points strictly inside Ω. If such a point lies inside a hole, we obtain no information at the point. Using the M^2 (or fewer) data points, we use the generalized collage theorem to solve the related inverse problem, seeking a diffusivity function of the form $K(x,y) = \lambda_0 + \lambda_1 x + \lambda_2 y$. The level curves are illustrated in Figure 4.

The results for $M = 9$, 49, and 99, are given in Table 2. We see that as the size of the holes decreases (even while the number increases), the solution to the inverse problem produces better estimates of the parameters. In addition, we see

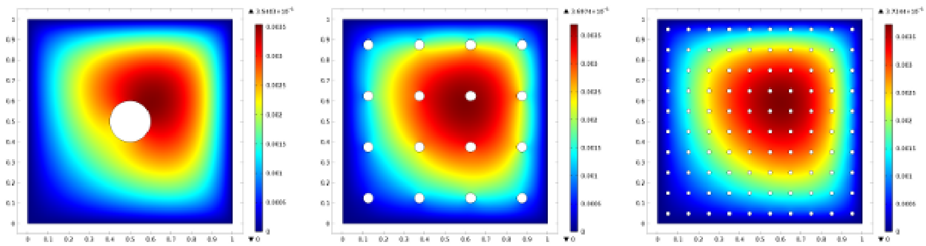

FIGURE 4. Level curves of solutions in Example 5.2, with $\varepsilon = 0.1$, 0.025, and 0.01.

that if a hole is too large, as in the $N = 1$ case, the estimates are very poor. In this case, the hole needs to be incorporated into the macroscopic-scale model, as it can't be considered part of the smaller-scale model. In the other cases of the table, the estimates are good.

			Recovered parameters		
ε	N_ε	M	λ_0	λ_1	λ_2
0.1	1	9	13.2068	−0.5921	0.6250
		49	13.2428	−0.5837	0.6346
		49	13.2419	−0.5798	0.6398
0.025	4	9	9.8434	1.8148	2.8119
		49	9.9758	1.6894	2.6875
		99	9.9787	1.6838	2.6820
0.01	10	9	9.9811	1.6221	2.6199
		49	10.0069	1.6041	2.6014
		99	10.0069	1.6039	2.6014

TABLE 2. Results for the inverse problem in Example 5.2. True values are $(\lambda_0, \lambda_1, \lambda_2) = (10, 2, 3)$.

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