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ON A SHARP CONDITION FOR THE EXISTENCE OF WEAK SOLUTIONS TO THE DIRICHLET PROBLEM FOR DEGENERATE NONLINEAR ELLIPTIC EQUATIONS WITH POWER WEIGHTS AND L^1-DATA

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Abstract. In this article, we establish a sharp condition for the existence of weak solutions to the Dirichlet problem for degenerate nonlinear elliptic second-order equations with L^1-data in a bounded open set Ω of $\mathbb{R}^n$ with $n \geqslant 2$. We assume that Ω contains the origin and assume that the growth and coercivity conditions on coefficients of the equations involve the weighted function $\mu(x) = |x|^\alpha$, where $\alpha \in (0, 1]$, and a parameter $p \in (1, n)$. We prove that if $p > 2 - (1 - \alpha)/n$, then the Dirichlet problem has weak solutions for every L^1-right-hand side. On the other hand, we find that if $p \leqslant 2-(1-\alpha)/n$, then there exists an L^1-datum such that the corresponding Dirichlet problem does not have weak solutions.

1. Introduction

It is known that the Dirichlet problem for nonlinear elliptic second-order equations in divergence form, whose principal coefficients grow with respect to the gradient of unknown function u as $|\nabla u|^{p-1}$, has weak solutions for every L^1-right-hand side only if $p > 2 - 1/n$ where n is the dimension of the set for which the problem is considered (see [3, 4, 5]). This fact concerns the equations whose coefficients are nondegenerate with respect to the spatial variable.

In this article, we establish an analogous fact for a class of degenerate nonlinear elliptic second-order equations with L^1-data in a bounded open set Ω of $\mathbb{R}^n$ with $n \geqslant 2$. We assume that Ω contains the origin and assume that the growth and coercivity conditions on coefficients of the equations involve the weighted function $\mu(x) = |x|^\alpha$, $x \in \Omega$, where $\alpha \in (0, 1]$, and a parameter $p \in (1, n)$. The following equation is a model representative of this class:

$$-\sum_{i=1}^{n} D_i(\mu|\nabla u|^{p-2} D_i u) = f \quad \text{in } \Omega$$

where $f \in L^1(\Omega)$.

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Using a general result from [10], we prove that if $p > 2 - (1-\alpha)/n$, then the Dirichlet problem for equations of the given class has weak solutions for every L^1-right-hand side (see Theorem 2.3). On the other hand, with the use of Banach-Steinhaus theorem we find that if $p \leqslant 2 - (1-\alpha)/n$, then there exists an L^1-datum such that the corresponding Dirichlet problem does not have weak solutions (see Theorem 2.4).

Let us mention some works close to the topic of this article. Regarding the solvability of nondegenerate elliptic equations with L^1-data and measures as data, additionally to [3, 4, 5], we also refer the readers to works [6, 7, 16]. Solvability of the Dirichlet problem for degenerate nonlinear elliptic second-order equations with L^1-data and measures as data was studied for instance in [1, 2, 8, 9, 10, 15].

We remark that in [1, 8], the existence of entropy solutions to the given problem was proved in the case of L^1-data. In [2], the existence of a renormalized solution of the problem was established for the same case. In [2, 9, 15], the existence of distributional solutions of the problem was obtained in the case of right-hand side measures.

Some general conditions for the existence of weak solutions to the Dirichlet problem for degenerate anisotropic elliptic second-order equations with L^1-right-hand sides were given in [10]. However, no results on the sharpness of conditions of the existence of weak solutions to the problem under consideration in the degenerate case were not given in the mentioned works.

Conditions of the existence of weak solutions to the Dirichlet problem for degenerate nonlinear elliptic high-order equations with a strengthened weighted coercivity and L^1-data were established in [11, 12]. Finally, we note that a condition of the nonexistence of weak solutions to the Dirichlet problem for nondegenerate nonlinear elliptic high-order equations with L^1-data was obtained in [13], and conditions of the nonexistence of weak solutions to the Dirichlet problem for nondegenerate nonlinear elliptic second- and high-order equations with data from Lebesgue classes close to L^1 were given in [14].

This article is organized as follows. In Section 2, we describe initial assumptions and give the statements of above-mentioned Theorems 2.3 and 2.4. Section 3 contains the proof of Theorem 2.3, and in Section 4, we expose the proof of Theorem 2.4. At last, in Section 5, we consider an example where conditions supposed for coefficients of the investigated equations are satisfied.

2. Initial assumptions and statement of results

Let $n \in \mathbb{N}$, $n \geqslant 2$, and let Ω be a bounded open set of $\mathbb{R}^n$. We assume that the origin is contained in Ω. Let $\alpha \in (0,1]$, and let $\mu : \Omega \to \mathbb{R}$ be the function such that for every $x \in \Omega$, $\mu(x) = |x|^\alpha$.

Next, let $p \in (1,n)$, $c_1, c_2 > 0$, and let $g, h : \Omega \to \mathbb{R}$ be functions such that $g, h \geqslant 0$ in Ω, $g, h \in L^1(\Omega)$ and $\mu g^p \in L^1(\Omega)$. Let for every $i \in \{1, \dots, n\}$, $a_i : \Omega \times \mathbb{R}^n \to \mathbb{R}$ be a Carathéodory function. We suppose that for almost every $x \in \Omega$ and for every $\xi \in \mathbb{R}^n$ the following inequalities hold:

$$\sum_{i=1}^n |a_i(x,\xi)| \leqslant c_1 \mu(x) |\xi|^{p-1} + \mu(x) g^{p-1}(x), \tag{2.1}$$

$$\sum_{i=1}^n a_i(x,\xi)\xi_i \geqslant c_2 \mu(x) |\xi|^p - h(x). \tag{2.2}$$

Moreover, we assume that for almost every $x \in \Omega$ and for every $\xi, \xi' \in \mathbb{R}^n$, $\xi \neq \xi'$,

$$\sum_{i=1}^{n} [a_i(x,\xi) - a_i(x,\xi')](\xi_i - \xi_i') > 0. \tag{2.3}$$

Definition 2.1. If $f \in L^1(\Omega)$, then $\mathcal{D}(f)$ is the set of all functions $u \in \mathring{W}^{1,1}(\Omega)$ such that

(i) for every $i \in \{1,\dots,n\}$, $a_i(x,\nabla u) \in L^1(\Omega)$;
(ii) for every function $\varphi \in C_0^\infty(\Omega)$,

$$\int_\Omega \Big\{ \sum_{i=1}^{n} a_i(x,\nabla u) D_i \varphi \Big\}\, dx = \int_\Omega f\varphi\, dx.$$

Definition 2.2. Let $f \in L^1(\Omega)$. We say that u is a weak solution to the Dirichlet problem

$$-\sum_{i=1}^{n} D_i a_i(x,\nabla u) = f \text{ in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega \tag{2.4}$$

if $u \in \mathcal{D}(f)$.

The latter definition corresponds to the definition of weak solution to the Dirichlet problem for nondegenerate elliptic second-order equations with L^1-data or measures as data (see for instance [4, 5]). In the next two sections we prove the following results.

Theorem 2.3. *Let $p > 2 - (1-\alpha)/n$. Then for every function $f \in L^1(\Omega)$ the set $\mathcal{D}(f)$ is nonempty.*

Theorem 2.4. *Let $p \leqslant 2 - (1-\alpha)/n$. Then there exists a function $f \in L^1(\Omega)$ such that the set $\mathcal{D}(f)$ is empty.*

Thus, by the above theorems, the condition $p > 2 - (1-\alpha)/n$ is a sharp requirement for guaranteeing the existence of weak solutions to problem (2.4) for every $f \in L^1(\Omega)$. The next result is a simple consequence of these theorems.

Corollary 2.5. *Suppose that $\alpha = 1$. Then the following assertions hold:*

(a) *if $p > 2$, then for every $f \in L^1(\Omega)$, $\mathcal{D}(f) \neq \emptyset$;*
(b) *if $p \leqslant 2$, then there exists $f \in L^1(\Omega)$ such that $\mathcal{D}(f) = \emptyset$;*
(c) *if $n = 2$, then there exists $f \in L^1(\Omega)$ such that $\mathcal{D}(f) = \emptyset$;*

Observe that the case $p > 2$ is possible only if $n > 2$.

3. Proof of Theorem 2.3

The proof is an application of a result in [10] on the existence of weak solutions to the Dirichlet problem for degenerate anisotropic elliptic second-order equations with L^1-data. Let us formulate this result.

Let for every $i \in \{1,\dots,n\}$, q_i be a number such that $1 < q_i < n$ and ν_i be a nonnegative function on Ω such that $\nu_i > 0$ a. e. in Ω,

$$\nu_i \in L^1_{\mathrm{loc}}(\Omega), \quad \Big(\frac{1}{\nu_i}\Big)^{1/(q_i-1)} \in L^1(\Omega). \tag{3.1}$$

We define

$$\overline{q} = \Big(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{q_i}\Big)^{-1}$$

and for every $m \in \mathbb{R}^n$ such that $m_i > 0$, $i = 1, \dots, n$, we set

$$p_m = n\Big(\sum_{i=1}^{n} \frac{1+m_i}{m_i q_i} - 1\Big)^{-1}.$$

Further, let $c', c'' > 0$, $g_1, g_2 \in L^1(\Omega)$, $g_1, g_2 \geqslant 0$ in Ω, and let for every $i \in \{1, \dots, n\}$, $b_i : \Omega \times \mathbb{R}^n \to \mathbb{R}$ be a Carathéodory function. We suppose that for almost every $x \in \Omega$ and for every $\xi \in \mathbb{R}^n$,

$$\sum_{i=1}^{n} (1/\nu_i)^{1/(q_i-1)}(x)|b_i(x,\xi)|^{q_i/(q_i-1)} \leqslant c' \sum_{i=1}^{n} \nu_i(x)|\xi_i|^{q_i} + g_1(x), \tag{3.2}$$

$$\sum_{i=1}^{n} b_i(x,\xi)\xi_i \geqslant c'' \sum_{i=1}^{n} \nu_i(x)|\xi_i|^{q_i} - g_2(x). \tag{3.3}$$

Moreover, we assume that for almost every $x \in \Omega$ and for every $\xi, \xi' \in \mathbb{R}^n$, $\xi \neq \xi'$,

$$\sum_{i=1}^{n} [b_i(x,\xi) - b_i(x,\xi')](\xi_i - \xi_i') > 0. \tag{3.4}$$

According to [10, Corollary 3.9], the following proposition holds.

Proposition 3.1. *Suppose that there exist $m, \sigma \in \mathbb{R}^n$ with positive coordinates such that the following conditions are satisfied:*

$$\forall i \in \{1, \dots, n\}, \quad \frac{\overline{q}}{p_m(\overline{q}-1)} < q_i - 1 - \frac{1}{m_i}, \quad \frac{1}{\nu_i} \in L^{m_i}(\Omega), \tag{3.5}$$

$$\forall i \in \{1, \dots, n\}, \quad \frac{1}{\sigma_i} < 1 - \frac{(q_i-1)\overline{q}}{p_m(\overline{q}-1)}, \quad \nu_i \in L^{\sigma_i}(\Omega). \tag{3.6}$$

Let $f \in L^1(\Omega)$. Then there exists a function $u \in \mathring{W}^{1,1}(\Omega)$ such that

(i) *for every $i \in \{1, \dots, n\}$, $b_i(x, \nabla u) \in L^1(\Omega)$;*
(ii) *for every function $\varphi \in C_0^1(\Omega)$,*

$$\int_\Omega \Big\{\sum_{i=1}^{n} b_i(x,\nabla u) D_i\varphi\Big\} dx = \int_\Omega f\varphi\, dx.$$

Now, let

$$p > 2 - \frac{1-\alpha}{n}. \tag{3.7}$$

To apply Proposition 3.1, for every $i \in \{1, \dots, n\}$ we set $q_i = p$, $\nu_i = \mu$ and $b_i = a_i$. Since $1 < p < n$, for every $i \in \{1, \dots, n\}$ we have $1 < q_i < n$. Obviously, for every $i \in \{1, \dots, n\}$, ν_i is a nonnegative function on Ω, $\nu_i > 0$ a.e. in Ω and the first inclusion of (3.1) holds. Furthermore, since, by (3.7), $\alpha/(p-1) < n$, the second inclusion of (3.1) holds for every $i \in \{1, \dots, n\}$. Setting $c' = (2c_1)^{p/(p-1)} n^{p+1}$, $c'' = c_2/n$, $g_1 = 2^{p/(p-1)} n\mu g^p$ and $g_2 = h$, we have $c', c'' > 0$, $g_1, g_2 \in L^1(\Omega)$, $g_1, g_2 \geqslant 0$ in Ω, and using (2.1) and (2.2), we obtain that for almost every $x \in \Omega$ and for every $\xi \in \mathbb{R}^n$ inequalities (3.2) and (3.3) hold. Moreover, from (2.3) it follows that for almost every $x \in \Omega$ and for every $\xi, \xi' \in \mathbb{R}^n$, $\xi \neq \xi'$, inequality (3.4) holds.

Next, since $\alpha \leqslant 1$ and $1 < p$, we have $\alpha < p$. Also in view of (3.7), $\alpha < n(p - 2 + 1/n)$. Therefore,

$$\max\Big\{\frac{n}{p}, \frac{1}{p-2+1/n}\Big\} < \frac{n}{\alpha}.$$

Taking this inequality into account, we fix a number t such that

$$\max\Big\{\frac{n}{p}\,,\,\frac{1}{p-2+1/n}\Big\} < t < \frac{n}{\alpha}\,, \tag{3.8}$$

and then we fix a number s such that

$$s > \frac{nt}{pt-n}\,. \tag{3.9}$$

Let $m, \sigma \in \mathbb{R}^n$ be elements such that for every $i \in \{1, \dots, n\}$, $m_i = t$ and $\sigma_i = s$. We have $\overline{q} = p$ and

$$\frac{1}{p_m} = \frac{1}{p} - \frac{1}{n} + \frac{1}{tp}\,.$$

Therefore, since, by (3.8), $1/t < p - 2 + 1/n$, we obtain

$$\frac{\overline{q}}{p_m(\overline{q}-1)} < p - 1 - \frac{1}{t}\,, \tag{3.10}$$

and using (3.9), we obtain

$$\frac{1}{s} < 1 - \frac{(p-1)\overline{q}}{p_m(\overline{q}-1)}\,. \tag{3.11}$$

Finally, since, in view of (3.8), $\alpha t < n$, we have $1/\mu \in L^t(\Omega)$, and it is obvious that $\mu \in L^s(\Omega)$. These inclusions along with (3.10) and (3.11) imply that conditions (3.5) and (3.6) are satisfied. Then, by Proposition 3.1, for every function $f \in L^1(\Omega)$ the set $\mathcal{D}(f)$ is nonempty. This completes the proof of the theorem.

4. Proof of Theorem 2.4

Let

$$p \leqslant 2 - \frac{1-\alpha}{n}\,. \tag{4.1}$$

Then taking into account that $\alpha \leqslant 1$ and $p > 1$, we have $0 \leqslant 2 - p < 1$. We define

$$r = \begin{cases} \frac{1}{2-p} & \text{if } p < 2, \\ +\infty & \text{if } p = 2. \end{cases}$$

Obviously, $r > 1$.

We denote by W the set of all functions $u \in L^1(\Omega)$ such that for every $i \in \{1, \dots, n\}$ there exists the weak derivative $D_i u$ and $\mu D_i u \in L^r(\Omega)$. W is a normed space with respect to the norm

$$\|u\| = \|u\|_{L^1(\Omega)} + \sum_{i=1}^{n} \|\mu D_i u\|_{L^r(\Omega)}\,.$$

Evidently, $C_0^\infty(\Omega) \subset W$. We denote by $\mathring{W}$ the closure of $C_0^\infty(\Omega)$ in W.

Proposition 4.1. *Assume that for every $f \in L^1(\Omega)$ the set $\mathcal{D}(f)$ is nonempty. Then $\mathring{W} \subset L^\infty(\Omega)$.*

Proof. Taking into account the assumption of the proposition, for every $f \in L^1(\Omega)$ we fix a function $u_f \in \mathcal{D}(f)$. Thus, if $f \in L^1(\Omega)$, then $u_f \in \mathring{W}^{1,1}(\Omega)$, for every $i \in \{1, \dots, n\}$ we have $a_i(x, \nabla u_f) \in L^1(\Omega)$, and

$$\forall \varphi \in C_0^\infty(\Omega), \quad \int_\Omega \Big\{ \sum_{i=1}^{n} a_i(x, \nabla u_f) D_i \varphi \Big\} dx = \int_\Omega f \varphi \, dx. \tag{4.2}$$

Observe that, in view of (2.1) and the inclusions $g \in L^1(\Omega)$ and $|\nabla u_f| \in L^1(\Omega)$ where $f \in L^1(\Omega)$, the following assertion holds:

$$\text{if } f \in L^1(\Omega) \text{ and } i \in \{1,\dots,n\}, \text{ then } (1/\mu)a_i(x,\nabla u_f) \in L^{1/(p-1)}(\Omega). \tag{4.3}$$

Next, let $f \in L^1(\Omega)$, $\varphi \in W$ and $i \in \{1,\dots,n\}$. It is clear that

$$|a_i(x,\nabla u_f)D_i\varphi| = |(1/\mu)a_i(x,\nabla u_f)| \cdot |\mu D_i\varphi| \quad \text{in } \Omega \setminus \{0\}. \tag{4.4}$$

Suppose that $p < 2$. Then using Young's inequality with the exponents $1/(p-1)$ and r, we obtain

$$|(1/\mu)a_i(x,\nabla u_f)| \cdot |\mu D_i\varphi| \leqslant |(1/\mu)a_i(x,\nabla u_f)|^{1/(p-1)} + |\mu D_i\varphi|^r . \tag{4.5}$$

Since $\varphi \in W$, we have $|\mu D_i\varphi|^r \in L^1(\Omega)$. This along with (4.3)–(4.5) implies that $a_i(x,\nabla u_f)D_i\varphi \in L^1(\Omega)$. Now, let $p = 2$. Then we have $\mu D_i\varphi \in L^\infty(\Omega)$. Therefore, by (4.4),

$$|a_i(x,\nabla u_f)D_i\varphi| \leqslant \|\mu D_i\varphi\|_{L^\infty(\Omega)}|(1/\mu)a_i(x,\nabla u_f)| \ \text{ a.e. in } \ \Omega.$$

This and (4.3) imply that $a_i(x,\nabla u_f)D_i\varphi \in L^1(\Omega)$.

Thus, the following assertion holds: if $f \in L^1(\Omega)$, $\varphi \in W$ and $i \in \{1,\dots,n\}$, then $a_i(x,\nabla u_f)D_i\varphi \in L^1(\Omega)$. Taking this assertion into account, for every $f \in L^1(\Omega)$ we define the functional $H_f : W \to \mathbb{R}$ by

$$\langle H_f,\varphi\rangle = \int_\Omega \Big\{ \sum_{i=1}^n a_i(x,\nabla u_f)D_i\varphi \Big\} dx, \quad \varphi \in W. \tag{4.6}$$

Let $f \in L^1(\Omega)$. Obviously, the functional H_f is linear. Moreover, if $\varphi \in W$, using (4.4), (4.3), the inclusions $\mu D_i\varphi \in L^r(\Omega)$, $i = 1,\dots,n$, and Hölder's inequality, we obtain

$$\begin{aligned}
|\langle H_f,\varphi\rangle| &\leqslant \sum_{i=1}^n \int_\Omega |(1/\mu)a_i(x,\nabla u_f)||\mu D_i\varphi| dx \\
&\leqslant \sum_{i=1}^n \|(1/\mu)a_i(x,\nabla u_f)\|_{L^{1/(p-1)}(\Omega)} \|\mu D_i\varphi\|_{L^r(\Omega)} \\
&\leqslant \Big\{ \sum_{i=1}^n \|(1/\mu)a_i(x,\nabla u_f)\|_{L^{1/(p-1)}(\Omega)} \Big\} \|\varphi\|.
\end{aligned}$$

Therefore, the functional H_f is continuous. Thus,

$$\forall f \in L^1(\Omega), \quad H_f \in W^*. \tag{4.7}$$

From (4.2) and (4.6) it follows that the following property holds:

$$\text{if } f \in L^1(\Omega) \text{ and } \varphi \in C_0^\infty(\Omega), \text{ then } \langle H_f,\varphi\rangle = \int_\Omega f\varphi \, dx. \tag{4.8}$$

Now, let us fix an arbitrary $\varphi \in \mathring{W}$, and let $F : L^1(\Omega) \to \mathbb{R}$ be the functional such that for every $f \in L^1(\Omega)$,

$$\langle F, f\rangle = \langle H_f,\varphi\rangle.$$

We shall show that $F \in (L^1(\Omega))^*$. To this end we fix a sequence $\{\varphi_k\} \subset C_0^\infty(\Omega)$ such that

$$\|\varphi_k - \varphi\| \to 0. \tag{4.9}$$

Taking $f_1, f_2 \in L^1(\Omega)$ and $\lambda_1, \lambda_2 \in \mathbb{R}$, owing to (4.8), for every $k \in \mathbb{N}$ we have

$$\langle H_{\lambda_1 f_1 + \lambda_2 f_2}, \varphi_k \rangle = \lambda_1 \langle H_{f_1}, \varphi_k \rangle + \lambda_2 \langle H_{f_2}, \varphi_k \rangle.$$

Hence, by (4.7) and (4.9), we deduce the equality

$$\langle H_{\lambda_1 f_1 + \lambda_2 f_2}, \varphi \rangle = \lambda_1 \langle H_{f_1}, \varphi \rangle + \lambda_2 \langle H_{f_2}, \varphi \rangle.$$

Therefore, in view of the definition of the functional F, we have

$$\langle F, \lambda_1 f_1 + \lambda_2 f_2 \rangle = \lambda_1 \langle F, f_1 \rangle + \lambda_2 \langle F, f_2 \rangle.$$

Thus, the functional F is linear. To prove the continuity of F, for every $k \in \mathbb{N}$ we define the functional $F_k : L^1(\Omega) \to \mathbb{R}$ by

$$\langle F_k, f \rangle = \langle H_f, \varphi_k \rangle, \quad f \in L^1(\Omega). \tag{4.10}$$

From (4.8) it follows that $\{F_k\} \subset (L^1(\Omega))^*$. Moreover, owing to (4.7) and (4.9), for every $f \in L^1(\Omega)$ the sequence of the numbers $\langle F_k, f \rangle$ is bounded. Therefore, by the Banach-Steinhaus theorem, there exists $C > 0$ such that

$$\forall k \in \mathbb{N}, \quad \|F_k\|_{(L^1(\Omega))^*} \leqslant C. \tag{4.11}$$

Using (4.10) and (4.11), we obtain that for every $f \in L^1(\Omega)$ and $k \in \mathbb{N}$,

$$|\langle H_f, \varphi_k \rangle| \leqslant C \|f\|_{L^1(\Omega)}.$$

This along with (4.7), (4.9) and the definition of the functional F implies that for every $f \in L^1(\Omega)$,

$$|\langle F, f \rangle| \leqslant C \|f\|_{L^1(\Omega)}.$$

Hence, taking into account the linearity of F, we have $F \in (L^1(\Omega))^*$. Therefore, there exists a function $\psi \in L^\infty(\Omega)$ such that

$$\forall f \in L^1(\Omega), \quad \langle F, f \rangle = \int_\Omega f \psi \, dx. \tag{4.12}$$

Let us show that $\varphi = \psi$ a.e. in Ω. In fact, let $f \in L^\infty(\Omega)$. Using (4.8), for every $k \in \mathbb{N}$ we have

$$\langle H_f, \varphi_k \rangle = \int_\Omega f \varphi \, dx + \int_\Omega f(\varphi_k - \varphi) dx.$$

This along with (4.7) and (4.9) implies that

$$\langle H_f, \varphi \rangle = \int_\Omega f \varphi \, dx. \tag{4.13}$$

On the other hand, by the definition of the functional F and (4.12), we have

$$\langle H_f, \varphi \rangle = \int_\Omega f \psi \, dx.$$

From this and (4.13) we derive that

$$\int_\Omega f(\varphi - \psi) dx = 0.$$

Hence, taking into account the arbitrariness of f in $L^\infty(\Omega)$, we obtain that $\varphi = \psi$ a.e. in Ω. Then $\varphi \in L^\infty(\Omega)$, and due to the arbitrariness of φ in $\mathring{W}$, we conclude that $\mathring{W} \subset L^\infty(\Omega)$. The proposition is proved. $\square$

Proposition 4.2. *The set $\mathring{W} \setminus L^\infty(\Omega)$ is nonempty.*

Proof. Let B be a closed ball of $\mathbb{R}^n$ with center at the origin such that $B \subset \Omega$. We fix a function $\varphi \in C_0^\infty(\Omega)$ such that $0 \leqslant \varphi \leqslant 1$ in Ω and $\varphi = 1$ in B, and also fix $M > (1 + \operatorname{diam}\Omega)e$.

Now, let $w : \Omega \to \mathbb{R}$ be the function such that $w(0) = 0$ and for every $x \in \Omega \setminus \{0\}$,

$$w(x) = \varphi(x) \ln\ln \frac{M}{|x|}\,.$$

It is easy to see that $w \in L^1(\Omega)$ and

$$w \notin L^\infty(\Omega). \tag{4.14}$$

Let us show that $w \in \mathring{W}$. For this purpose, for every $j \in \mathbb{N}$ we define the function $w_j : \Omega \to \mathbb{R}$ by

$$w_j(x) = \varphi(x) \ln\ln \frac{M}{(|x|^2 + 1/j)^{1/2}}\,, \quad x \in \Omega.$$

We have

$$\{w_j\} \subset C_0^\infty(\Omega), \tag{4.15}$$

$$w_j \to w \quad \text{in } \Omega \setminus \{0\}, \tag{4.16}$$

$$\forall j \in \mathbb{N}, \quad 0 \leqslant w_j \leqslant w \quad \text{in } \Omega \setminus \{0\}. \tag{4.17}$$

Using (4.16), (4.17), the inclusion $w \in L^1(\Omega)$ and Dominated Convergence Theorem, we obtain that

$$w_j \to w \quad \text{strongly in } L^1(\Omega). \tag{4.18}$$

Next, let us fix $i \in \{1, \dots, n\}$, and let $z_i : \Omega \to \mathbb{R}$ be the function such that $z_i(0) = 0$ and for every $x \in \Omega \setminus \{0\}$,

$$z_i(x) = -\varphi(x) \frac{x_i}{|x|^2} \Big(\ln \frac{M}{|x|} \Big)^{-1} + (D_i\varphi(x)) \ln\ln \frac{M}{|x|}\,.$$

Obviously, $z_i \in L^1(\Omega)$. For every $j \in \mathbb{N}$ and $x \in \Omega$ we have

$$\begin{aligned} D_i w_j(x) = {} & -\varphi(x) \frac{x_i}{|x|^2 + 1/j} \Big(\ln \frac{M}{(|x|^2 + 1/j)^{1/2}} \Big)^{-1} \\ & + (D_i\varphi(x)) \ln\ln \frac{M}{(|x|^2 + 1/j)^{1/2}}\,. \end{aligned} \tag{4.19}$$

Evidently,

$$D_i w_j \to z_i \quad \text{in } \Omega \setminus \{0\}. \tag{4.20}$$

Moreover, by (4.19), for every $j \in \mathbb{N}$ and $x \in \Omega \setminus \{0\}$ we have

$$|D_i w_j(x)| \leqslant \Big(1 + M \max_{\Omega} |D_i\varphi|\Big) \frac{1}{|x|}\,.$$

Using this fact, the inclusion $z_i \in L^1(\Omega)$, (4.20) and Dominated Convergence Theorem, we conclude that

$$D_i w_j \to z_i \quad \text{strongly in } L^1(\Omega). \tag{4.21}$$

In turn, using (4.18) and (4.21), in a standard way we establish that there exists the weak derivative $D_i w$ and

$$D_i w = z_i \quad \text{a.e. in } \Omega. \tag{4.22}$$

From (4.20) and (4.22) it follows that

$$D_i w_j \to D_i w \quad \text{a.e. in } \Omega. \tag{4.23}$$

Let us show that

$$\mu D_i w \in L^r(\Omega), \tag{4.24}$$

$$\|\mu D_i(w_j - w)\|_{L^r(\Omega)} \to 0. \tag{4.25}$$

At first we suppose that $p < 2$. Then $r = 1/(2-p)$, and from (4.1) we infer that

$$(1-\alpha)r \leqslant n. \tag{4.26}$$

We set $M_i = \max_\Omega |D_i\varphi|$, and let $\psi : \Omega \to \mathbb{R}$ be the function such that $\psi(0) = 0$ and for every $x \in \Omega \setminus \{0\}$,

$$\psi(x) = \Big(\frac{M}{|x|}\Big)^n \Big(\ln \frac{M}{|x|}\Big)^{-r}.$$

Since $r > 1$, we have $\psi \in L^1(\Omega)$. Fixing an arbitrary $x \in \Omega \setminus \{0\}$, from the definition of the function z_i we obtain

$$|z_i(x)| \leqslant \frac{1}{|x|}\Big(\ln \frac{M}{|x|}\Big)^{-1} + M_i \ln\ln \frac{M}{|x|}. \tag{4.27}$$

It is easy to see that

$$\ln\ln \frac{M}{|x|} < \ln \frac{M}{|x|} < \frac{4M}{|x|}\Big(\ln \frac{M}{|x|}\Big)^{-1}.$$

This and (4.27) imply that

$$|\mu z_i|^r(x) \leqslant \frac{(1+4MM_i)^r}{|x|^{(1-\alpha)r}}\Big(\ln \frac{M}{|x|}\Big)^{-r}.$$

Then, taking into account (4.22) and (4.26), we find that

$$|\mu D_i w|^r \leqslant (1+4MM_i)^r \psi \quad \text{a.e. in } \Omega. \tag{4.28}$$

Hence, in view of the inclusion $\psi \in L^1(\Omega)$, we obtain that inclusion (4.24) holds. Besides, starting from (4.19), by analogy with (4.28), we establish that for every $j \in \mathbb{N}$,

$$|\mu D_i w_j|^r \leqslant (1+4MM_i)^r \psi \quad \text{in } \Omega \setminus \{0\}. \tag{4.29}$$

Using (4.23), (4.28), (4.29), the inclusion $\psi \in L^1(\Omega)$ and Dominated Convergence Theorem, we obtain that assertion (4.25) holds.

Now, let $p = 2$. Then from the initial assumption $\alpha \leqslant 1$ and (4.1) it follows that $\alpha = 1$. Moreover, $r = +\infty$. Taking into consideration the equality $\alpha = 1$ and the definitions of the functions μ and z_i, we find that for every $x \in \Omega \setminus \{0\}$, $\mu(x)|z_i(x)| \leqslant 1 + 4MM_i$. This along with (4.22) and the equality $r = +\infty$ implies that inclusion (4.24) holds. In order to prove the validity of assertion (4.25) in the case under consideration, for every $j \in \mathbb{N}$ and $x \in \Omega \setminus \{0\}$ we set

$$\beta_j(x) = \frac{1}{j|x|^2 + 1}\Big(\ln \frac{M}{(|x|^2 + 1/j)^{1/2}}\Big)^{-1},$$

$$\gamma_j(x) = \Big(\ln \frac{M}{(|x|^2 + 1/j)^{1/2}}\Big)^{-1} - \Big(\ln \frac{M}{|x|}\Big)^{-1},$$

$$\lambda_j(x) = |x|\Big\{\ln\ln \frac{M}{|x|} - \ln\ln \frac{M}{(|x|^2 + 1/j)^{1/2}}\Big\}.$$

Using (4.19), the definitions of the functions z_i and μ and the equality $\alpha = 1$, we find that for every $j \in \mathbb{N}$ and $x \in \Omega \setminus \{0\}$,

$$\mu(x)|D_i w_j(x) - z_i(x)| \leqslant \beta_j(x) + \gamma_j(x) + M_i \lambda_j(x). \tag{4.30}$$

Now, let $\varepsilon \in (0,1)$ and

$$\varepsilon_1 = \max\Big\{2e^{1/\varepsilon},\ \Big(\frac{2M}{\varepsilon}\Big)^2\Big\}.$$

We fix numbers ε_2 and ε_3 such that $0 < \varepsilon_3 < \varepsilon_2 < 1$ and

$$\ln(1+\varepsilon_2) \leqslant \frac{\varepsilon}{1+\operatorname{diam}\Omega}\,, \quad \ln(1+\varepsilon_3) \leqslant \varepsilon_2, \tag{4.31}$$

and then fix an arbitrary $j \in \mathbb{N}$ such that

$$j \geqslant \Big(\frac{\varepsilon_1}{M\varepsilon_3}\Big)^2 \ln\frac{\varepsilon_1}{2}\,.$$

Let $x \in \Omega \setminus \{0\}$, and assume that $|x| \leqslant M/\varepsilon_1$. Then, taking into account that $\varepsilon_1 \geqslant 2e^{1/\varepsilon}$ and $j \geqslant (\varepsilon_1/M)^2$, we obtain

$$\beta_j(x) \leqslant \Big(\ln\frac{M}{(|x|^2+1/j)^{1/2}}\Big)^{-1} \leqslant \Big(\ln\frac{\varepsilon_1}{2}\Big)^{-1} \leqslant \varepsilon,$$

$$\gamma_j(x) \leqslant \Big(\ln\frac{M}{(|x|^2+1/j)^{1/2}}\Big)^{-1} \leqslant \varepsilon.$$

Moreover, since $\varepsilon_1 \geqslant (2M/\varepsilon)^2$ and

$$\ln\ln\frac{M}{|x|} < \ln\frac{M}{|x|} < 2\Big(\frac{M}{|x|}\Big)^{1/2},$$

we obtain

$$\lambda_j(x) \leqslant |x|\ln\ln\frac{M}{|x|} \leqslant 2(M|x|)^{1/2} \leqslant \frac{2M}{\varepsilon_1^{1/2}} \leqslant \varepsilon.$$

Therefore, if $|x| \leqslant M/\varepsilon_1$, then

$$\beta_j(x) + \gamma_j(x) + M_i\lambda_j(x) \leqslant (2+M_i)\varepsilon. \tag{4.32}$$

Suppose now that $|x| > M/\varepsilon_1$. Then, taking into account that $\varepsilon_1 \geqslant 2e^{1/\varepsilon}$ and $j \geqslant (\varepsilon_1/M)^2\ln(\varepsilon_1/2)$, we obtain

$$\beta_j(x) \leqslant \frac{1}{j|x|^2} \leqslant \frac{1}{j}\Big(\frac{\varepsilon_1}{M}\Big)^2 \leqslant \Big(\ln\frac{\varepsilon_1}{2}\Big)^{-1} \leqslant \varepsilon.$$

Moreover, since $\varepsilon_3 < \varepsilon_2$ and $j \geqslant (\varepsilon_1/M\varepsilon_3)^2$, using the first inequality of (4.31), we obtain

$$\begin{aligned}\gamma_j(x) &\leqslant \ln\frac{M}{|x|} - \ln\frac{M}{(|x|^2+1/j)^{1/2}} \leqslant \ln\Big(1+\frac{1}{j^{1/2}|x|}\Big)\\ &\leqslant \ln\Big(1+\frac{\varepsilon_1}{j^{1/2}M}\Big) \leqslant \ln(1+\varepsilon_3) \leqslant \varepsilon.\end{aligned}$$

Hence,

$$\Big(\ln\frac{M}{|x|}\Big)\Big(\ln\frac{M}{(|x|^2+1/j)^{1/2}}\Big)^{-1} \leqslant 1+\ln(1+\varepsilon_3).$$

This along with inequalities (4.31) implies that

$$\lambda_j(x) \leqslant |x|\ln(1+\ln(1+\varepsilon_3)) \leqslant |x|\ln(1+\varepsilon_2) \leqslant \varepsilon.$$

Therefore, if $|x| > M/\varepsilon_1$, then inequality (4.32) also holds. From the result obtained and (4.30) we deduce that for every $x \in \Omega \setminus \{0\}$,

$$\mu(x)|D_iw_j(x) - z_i(x)| \leqslant (2+M_i)\varepsilon.$$

Then, taking into account (4.22) and the equality $r = +\infty$, we obtain

$$\|\mu D_i(w_j - w)\|_{L^r(\Omega)} \leqslant (2+M_i)\varepsilon.$$

Hence, we obtain that assertion (4.25) holds.

Using the inclusions $w \in L^1(\Omega)$ and (4.24) along with (4.18) and (4.25), we conclude that $w \in W$ and $\|w_j - w\| \to 0$. This and (4.15) imply that $w \in \mathring{W}$. Then, in view of (4.14), we obtain the inclusion $w \in \mathring{W} \setminus L^\infty(\Omega)$. Therefore, the set $\mathring{W} \setminus L^\infty(\Omega)$ is nonempty. The proposition is proved. □

From Propositions 4.1 and 4.2 we deduce that there exists a function $f \in L^1(\Omega)$ such that the set $\mathcal{D}(f)$ is empty. This completes the proof of the theorem.

5. An example

In this section, we consider an example where conditions (2.1)–(2.3) are satisfied. Let $\nu : \Omega \to \mathbb{R}$ be a nonnegative function such that

$$\nu^{1/(p-1)}(1/\mu)^{1/(p-1)} \in L^1(\Omega), \quad \nu^{p/(p-1)}(1/\mu)^{1/(p-1)} \in L^1(\Omega), \tag{5.1}$$

and let $\beta : \mathbb{R} \to \mathbb{R}$ be a nondecreasing bounded and continuous function. We set

$$c = \sup_{s\in\mathbb{R}} |\beta(s)|, \quad c_1 = n, \quad c_2 = \frac{p-1}{p}\,,$$
$$g = (cn)^{1/(p-1)}\nu^{1/(p-1)}(1/\mu)^{1/(p-1)}, \quad h = (cn)^{p/(p-1)}\nu^{p/(p-1)}(1/\mu)^{1/(p-1)}.$$

Obviously, $c_1, c_2 > 0$, $g, h \geqslant 0$ in Ω, and by virtue of (5.1), we have $g, h \in L^1(\Omega)$ and $\mu g^p \in L^1(\Omega)$.

For every $i \in \{1, \dots, n\}$ and for every $(x, \xi) \in \Omega \times \mathbb{R}^n$, let

$$a_i(x, \xi) = \mu(x)|\xi|^{p-2}\xi_i + \nu(x)\beta(\xi_i). \tag{5.2}$$

It is easy to verify that for every $x \in \Omega \setminus \{0\}$ and for every $\xi \in \mathbb{R}^n$ inequalities (2.1) and (2.2) hold, and for every $x \in \Omega \setminus \{0\}$ and for every $\xi, \xi' \in \mathbb{R}^n$, $\xi \neq \xi'$, inequality (2.3) holds.

Observe that the continuity of the function β guarantees the continuity on $\mathbb{R}^n$ of the functions $a_i(x, \cdot)$ for every $i \in \{1, \dots, n\}$ and for every $x \in \Omega$.

Moreover, we remark that inclusions (5.1) hold if $\nu = \mu$ or, more generally, if for every $x \in \Omega\setminus\{0\}$, $\nu(x) = |x|^\gamma$ where $\gamma > \alpha - n(p-1)$ and $\gamma > (\alpha - n(p-1))/p$. Some suitable examples of the function β in (5.2) are as follows: 1) $\beta(s) = s/(1 + |s|)$; 2) $\beta(s) = -1$ if $s < 0$ and $\beta(s) = -1/(1 + s)$ if $s \geqslant 0$. In both cases the function β is nondecreasing, bounded and continuous.

Finally, let us note that the requirements $g \in L^1(\Omega)$ and $\mu g^p \in L^1(\Omega)$, given in the beginning of Section 2, are independent one of other. The same concerns inclusions (5.1). For instance, if $p < 1 + \alpha/n$, we have $n < (n + \alpha)/p$. Then, taking γ such that $n \leqslant \gamma < (n + \alpha)/p$ and $g : \Omega \to \mathbb{R}$ such that $g(x) = |x|^{-\gamma}$, $x \in \Omega \setminus \{0\}$, we obtain $g \notin L^1(\Omega)$ but $\mu g^p \in L^1(\Omega)$. On the other hand, if $p > 1 + \alpha/n$, we have $(n + \alpha)/p < n$. Then, fixing γ such that $(n + \alpha)/p \leqslant \gamma < n$ and taking $g : \Omega \to \mathbb{R}$ depending on γ as above, we obtain $g \in L^1(\Omega)$ but $\mu g^p \notin L^1(\Omega)$. Analogously, if $p < 1 + \alpha/n$, $n \leqslant \gamma < (n + \alpha)/p$ and for every $x \in \Omega \setminus \{0\}$, $\nu(x) = |x|^{\alpha - \gamma(p-1)}$, then the first inclusion of (5.1) does not hold but the second inclusion of (5.1) is valid. If $p > 1 + \alpha/n$, $(n + \alpha)/p \leqslant \gamma < n$ and ν is the same as in the previous case, then the first inclusion of (5.1) is valid but the second inclusion of (5.1) does not hold.

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