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WEAK SOLUTIONS TO THE LANDAU-LIFSHITZ-MAXWELL SYSTEM WITH NONLINEAR NEUMANN BOUNDARY CONDITIONS ARISING FROM SURFACE ENERGIES

GILLES CARBOU, PIERRE FABRIE, KÉVIN SANTUGINI

Abstract. We study the Landau-Lifshitz system associated with Maxwell equations in a bilayered ferromagnetic body when super-exchange and surface anisotropy interactions are present in the spacer in-between the layers. In the presence of these surface energies, the Neumann boundary condition becomes nonlinear. We prove, in three dimensions, the existence of global weak solutions to the Landau-Lifshitz-Maxwell system with nonlinear Neumann boundary conditions.

1. Introduction

Ferromagnetic materials are widely used in the industrial world. Their four main applications are data storage (hard drives), radar stealth, communications (wave circulator), and energy (transformers). For an introduction to ferromagnetism, see Aharoni[2] or Brown[5].

The state of a ferromagnetic body is characterized by its magnetization $\mathbf{m}$, a vector field whose norm is equal to 1 inside the ferromagnetic body and null outside. The evolution of $\mathbf{m}$ can be modeled by the Landau-Lifshitz equation

$$\frac{\partial \mathbf{m}}{\partial t} = -\mathbf{m} \wedge \mathbf{h}_{\mathrm{tot}} - \alpha \mathbf{m} \wedge (\mathbf{m} \wedge \mathbf{h}_{\mathrm{tot}}),$$

where $\mathbf{h}_{\mathrm{tot}}$ depends on $\mathbf{m}$ and contains various contributions. In particular, in this paper, $\mathbf{h}_{\mathrm{tot}}$ includes various volume and surface energy densities, among which the solution to Maxwell equations and several surface terms such as super-exchange and surface anisotropy.

Alouges and Soyeur [3] established the existence and the non-uniqueness of weak solutions to the Landau-Lifshitz system when only exchange is present, *i.e.* when $\mathbf{h}_{\mathrm{tot}} = \Delta \mathbf{m}$, see also Visintin [14]. Labbé [8, Ch. 10] extended the existence result in the presence of the magnetostatic field. In the absence of the exchange interaction, Joly, Métivier and Rauch obtain global existence and uniqueness results in [7]. Carbou and Fabrie [6] proved the existence of weak solutions when the Landau-Lifshitz equation is associated with Maxwell equations. Santugini proved

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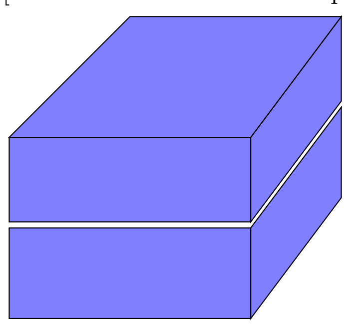
in [12], see also [11, chap. 6], the existence of weak solutions globally in time to the magnetostatic Landau-Lifshitz system in the presence of surface energies that cause the Neumann boundary conditions to become nonlinear. In this paper, we prove the existence of weak solutions to the full Landau-Lifshitz-Maxwell system with the nonlinear Neumann boundary conditions arising from the super-exchange and the surface anisotropy energies. In addition, we address the long time behavior by describing the ω-limit set of the trajectories.

The plan of the article is the following. In §2, we introduce several notations we use throughout this paper. In §3, we recall the micromagnetic model. In §4, we state our main theorems. Theorem 4.2 states the global existence in time of weak solutions to the Landau-Lifshitz system with the nonlinear Neumann Boundary conditions arising from the super-exchange and the surface anisotropy energies. Theorem 4.4 describes the ω-limit set of a solution given by the previous theorem. In §5, before starting the proofs, we recall technical results on Sobolev Spaces. We prove Theorem 4.2 in §6 and Theorem 4.4 in §7.

Notation. Throughout the paper, $\|\cdot\|$ denotes the Euclidean norm over $\mathbb{R}^d$ where d is a positive integer, often equal to 3. We denote by $\cdot$ the associated scalar product. The L^2 norm over a measurable set A is denoted by $\|\cdot\|_{\mathrm{L}^2(A)}$.

2. Geometry of spacers and related notation

In this paper, we consider a ferromagnetic domain with spacer. We denote by $\Omega = B \times \mathcal{I}$ this domain, where B is a bounded domain of $\mathbb{R}^2$ with smooth boundary and $\mathcal{I} =]-L^-, 0[\cup]0, L^+[$ where L^+ and L^- are two positive real numbers.



On the common boundary $\Gamma = B \times \{0\}$ (the spacer), γ^+ is the trace map from above that sends the restriction $\mathbf{m}_{|B\times]0,L^+[}$ to $\gamma^+\mathbf{m}$ on Γ, and γ^- is the trace map from below that sends the restriction $\mathbf{m}_{|B\times]-L^-,0[}$ to $\gamma^-\mathbf{m}$ on Γ. To simplify notations, we consider Γ has two sides: $\Gamma^+ = B \times \{0^+\}$ and $\Gamma^- = B \times \{0^-\}$. By $\Gamma^\pm$, we denote the union of these two sides $\Gamma^+ \cup \Gamma^-$. In this paper, integrating over $\Gamma^\pm$ means integrating over both sides, while integrating over Γ means integrating only once. On $\Gamma^\pm$, γ is the map that sends $\mathbf{m}$ to its trace on both sides. The trace map γ^* is the trace map that exchange the two sides of Γ: it maps $\mathbf{m}$ to $\gamma(\mathbf{m} \circ s)$ where s is the application that sends (x, y, z, t) to $(x, y, -z, t)$.

For convenience, we denote by $\boldsymbol{\nu}$ the extension to Ω of the unitary exterior normal defined on $\Gamma^\pm$, thus $\boldsymbol{\nu}(\mathbf{x}) = -\mathbf{e}_z$ if $z > 0$ or if $\mathbf{x}$ belongs to Γ^+, and $\boldsymbol{\nu}(\mathbf{x}) = \mathbf{e}_z$ if $z < 0$ or if $\mathbf{x}$ belongs to Γ^-.

In this article, $\mathbb{H}^1(\Omega)$ denotes $\mathrm{H}^1(\Omega; \mathbb{R}^3)$, and $\mathbb{L}^2(\Omega)$ denotes $\mathrm{L}^2(\Omega; \mathbb{R}^3)$. By $\mathcal{C}_c^\infty(\Omega)$, we denote the set of $\mathcal{C}^\infty$ functions that have compact support in Ω. By

$\mathcal{C}_c^\infty([0,T]\times\Omega)$, we denote the set of $\mathcal{C}^\infty$ functions that have compact support in $[0,T]\times\Omega$.

3. The micromagnetic model

In the micromagnetic model, introduced by W.F Brown[5], the magnetization $\mathbf{M}$ is the mean at the mesoscopic scale of the microscopic magnetization. It has constant norm M_s in the ferromagnetic material and is null outside. In this paper, we only work with the dimensionless magnetization $\mathbf{m} = \mathbf{M}/M_s$.

The variations of $\mathbf{m}$ are described by a phenomenological partial differential equation introduced in Landau-Lifshitz [10], the Landau-Lifshitz equation:

$$\frac{\partial \mathbf{m}}{\partial t} = -\mathbf{m}\wedge \mathbf{h}_{\text{tot}} - \alpha\mathbf{m}\wedge(\mathbf{m}\wedge\mathbf{h}_{\text{tot}}),$$

where the magnetic effective field $\mathbf{h}_{\text{tot}}$ is the Fréchet derivative of the micromagnetic energy. This micromagnetic energy is the sum of several contributions. Its minimizers under the constraint $\|\mathbf{m}\| = 1$ are the steady states of the magnetization. Let us describe now the contributions of the energy.

3.1. **Volume energies.**

3.1.1. *Exchange.* Exchange is essential in the micromagnetic theory. Without exchange, there would be no ferromagnetic materials. This interaction aligns the magnetization over short distances. In the isotropic and homogenous case, the exchange energy may be modeled by the following energy

$$\mathrm{E}_e(\mathbf{m}) = \frac{A}{2}\int_\Omega \|\nabla\mathbf{m}\|^2\,\mathrm{d}\mathbf{x},$$

where the constant A is called exchange coefficient and depends on the material. The associated exchange operator is $\mathcal{H}_e(\mathbf{m}) = -A\Delta\mathbf{m}$.

3.1.2. *Anisotropy.* Many ferromagnetic materials have a crystalline structure. This crystalline structure can penalize some directions of magnetization and favor others. Anisotropy can be modeled by

$$\mathrm{E}_a(\mathbf{m}) = \frac{1}{2}\int_\Omega (\mathbf{K}(\mathbf{x})\mathbf{m}(\mathbf{x}))\cdot\mathbf{m}(\mathbf{x})\,\mathrm{d}\mathbf{x}.$$

where $\mathbf{K}$ is a positive symmetric matrix field. The associated anisotropy operator is $\mathcal{H}_a(\mathbf{m}) = -\mathbf{K}\mathbf{m}$.

3.1.3. *Maxwell.* This is the magnetic interaction that comes from Maxwell equations. The constitutive relations in the ferromagnetic medium are given by:

$$\begin{gathered} B = \mu_0(\mathbf{h} + \overline{\mathbf{m}}),\\ D = \varepsilon_0\mathbf{e}, \end{gathered}$$

where $\overline{\mathbf{m}}$ is the extension of $\mathbf{m}$ by zero outside Ω, and where μ_0 and ε_0 are the vacuum permeability and permittivity.

Starting from the Maxwell equations, the magnetic excitation $\mathbf{h}$ and the electric field $\mathbf{e}$ are solutions to the following system:

$$\begin{gathered} \mu_0\frac{\partial(\mathbf{h}+\overline{\mathbf{m}})}{\partial t} + \operatorname{curl}\mathbf{e} = 0,\\ \mu_0\frac{\partial \mathbf{e}}{\partial t} + \sigma(\mathbf{e}+\mathbf{f})\mathbb{1}_\Omega - \operatorname{curl}\mathbf{h} = 0, \end{gathered}$$

where $\sigma \geq 0$ is the conductivity of the material and f is a source term modeling an applied electric field.

As these are evolution equations, initial conditions are needed to complete the system. The energy associated with the Maxwell interaction is

$$\mathrm{E}_{\mathrm{maxw}}(\mathbf{h},\mathbf{e}) = \frac{1}{2}\|\mathbf{h}\|^2_{\mathbb{L}^2(\mathbb{R}^3)} + \frac{\varepsilon_0}{2\mu_0}\|\mathbf{e}\|^2_{\mathbb{L}^2(\mathbb{R}^3)}.$$

We recall the Law of Faraday: $\operatorname{div} B = 0$. Here, the constitutive relation reads $B = \mu_0(\mathbf{h} + \overline{\mathbf{m}})$. Therefore, in order to satisfy the law of Faraday, we must assume that it is satisfied at initial time. For positive times, by taking the divergence of the first Maxwell's equation, we remark that the divergence free condition is propagated by the system.

3.1.4. *Volumic effective field.* The volumic effective field is the sum of the previous volumic contributions:

$$\mathbf{h}^{\mathrm{vol}}_{\mathrm{tot}} = \mathbf{h} - \mathbf{K}\mathbf{m} + A\Delta\mathbf{m}. \tag{3.1}$$

3.2. **Surface energies.** When a spacer is present inside a ferromagnetic material, new physical phenomena may appear in the spacer. These phenomena are modeled by surface energies, see Labrune and Miltat [9].

3.2.1. *Super-exchange.* This surface energy penalizes the jump of the magnetization across the spacer. It is modeled by a quadratic and a biquadratic term:

$$\mathrm{E}_{se}(\mathbf{m}) = \frac{J_1}{2}\int_\Gamma \|\gamma^+\mathbf{m} - \gamma^-\mathbf{m}\|^2 \,\mathrm{d}S(\hat{\mathbf{x}}) + J_2\int_\Gamma \|\gamma^+\mathbf{m} \wedge \gamma^-\mathbf{m}\|^2 \,\mathrm{d}S(\hat{\mathbf{x}}). \tag{3.2}$$

The magnetic excitation associated with super-exchange is:

$$\mathcal{H}_{se}(\mathbf{m}) = \Big(J_1(\gamma^*\mathbf{m} - \gamma\mathbf{m}) + 2J_2\big((\gamma\mathbf{m}\cdot\gamma^*\mathbf{m})\gamma^*\mathbf{m} - \|\gamma^*\mathbf{m}\|^2\gamma\mathbf{m}\big)\Big)\,\mathrm{d}S(\Gamma^+\cup\Gamma^-),$$

where γ^* is defined in §3. Integration over $\mathrm{d}S(\Gamma^+\cup\Gamma^-)$ should be understood as integrating over both faces of the surface Γ.

3.2.2. *Surface anisotropy.* Surface anisotropy penalizes magnetization that is orthogonal on the boundary. In the micromagnetic model, it is modeled by a surface energy:

$$\begin{aligned}\mathrm{E}_{sa}(\mathbf{m}) &= \frac{K_s}{2}\int_{\Gamma^+}\|\gamma\mathbf{m}\wedge\boldsymbol{\nu}\|^2\,\mathrm{d}S(\hat{\mathbf{x}}) + \frac{K_s}{2}\int_{\Gamma^-}\|\gamma\mathbf{m}\wedge\boldsymbol{\nu}\|^2\,\mathrm{d}S(\hat{\mathbf{x}}) \\ &= \frac{K_s}{2}\int_{\Gamma^\pm}\|\gamma\mathbf{m}\wedge\boldsymbol{\nu}\|^2\,\mathrm{d}S(\hat{\mathbf{x}}).\end{aligned} \tag{3.3}$$

The magnetic excitation associated with surface anisotropy is:

$$\mathcal{H}_{sa}(\mathbf{m}) = K_s\big((\gamma\mathbf{m}\cdot\boldsymbol{\nu})\boldsymbol{\nu} - \gamma\mathbf{m}\big)\,\mathrm{d}S(\Gamma^+\cup\Gamma^-).$$

3.2.3. *New boundary conditions.* Without surface energies, the standard boundary condition is the homogenous Neumann condition. When surface energies are present, the boundary conditions are the ones arising from the stationarity conditions on the total magnetic energy:

$$A\gamma\mathbf{m}\wedge\frac{\partial\mathbf{m}}{\partial\boldsymbol{\nu}} = K_s(\boldsymbol{\nu}\cdot\gamma\mathbf{m})\gamma\mathbf{m}\wedge\boldsymbol{\nu} + J_1\gamma\mathbf{m}\wedge\gamma^*\mathbf{m} + 2J_2(\gamma\mathbf{m}\cdot\gamma^*\mathbf{m})\gamma\mathbf{m}\wedge\gamma^*\mathbf{m}$$

on the interface $\Gamma^{\pm}$. A more convincing justification for these boundary conditions is that they are the ones needed to recover formally the energy inequality. These boundary conditions are nonlinear.

4. Landau-Lifshitz system

We consider the following Landau-Lifshitz-Maxwell system:

$$\frac{\partial \mathbf{m}}{\partial t} = -\mathbf{m} \wedge \mathbf{h}_{\mathrm{tot}}^{\mathrm{vol}} - \alpha \mathbf{m} \wedge (\mathbf{m} \wedge \mathbf{h}_{\mathrm{tot}}^{\mathrm{vol}}) \quad \text{in } \mathbb{R}^+ \times \Omega, \tag{4.1a}$$

$$\mathbf{m}(0,\cdot) = \mathbf{m}_0 \quad \text{in } \Omega, \tag{4.1b}$$

$$\|\mathbf{m}\| = 1 \quad \text{in } \mathbb{R}^+ \times \Omega, \tag{4.1c}$$

$$\frac{\partial \mathbf{m}}{\partial \boldsymbol{\nu}} = 0 \quad \text{on } \partial\Omega \setminus \Gamma^{\pm}, \tag{4.1d}$$

$$\begin{aligned}\frac{\partial \mathbf{m}}{\partial \boldsymbol{\nu}} = {}& \frac{Ks}{A}(\boldsymbol{\nu} \cdot \gamma\mathbf{m})(\boldsymbol{\nu} - (\boldsymbol{\nu} \cdot \gamma\mathbf{m})\gamma\mathbf{m}) + \frac{J_1}{A}(\gamma^*\mathbf{m} - (\gamma\mathbf{m} \cdot \gamma^*\mathbf{m})\gamma\mathbf{m}) \\ & + 2\frac{J_2}{A}(\gamma\mathbf{m} \cdot \gamma^*\mathbf{m})(\gamma^*\mathbf{m} - (\gamma\mathbf{m} \cdot \gamma^*\mathbf{m})\gamma\mathbf{m}) \quad \text{on } \mathbb{R}^+ \times \Gamma^{\pm},\end{aligned} \tag{4.1e}$$

where $\mathbf{h}_{\mathrm{tot}}^{\mathrm{vol}}$ is given by (3.1) and $(\mathbf{e}, \mathbf{h})$ is solution to Maxwell equations:

$$\mu_0 \frac{\partial(\overline{\mathbf{m}} + \mathbf{h})}{\partial t} + \operatorname{curl} \mathbf{e} = 0 \quad \text{in } \mathbb{R}^+ \times \mathbb{R}^3, \tag{4.2a}$$

$$\varepsilon_0 \frac{\partial \mathbf{e}}{\partial t} + \sigma(\mathbf{e} + \mathbf{f})\mathbb{1}_{\Omega} - \operatorname{curl} \mathbf{h} = 0 \quad \text{in } \mathbb{R}^+ \times \mathbb{R}^3, \tag{4.2b}$$

$$\mathbf{e}(0,\cdot) = \mathbf{e}_0 \quad \text{in } \mathbb{R}^3, \tag{4.2c}$$

$$\mathbf{h}(0,\cdot) = \mathbf{h}_0 \quad \text{in } \mathbb{R}^3. \tag{4.2d}$$

We first begin by defining the concept of weak solution to the Landau-Lifshitz-Maxwell system with surface energies. This concept of weak solutions is present in [3, 6, 8, 12]. The key point is that the Landau-Lifshitz equation (4.1a) is formally equivalent to the following Landau-Lifshitz-Gilbert equation:

$$\frac{\partial \mathbf{m}}{\partial t} - \alpha \mathbf{m} \wedge \frac{\partial \mathbf{m}}{\partial t} = -(1 + \alpha^2)\mathbf{m} \wedge \mathbf{h}_{\mathrm{tot}}^{\mathrm{vol}},$$

which is more convenient to obtain the weak formulation defined as

Definition 4.1 (Weak solutions to Landau-Lifshitz-Maxwell with surface energies)**.** Let $\mathbf{m}$ be in $\mathrm{L}^\infty(]0,+\infty[;\mathbb{H}^1(\Omega))$, $\mathbf{e}$ and $\mathbf{h}$ be in $\mathrm{L}^\infty(\mathbb{R}^+;\mathbb{L}^2(\mathbb{R}^3))$. We say that $(\mathbf{m}, \mathbf{e}, \mathbf{h})$ is a weak solutions to the Landau-Lifshitz Maxwell system with surface energies if

(1) $\|\mathbf{m}\| = 1$ almost everywhere in $]0,T[\times\Omega$.
(2) $\frac{\partial \mathbf{m}}{\partial t} \in \mathbb{L}^2(\mathbb{R}^+ \times \Omega)$.

(3) For all $T>0$ and $\boldsymbol{\phi}$ in $\mathbb{H}^1(]0,T[\times\Omega)$,

$$
\begin{aligned}
&\iint_{]0,T[\times\Omega} \frac{\partial \mathbf{m}}{\partial t}(t,\mathbf{x})\cdot\boldsymbol{\phi}(t,\mathbf{x})\,\mathrm{d}\mathbf{x}\,\mathrm{d}t\\
&-\alpha\iint_{]0,T[\times\Omega}\Big(\mathbf{m}(t,\mathbf{x})\wedge\frac{\partial \mathbf{m}}{\partial t}(t,\mathbf{x})\Big)\cdot\boldsymbol{\phi}(t,\mathbf{x})\,\mathrm{d}\mathbf{x}\,\mathrm{d}t\\
&=(1+\alpha^2)A\iint_{]0,T[\times\Omega}\sum_{i=1}^{3}\Big(\mathbf{m}(t,\mathbf{x})\wedge\frac{\partial \mathbf{m}}{\partial x_i}(t,\mathbf{x})\Big)\cdot\frac{\partial\boldsymbol{\phi}}{\partial x_i}(t,\mathbf{x})\,\mathrm{d}\mathbf{x}\,\mathrm{d}t\\
&\quad+(1+\alpha^2)\iint_{]0,T[\times\Omega}(\mathbf{m}(t,\mathbf{x})\wedge\mathbf{K}(\mathbf{x})\mathbf{m}(t,\mathbf{x}))\cdot\boldsymbol{\phi}(t,\mathbf{x})\,\mathrm{d}\mathbf{x}\,\mathrm{d}t\\
&\quad-(1+\alpha^2)\iint_{]0,T[\times\Omega}(\mathbf{m}(t,\mathbf{x})\wedge\mathbf{h}(t,\mathbf{x}))\cdot\boldsymbol{\phi}(t,\mathbf{x})\,\mathrm{d}\mathbf{x}\,\mathrm{d}t\\
&\quad-(1+\alpha^2)K_s\iint_{]0,T[\times\Gamma^\pm}(\boldsymbol{\nu}\cdot\gamma\mathbf{m})(\gamma\mathbf{m}\wedge\boldsymbol{\nu})\cdot\gamma\boldsymbol{\phi}\,\mathrm{d}S(\hat{\mathbf{x}})\,\mathrm{d}t\\
&\quad-(1+\alpha^2)J_1\iint_{]0,T[\times\Gamma^\pm}(\gamma\mathbf{m}\wedge\gamma^*\mathbf{m})\cdot\gamma\boldsymbol{\phi}\,\mathrm{d}S(\hat{\mathbf{x}})\,\mathrm{d}t\\
&\quad-2(1+\alpha^2)J_2\iint_{]0,T[\times\Gamma^\pm}(\gamma\mathbf{m}\cdot\gamma^*\mathbf{m})(\gamma\mathbf{m}\wedge\gamma^*\mathbf{m})\cdot\gamma\boldsymbol{\phi}\,\mathrm{d}S(\hat{\mathbf{x}})\,\mathrm{d}t.
\end{aligned}\tag{4.3a}
$$

(4) In the sense of traces, $\mathbf{m}(0,\cdot)=\mathbf{m}_0$.

(5) For all $\boldsymbol{\psi}$ in $\mathcal{C}_c^\infty([0,+\infty[\times\mathbb{R}^3;\mathbb{R}^3)$:

$$
\begin{aligned}
&-\mu_0\iint_{\mathbb{R}^+\times\mathbb{R}^3}(\mathbf{h}+\mathbf{m})\cdot\frac{\partial\boldsymbol{\psi}}{\partial t}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t+\iint_{\mathbb{R}^+\times\mathbb{R}^3}\mathbf{e}\cdot\operatorname{curl}\boldsymbol{\psi}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t\\
&=\mu_0\int_{\mathbb{R}^3}(\mathbf{h}_0+\mathbf{m}_0)\cdot\boldsymbol{\psi}_0\,\mathrm{d}\mathbf{x}
\end{aligned}\tag{4.3b}
$$

(6) For all $\boldsymbol{\Theta}$ in $\mathcal{C}_c^\infty([0,+\infty[\times\mathbb{R}^3;\mathbb{R}^3)$:

$$
\begin{aligned}
&-\varepsilon_0\iint_{\mathbb{R}^+\times\mathbb{R}^3}\mathbf{e}\cdot\frac{\partial\boldsymbol{\Theta}}{\partial t}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t-\iint_{\mathbb{R}^+\times\mathbb{R}^3}\mathbf{h}\cdot\operatorname{curl}\boldsymbol{\Theta}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t\\
&+\sigma\iint_{\mathbb{R}^+\times\Omega}(\mathbf{e}+\mathbf{f})\cdot\boldsymbol{\Theta}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t\\
&=\varepsilon_0\int_{\mathbb{R}^3}\mathbf{e}_0\cdot\boldsymbol{\Theta}_0\,\mathrm{d}\mathbf{x}.
\end{aligned}\tag{4.3c}
$$

(7) The following energy inequality holds for almost all $T>0$,

$$
\begin{aligned}
&\mathrm{E}(\mathbf{m}(T),\mathbf{h}(T),\mathbf{e}(T))+\frac{\alpha}{1+\alpha^2}\iint_{]0,T[\times\Omega}\|\frac{\partial\mathbf{m}}{\partial t}\|^2\,\mathrm{d}\mathbf{x}\,\mathrm{d}t\\
&+\frac{\sigma}{\mu_0}\int_0^T\|\mathbf{e}\|^2_{\mathbb{L}^2(\Omega)}\,\mathrm{d}t+\frac{\sigma}{\mu_0}\iint_{]0,T[\times\Omega}\mathbf{e}\cdot\mathbf{f}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t\\
&\le\mathrm{E}(\mathbf{m}_0,\mathbf{h}_0,\mathbf{e}_0),
\end{aligned}\tag{4.3d}
$$

where

$$
\begin{aligned}
\mathrm{E}(\mathbf{m},\mathbf{h},\mathbf{e})&=\frac{A}{2}\int_\Omega\|\nabla\mathbf{m}\|^2\,\mathrm{d}\mathbf{x}+\frac12\int_\Omega(\mathbf{K}(\mathbf{x})\mathbf{m}(\mathbf{x}))\cdot\mathbf{m}(\mathbf{x})\,\mathrm{d}\mathbf{x}\\
&\quad+\frac{\varepsilon_0}{2\mu_0}\int_{\mathbb{R}^3}\|\mathbf{e}(\mathbf{x})\|^2+\frac12\int_{\mathbb{R}^3}\|\mathbf{h}(\mathbf{x})\|^2+\frac{K_s}{2}\int_{\Gamma^+\cup\Gamma^-}\|\gamma^+\mathbf{m}\wedge\boldsymbol{\nu}\|^2\,\mathrm{d}S(\mathbf{x})
\end{aligned}
$$

$$+\frac{J_1}{2}\int_\Gamma \|\gamma^+\mathbf{m}-\gamma^-\mathbf{m}\|^2\,\mathrm{d}\mathbf{x}+J_2\int_\Gamma \|\gamma^+\mathbf{m}\wedge\gamma^-\mathbf{m}\|^2\,\mathrm{d}\mathbf{x}.$$

Our first result states the existence of a global in time weak solution to the Laudau-Lifshitz-Maxwell system.

Theorem 4.2. *Let $\mathbf{m}_0$ be in $\mathbb{H}^1(\Omega)$ such that $\|\mathbf{m}_0\|=1$ almost everywhere in Ω. Let $\mathbf{h}_0$ and $\mathbf{e}_0$ be in $\mathbb{L}^2(\Omega)$. Let $\mathbf{f}$ be in $\mathbb{L}^2(\mathbb{R}^+\times\Omega)$. Suppose $\operatorname{div}(\mathbf{h}_0+\overline{\mathbf{m}_0})=0$ in $\mathbb{R}^3$, where $\overline{\mathbf{m}_0}$ is the extension of $\mathbf{m}_0$ by 0 outside Ω. Then, there exists at least one weak solution to the Landau-Lifshitz-Maxwell system in the sense of Definition 4.1.*

Uniqueness is unlikely as the solution is not unique when only the exchange energy is present, see [3]. In our second result we characterize the ω-limit set of a trajectory. The definition is the following.

Definition 4.3. Let $(\mathbf{m},\mathbf{h},\mathbf{e})$ be a weak solution of the Landau-Lifshitz-Maxwell system given by Theorem 4.2. We call ω-limit set of this trajectory the set:

$$\begin{aligned}&\omega(\mathbf{m},\mathbf{h},\mathbf{e})\\ &=\big\{v\in\mathbb{H}^1(\Omega),\exists(t_n)_n,\ \lim_{n\to+\infty}t_n=+\infty,\ \mathbf{m}(t_n,\cdot)\rightharpoonup v \text{ weakly in } \mathbb{H}^1(\Omega)\big\}.\end{aligned}$$

We remark that $m\in \mathrm{L}^\infty(]0,+\infty[;\mathbb{H}^1(\Omega))$ so that $\omega(\mathbf{m},\mathbf{h},\mathbf{e})$ is non empty.

Theorem 4.4. *Let $(\mathbf{m},\mathbf{e},\mathbf{h})$ be a weak solution of the Landau-Lifshitz-Maxwell system given by Theorem 4.2. Let $\mathbf{u}\in\omega(\mathbf{m},\mathbf{h},\mathbf{e})$. Then $\mathbf{u}$ satisfies:*

1. *$\mathbf{u}\in\mathbb{H}^1(\Omega)$, $\|\mathbf{u}\|=1$ almost everywhere,*
2. *for all $\boldsymbol{\varphi}\in\mathbb{H}^1(\Omega)$,*

$$\begin{aligned}0&=A\int_\Omega\sum_{i=1}^3\Big(\mathbf{u}(\mathbf{x})\wedge\frac{\partial\mathbf{u}}{\partial x_i}(\mathbf{x})\Big)\cdot\frac{\partial\boldsymbol{\varphi}}{\partial x_i}(\mathbf{x})\,\mathrm{d}\mathbf{x}+\int_\Omega(\mathbf{u}(\mathbf{x})\wedge\mathbf{K}(\mathbf{x})\mathbf{u}(\mathbf{x}))\cdot\boldsymbol{\varphi}(\mathbf{x})\,\mathrm{d}\mathbf{x}\\ &\quad-\int_\Omega(\mathbf{u}(\mathbf{x})\wedge\mathbf{H}(\mathbf{x}))\cdot\boldsymbol{\varphi}(\mathbf{x})\,\mathrm{d}\mathbf{x}-K_s\int_{(\Gamma^\pm)}(\boldsymbol{\nu}\cdot\gamma\mathbf{u})(\gamma\mathbf{u}\wedge\boldsymbol{\nu})\cdot\gamma\boldsymbol{\varphi}\,\mathrm{d}S(\hat{\mathbf{x}})\\ &\quad-J_1\int_{(\Gamma^\pm)}(\gamma\mathbf{u}\wedge\gamma^*\mathbf{m})\cdot\gamma\boldsymbol{\varphi}\,\mathrm{d}S(\hat{\mathbf{x}})\\ &\quad-2J_2\int_{\Gamma^\pm}(\gamma\mathbf{u}\cdot\gamma^*\mathbf{u})(\gamma\mathbf{u}\wedge\gamma^*\mathbf{u})\cdot\gamma\boldsymbol{\varphi}\,\mathrm{d}S(\hat{\mathbf{x}}).\end{aligned}\tag{4.4}$$

3. *$\mathbf{H}$ is deduced from $\mathbf{u}$ by the relations:*

$$\operatorname{div}(\mathbf{H}+\overline{\mathbf{u}})=0 \text{ and } \operatorname{curl}\mathbf{H}=0\quad \text{in } \mathcal{D}'(\mathbb{R}^3).\tag{4.5}$$

Remark 4.5. Equation (4.4) is the weak formulation of the following problem:

$$\mathbf{u}\wedge(A\Delta\mathbf{u}-\mathbf{K}\mathbf{u}+\mathbf{H})=0\quad\text{in }\Omega,$$

where $\mathbf{H}$, called the demagnetizing field, satisfies (4.5),

$$\frac{\partial\mathbf{m}}{\partial\boldsymbol{\nu}}=0\quad\text{on }\partial\Omega\setminus\Gamma^\pm,$$

$$\begin{aligned}\frac{\partial\mathbf{m}}{\partial\boldsymbol{\nu}}&=\frac{Ks}{A}(\boldsymbol{\nu}\cdot\gamma\mathbf{m})(\boldsymbol{\nu}-(\boldsymbol{\nu}\cdot\gamma\mathbf{m})\gamma\mathbf{m})+\frac{J_1}{A}(\gamma^*\mathbf{m}-(\gamma\mathbf{m}\cdot\gamma^*\mathbf{m})\gamma\mathbf{m})\\ &\quad+2\frac{J_2}{A}(\gamma\mathbf{m}\cdot\gamma^*\mathbf{m})(\gamma^*\mathbf{m}-(\gamma\mathbf{m}\cdot\gamma^*\mathbf{m})\gamma\mathbf{m})\quad\text{on }\mathbb{R}^+\times\Gamma^\pm,\end{aligned}$$

5. Technical prerequisite results on Sobolev Spaces

In this section, we remind the reader about some useful previously known results on Sobolev Spaces that we use in this paper. In the whole section $\mathcal{O}$ is any bounded open set of $\mathbb{R}^3$, regular enough for the usual embeddings result to hold. For example, it is enough that $\mathcal{O}$ satisfy the cone property, see[1, §4.3].

We start with Aubin's lemma [4], as extended in [13, Corollary 4].

Lemma 5.1 (Aubin's lemma). *Let $X \Subset B \subset Y$ be Banach spaces. Let F be bounded in $\mathrm{L}^p(]0,T[;X)$. Suppose $\{\partial_t u, u \in F\}$ is bounded in $\mathrm{L}^r(]0,T[;Y)$. Suppose for all t,*

- *If $r \geq 1$ and $1 \leq p < +\infty$, then F is a compact subset of $\mathrm{L}^p(]0,T[;B)$.*
- *If $r > 1$ and $p = +\infty$, then F is a compact subset of $\mathcal{C}(0,T;B)$.*

Lemma 5.2. *For all $T > 0$, the imbedding from $\mathrm{H}^1(]0,T[\times\mathcal{O})$ to $\mathcal{C}([0,T],\mathrm{L}^2(\mathcal{O}))$ is compact.*

Proof. Use the Aubin's lemma, see [13, Corollary 4], extended to the case $p = +\infty$, with $X = \mathrm{H}^1(\mathcal{O})$ and $B = Y = \mathrm{L}^2(\Omega)$. □

Lemma 5.3. *Let u be in $\mathrm{H}^1(]0,T[\times\mathcal{O}) \cap \mathrm{L}^\infty(]0,T[;\mathrm{H}^1(\mathcal{O}))$. Then u belongs to $\mathcal{C}([0,T];\mathbb{H}^1_\omega(\mathcal{O}))$ where $\mathrm{H}^1_\omega(\mathcal{O})$ is the space $\mathrm{H}^1(\mathcal{O})$ but with the weak topology.*

Proof. The function u belongs to $\mathcal{C}([0,T],\mathrm{L}^2(\mathcal{O}))$. Let now $(t_n)_n$ be a sequence in $[0,T]$ converging to t. Then, $u(t_n,\cdot)$ converges to $u(t,\cdot)$ in $\mathrm{L}^2(\mathcal{O})$. Also, the sequence $(u(t_n,\cdot))_{n\in\mathbb{N}}$ is bounded in $\mathrm{H}^1(\mathcal{O})$, therefore from any subsequence of $(u(t_n,\cdot))_{n\in\mathbb{N}}$, one can extract a subsequence that converges weakly in $\mathrm{H}^1(\mathcal{O})$. The only possible limit is $u(t,\cdot)$ therefore the whole sequence converges weakly in $\mathrm{H}^1(\mathcal{O})$. □

Lemma 5.4. *Let $(u_n)_{n\in\mathbb{N}}$ be bounded in $\mathrm{H}^1(]0,T[\times\mathcal{O})$ and in $\mathrm{L}^\infty(]0,T[;\mathrm{H}^1(\mathcal{O}))$. Let $(u_{n_k})_{k\in\mathbb{N}}$ be a subsequence which converges weakly to some u in $\mathrm{H}^1(]0,T[\times\mathcal{O})$. Then, for all t in $[0,T]$, the same subsequence $u_{n_k}(t,\cdot)$ converges weakly to $u(t,\cdot)$ in $\mathrm{H}^1(\mathcal{O})$.*

Proof. For all t in $[0,T]$, $u_{n_k}(t,\cdot)$ converges strongly to $u(t,\cdot)$ in $\mathrm{L}^2(\mathcal{O})$. Therefore, any subsequence $u_{n_{k_j}}(t,\cdot)$ that converges weakly in $\mathrm{H}^1(\mathcal{O})$ has $u(t,\cdot)$ for limit. Since $u_{n_k}(t,\cdot)$ is bounded in $\mathrm{H}^1(\mathcal{O})$, from any subsequence of $u_{n_k}(t,\cdot)$, one can extract a further subsequence that converges weakly in $\mathrm{H}^1(\mathcal{O})$, therefore, for all t in $[0,T]$, the whole subsequence $u_{n_k}(t,\cdot)$ converges weakly to $u(t,\cdot)$ in $\mathrm{H}^1(\mathcal{O})$. □

6. Proof of Theorem 4.2

6.1. **Main idea of the proof.** We proceed as in [6] and [12] and combine the ideas of both papers. We start by extending the surface energies to a thin layer of thickness $2\eta > 0$.

As in [12], let $\mathcal{I}_\eta =]-L^-,-\eta[\cup]\eta,L^+[$. We consider the operator

$$
\begin{gathered}
\mathcal{H}_s^\eta : \mathbb{H}^1(\Omega) \cap \mathbb{L}^\infty(\Omega) \to \mathbb{H}^1(\Omega) \cap \mathbb{L}^\infty(\Omega) \\
\mathbf{m} \mapsto \frac{1}{2\eta}
\begin{cases}
0 & \text{in } \mathbb{R}^3 \setminus (B \times (\mathcal{I} \setminus \mathcal{I}_\eta)), \\
2K_s((\mathbf{m}\cdot\boldsymbol{\nu})\boldsymbol{\nu} - \mathbf{m}) + 2J_1(\mathbf{m}^* - \mathbf{m}) & \\
+4J_2\big((\mathbf{m}\cdot\mathbf{m}^*)\mathbf{m}^* - \|\mathbf{m}^*\|^2\mathbf{m}\big) & \text{in } B \times (\mathcal{I} \setminus \mathcal{I}_\eta),
\end{cases}
\end{gathered}
\tag{6.1}
$$

where $\mathbf{m}^*$ is the reflection of $\mathbf{m}$, *i.e.* $\mathbf{m}^*(x,y,z,t) = \mathbf{m}(x,y,-z,t)$, see Figure 1.

FIGURE 1. Artificial boundary layer

The associated energy is:

$$\begin{aligned} \mathrm{E}_s^\eta(\mathbf{m}) = \frac{K_s}{2\eta} \int_{B\times(\mathcal{I}\setminus\mathcal{I}_\eta)} \left(\|\mathbf{m}\|^2 - (\mathbf{m}\cdot\boldsymbol{\nu})^2\right) \,\mathrm{d}\mathbf{x} \\ + \frac{J_1}{2\eta} \int_{B\times(\mathcal{I}\setminus\mathcal{I}_\eta)} \Big(\frac{\|\mathbf{m}\|^2 + \|\mathbf{m}^*\|^2}{2} - (\mathbf{m}\cdot\mathbf{m}^*)\Big) \,\mathrm{d}\mathbf{x} \\ + \frac{J_2}{2\eta} \int_{B\times\mathcal{I}\setminus\mathcal{I}_\eta} \left(\|\mathbf{m}^*\|^2\|\mathbf{m}\|^2 - (\mathbf{m}\cdot\mathbf{m}^*)^2\right) \,\mathrm{d}\mathbf{x}. \end{aligned} \tag{6.2}$$

This energy will replace the surface terms (3.2) and (3.3). We consider the doubly penalized problem:

$$\begin{aligned} \alpha\frac{\partial \mathbf{m}_{k,\eta}}{\partial t} + \mathbf{m}_{k,\eta} \wedge \frac{\partial \mathbf{m}_{k,\eta}}{\partial t} = (1+\alpha^2)(A\Delta\mathbf{m} - \mathbf{K}\mathbf{m} + \mathbf{h}_{k,\eta} + \mathcal{H}_s^\eta(\mathbf{m}_{k,\eta})) \\ - k(1+\alpha^2)((\|\mathbf{m}_{k,\eta}\|^2 - 1)\mathbf{m}_{k,\eta}), \end{aligned} \tag{6.3a}$$

$$\frac{\partial \mathbf{m}_{k,\eta}}{\partial \boldsymbol{\nu}} = 0 \quad \text{on } \partial\Omega, \tag{6.3b}$$

$$\mathbf{m}_{k,\eta}(0,\cdot) = \mathbf{m}_0, \tag{6.3c}$$

with Maxwell equations:

$$\varepsilon_0 \frac{\partial \mathbf{e}_{k,\eta}}{\partial t} + \sigma(\mathbf{e}_{k,\eta} + \mathbf{f})\mathbb{1}_\Omega - \operatorname{curl}\mathbf{h}_{k,\eta} = 0, \tag{6.4a}$$

$$\mu_0 \frac{\partial(\mathbf{m}_{k,\eta} + \mathbf{h}_{k,\eta})}{\partial t} + \operatorname{curl}\mathbf{e}_{k,\eta} = 0, \tag{6.4b}$$

$$\mathbf{e}_{k,\eta}(0,\cdot) = \mathbf{e}_0, \tag{6.4c}$$

$$\mathbf{h}_{k,\eta}(0,\cdot) = \mathbf{h}_0. \tag{6.4d}$$

The idea is to prove the existence of weak solutions to the penalized problem via Galerkin, then have k tend to $+\infty$ to satisfy the local norm constraint on the magnetization, then have η tend to 0 to transform the homogenous Neumann boundary condition into the nonlinear condition (4.1e).

6.2. **First Step of Galerkin's method.** As in [3] we consider the eigenvectors $(v_j)_{j\geq 1}$ of the Laplace operator with Neumann homogenous conditions. This basis is, up to a renormalisation, an Hilbertian basis for the spaces $\mathrm{L}^2(\Omega)$, $\mathrm{H}^1(\Omega)$, and $\{u \in \mathrm{H}^2(\Omega), \frac{\partial u}{\partial \boldsymbol{\nu}} = 0\}$. The eigenvectors v_k all belong to $\mathcal{C}^\infty(\overline{\Omega})$. We call V_n the space spanned by $(v_j)_{1\leq j\leq n}$. As in [6], we consider an Hilbertian basis $(\omega_j)_{j\geq 1}$ of $\mathrm{L}^2(\mathbb{R}^3;\mathbb{R}^3)$ such that every ω_j belongs to $\mathcal{C}_c^\infty(\mathbb{R}^3;\mathbb{R}^3)$. We call W_n the space spanned by $(\omega_j)_{0\leq j\leq n}$.

Set $n \geq 1$, $\eta > 0$ and $k > 0$. We search for $\mathbf{m}_{n,k,\eta}$ in $\mathrm{H}^1(\mathbb{R}^+;(V_n)^3)$, $\mathbf{h}_{n,k,\eta}$ in $\mathrm{H}^1(\mathbb{R}^+;W_n)$, and $\mathbf{e}_{n,k,\eta}$ in $\mathrm{H}^1(\mathbb{R}^+;W_n)$ such that

$$\begin{aligned}\alpha\frac{\mathrm{d}\mathbf{m}_{n,k,\eta}}{\mathrm{d}t} &= -\mathcal{P}_{V_n}(\mathbf{m}_{n,k,\eta}\wedge\frac{\mathrm{d}\mathbf{m}_{n,k,\eta}}{\mathrm{d}t})\\ &\quad+(1+\alpha^2)\mathcal{P}_{V_n}(A\Delta\mathbf{m}_{n,k,\eta}-\mathbf{K}\mathbf{m}_{n,k,\eta})\\ &\quad+(1+\alpha^2)\mathcal{P}_{V_n}(\mathbf{h}_{n,k,\eta}+\mathcal{H}_s^\eta(\mathbf{m}_{n,k,\eta}))\\ &\quad-(1+\alpha^2)k\mathcal{P}_{V_n}((\|\mathbf{m}_{n,k,\eta}\|^2-1)\mathbf{m}_{n,k,\eta}),\end{aligned} \tag{6.5a}$$

and

$$\mu_0\frac{\mathrm{d}\mathbf{h}_{n,k,\eta}}{\mathrm{d}t} = -\mu_0\mathcal{P}_{W_n}\Big(\frac{\mathrm{d}\mathbf{m}_{n,k,\eta}}{\mathrm{d}t}\Big)+\mathcal{P}_{W_n}(\operatorname{curl}\mathbf{e}_{n,k,\eta}). \tag{6.5b}$$

and

$$\varepsilon_0\frac{\mathrm{d}\mathbf{e}_{n,k,\eta}}{\mathrm{d}t} = -\mathcal{P}_{W_n}(\operatorname{curl}\mathbf{h}_{n,k,\eta})-\mathcal{P}_{W_n}(\mathbb{1}_\Omega(\mathbf{e}_{n,k,\eta}+\mathbf{f})), \tag{6.5c}$$

with the initial conditions:

$$\mathbf{m}_{n,k,\eta}(0,\cdot)=\mathcal{P}_{V_n}(\mathbf{m}_0), \tag{6.6a}$$
$$\mathbf{h}_{n,k,\eta}(0,\cdot)=\mathcal{P}_{W_n}(\mathbf{h}_0), \tag{6.6b}$$
$$\mathbf{e}_{n,k,\eta}(0,\cdot)=\mathcal{P}_{W_n}(\mathbf{e}_0), \tag{6.6c}$$

where $\mathcal{P}_{V_n}$ is the orthogonal projection on $(V_n)^3$ in $\mathbb{L}^2(\Omega)$ and $\mathcal{P}_{W_n}$ is the orthogonal projection on W_n in $\mathbb{L}^2(\Omega)$. Let $\mathbf{a}(t) = (\mathbf{a}_i(t))_{1\leq i\leq n}$, $\mathbf{b}(t) = (b_i(t))_{1\leq i\leq n}$ and $\mathbf{c}(t) = (c_i(t))_{1\leq i\leq n}$ be the coefficients of $\mathbf{m}_{n,k,\eta}(t,\cdot)$, $\mathbf{h}_{n,k,\eta}(t,\cdot)$ and $\mathbf{e}_{n,k,\eta}(t,\cdot)$ in the decomposition

$$\mathbf{m}_{n,k,\eta}(t,\cdot)=\sum_{i=1}^n\mathbf{a}_\mathbf{i}(t)v_i,\quad \mathbf{h}_{n,k,\eta}(t,\cdot)=\sum_{i=1}^n b_i(t)\omega_i,\quad \mathbf{e}_{n,k,\eta}(t,\cdot)=\sum_{i=1}^n c_i(t)\omega_i.$$

Then, System (6.5) is equivalent to

$$\frac{\mathrm{d}\mathbf{a}}{\mathrm{d}t}+\boldsymbol{\phi}(\mathbf{a},\frac{\mathrm{d}\mathbf{a}}{\mathrm{d}t})=F_\mathbf{m}(\mathbf{a},\mathbf{b}), \tag{6.7a}$$
$$\frac{\mathrm{d}(\mathbf{b}+L\mathbf{a})}{\mathrm{d}t}=F_\mathbf{h}(\mathbf{c}), \tag{6.7b}$$
$$\frac{\mathrm{d}\mathbf{c}}{\mathrm{d}t}=F_\mathbf{e}(\mathbf{h}_{n,k,\eta},\mathbf{e}_{n,k,\eta})+\mathbf{f}^*, \tag{6.7c}$$

where L is linear, $F_\mathbf{m}$, $F_\mathbf{h}$ and $F_\mathbf{e}$ are polynomial thus of class $\mathcal{C}^\infty$, and $\mathbf{f}^*$ is in $\mathrm{L}^2(\mathbb{R}^+;\mathbb{R}^n)$. These are supplemented by initial conditions

$$\mathbf{a}(0,\cdot)=\mathbf{a}_0,\quad \mathbf{b}(0,\cdot)=\mathbf{b}_0,\quad \mathbf{c}(0,\cdot)=\mathbf{c}_0, \tag{6.8}$$

where $\mathbf{a}_0$, $\mathbf{b}_0$, and $\mathbf{c}_0$ are obtained from (6.6). As $\boldsymbol{\phi}(\cdot,\cdot)$ is bilinear continuous and $\boldsymbol{\phi}(\mathbf{a},\cdot)$ is antisymmetric, the linear application $\mathrm{Id}-\boldsymbol{\phi}(\mathbf{a},\cdot)$ is invertible. Therefore, by the Carathéorody theorem, System (6.7) has local solutions with initial conditions (6.8). Therefore, there exists $T^*>0$ and $\mathbf{m}_{n,k,\eta}$ in $\mathrm{H}^1(]0,T^*[;(V_n)^3)$, $\mathbf{h}_{n,k,\eta}$ in $\mathrm{H}^1(]0,T^*[;W_n)$ and $\mathbf{e}_{n,k,\eta}$ in $\mathrm{H}^1(]0,T^*[;W_n)$ that satisfy (6.5) and (6.6).

Multiplying (6.5) by test functions and integrating by part yields:

$$
\begin{aligned}
&\alpha \iint_{]0,T[\times\Omega} \frac{\partial \mathbf{m}_{n,k,\eta}}{\partial t}\cdot \boldsymbol{\phi}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t + \iint_{]0,T[\times\Omega} \Big(\mathbf{m}_{n,k,\eta}\wedge \frac{\partial \mathbf{m}_{n,k,\eta}}{\partial t}\Big)\cdot \boldsymbol{\phi}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\
&= -(1+\alpha^2)A \iint_{]0,T[\times\Omega} \sum_{i=1}^{3} \frac{\partial \mathbf{m}_{n,k,\eta}}{\partial x_i}\cdot \frac{\partial \boldsymbol{\phi}}{\partial x_i}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\
&\quad - (1+\alpha^2) \iint_{]0,T[\times\Omega} (\mathbf{K}(\mathbf{x})\mathbf{m}_{n,k,\eta}(\mathbf{x}))\cdot \boldsymbol{\phi}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\
&\quad + (1+\alpha^2) \iint_{]0,T[\times\Omega} \mathbf{h}_{n,k,\eta}\cdot \boldsymbol{\phi}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\
&\quad - (1+\alpha^2)k \iint_{]0,T[\times\Omega} (\|\mathbf{m}_{n,k,\eta}\|^2-1)\mathbf{m}_{n,k,\eta}\cdot \boldsymbol{\phi}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\
&\quad + (1+\alpha^2)\frac{K_s}{\eta} \iint_{]0,T[\times(B\times]-\eta,\eta[)} ((\boldsymbol{\nu}\cdot \mathbf{m}_{n,k,\eta})\boldsymbol{\nu} - \mathbf{m}_{n,k,\eta})\cdot \boldsymbol{\phi}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\
&\quad + (1+\alpha^2)\frac{J_1}{\eta} \iint_{]0,T[\times(B\times]-\eta,\eta[)} (\mathbf{m}^*_{n,k,\eta} - \mathbf{m}_{n,k,\eta})\cdot \boldsymbol{\phi}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\
&\quad + 2(1+\alpha^2)\frac{J_2}{\eta} \iint_{]0,T[\times(B\times]-\eta,\eta[)} \Big((\mathbf{m}_{n,k,\eta}\cdot \mathbf{m}^*_{n,k,\eta})\mathbf{m}^*_{n,k,\eta} \\
&\qquad - \|\mathbf{m}^*_{n,k,\eta}\|^2 \mathbf{m}_{n,k,\eta}\Big)\cdot \boldsymbol{\phi}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t,
\end{aligned}
\tag{6.9a}
$$

for all $\boldsymbol{\phi}$ in $\mathcal{C}^\infty([0,T^*],V_n^3)$. And

$$
\begin{aligned}
&\mu_0 \iint_{]0,T[\times\mathbb{R}^3} \Big(\frac{\partial \mathbf{h}_{n,k,\eta}}{\partial t} + \frac{\partial \mathbf{m}_{n,k,\eta}}{\partial t}\Big)\cdot \boldsymbol{\psi}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\
&+ \iint_{]0,T[\times\mathbb{R}^3} \operatorname{curl} \mathbf{e}_{n,k,\eta}\cdot \boldsymbol{\psi}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t = 0,
\end{aligned}
\tag{6.9b}
$$

for all $\boldsymbol{\psi}$ in $\mathcal{C}^\infty([0,T^*],W_n)$. And

$$
\begin{aligned}
&\varepsilon_0 \iint_{]0,T[\times\mathbb{R}^3} \frac{\partial \mathbf{e}_{n,k,\eta}}{\partial t}\cdot \boldsymbol{\Theta}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t - \iint_{]0,T[\times\mathbb{R}^3} \operatorname{curl} \mathbf{h}_{n,k,\eta}\cdot \boldsymbol{\Theta}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\
&+ \sigma \iint_{]0,T[\times\Omega} (\mathbf{e}_{n,k,\eta} + \mathbf{f})\cdot \boldsymbol{\Theta}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t = 0,
\end{aligned}
\tag{6.9c}
$$

for all $\boldsymbol{\Theta}$ in $\mathcal{C}_c^\infty([0,T^*],W_n)$.

By density, (6.9) also holds if $\boldsymbol{\phi}$ belongs to the space $\mathrm{L}^2(]0,T^*[;V_n^3)$, $\boldsymbol{\psi}$ belongs to $\mathrm{L}^2(]0,T^*[,W_n)$, and $\boldsymbol{\Theta}$ belongs to $\mathrm{L}^2(]0,T^*[,W_n)$. As in [6], set $\boldsymbol{\phi} = \frac{\partial \mathbf{m}_{n,k,\eta}}{\partial t}$ in (6.9a), we obtain

$$
\begin{aligned}
&\frac{A}{2}\int_\Omega \|\nabla \mathbf{m}_{n,k,\eta}(T,\mathbf{x})\|^2\,\mathrm{d}\mathbf{x} + \frac{1}{2}\int_\Omega (\mathbf{K}(\mathbf{x})\mathbf{m}_{n,k,\eta}(T,\mathbf{x}))\cdot \mathbf{m}(T,\mathbf{x})\,\mathrm{d}\mathbf{x} \\
&+ \frac{k}{4}\int_\Omega (\|\mathbf{m}_{n,k,\eta}(T,\mathbf{x}))\|^2-1)^2\,\mathrm{d}\mathbf{x} - \iint_{]0,T[\times\Omega} \mathbf{h}_{n,k,\eta}\cdot \frac{\partial \mathbf{m}_{n,k,\eta}}{\partial t}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\
&+ \mathrm{E}_s^\eta(\mathbf{m}_{n,k,\eta}(T,\cdot)) + \frac{\alpha}{1+\alpha^2}\iint_{]0,T[\times\Omega} \|\frac{\partial \mathbf{m}_{n,k,\eta}}{\partial t}\|^2\,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\
&\le \frac{A}{2}\int_\Omega \|\nabla \mathcal{P}_{V_n}(\mathbf{m}_0)\|^2\,\mathrm{d}\mathbf{x} + \frac{1}{2}\int_\Omega (\mathbf{K}(\mathbf{x})\mathcal{P}_{V_n}(\mathbf{m}_0))\cdot \mathcal{P}_{V_n}(\mathbf{m}_0)\,\mathrm{d}\mathbf{x}
\end{aligned}
$$

$$+\frac{k}{4}\int_\Omega (\|\mathcal{P}_{V_n}(\mathbf{m}_0))\|^2-1)^2\,\mathrm{d}\mathbf{x}+\mathrm{E}_s^\eta(\mathcal{P}_{V_n}(\mathbf{m}_0)).$$

Set $\boldsymbol{\psi}=\mathbf{h}_{n,k,\eta}$ in (6.9b), we obtain

$$\begin{aligned}
&\frac{\mu_0}{2}\int_{\mathbb{R}^3}\|\mathbf{h}_{n,k,\eta}(T,\mathbf{x})\|^2\,\mathrm{d}\mathbf{x}\,\mathrm{d}t+\mu_0\iint_{]0,T[\times\Omega}\frac{\partial \mathbf{m}_{n,k,\eta}}{\partial t}\cdot\mathbf{h}_{n,k,\eta}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t\\
&+\iint_{]0,T[\times\mathbb{R}^3}\mathbf{h}_{n,k,\eta}\cdot\operatorname{curl}\mathbf{e}_{n,k,\eta}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t\\
&\le\frac{\mu_0}{2}\int_{\mathbb{R}^3}\|\mathcal{P}_{W_n}(\mathbf{h}_0)\|^2\,\mathrm{d}\mathbf{x},
\end{aligned}$$

Set $\boldsymbol{\Theta}=\mathbf{e}_{n,k,\eta}$ in (6.9c), we obtain

$$\begin{aligned}
&\frac{\varepsilon_0}{2}\iint_{\mathbb{R}^3}\|\mathbf{e}_{n,k,\eta}(T,\cdot)\|^2-\iint_{]0,T[\times\mathbb{R}^3}\mathbf{e}_{n,k,\eta}\cdot\operatorname{curl}\mathbf{h}_{n,k,\eta}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t\\
&+\sigma\iint_{]0,T[\times\mathbb{R}^3}\|\mathbf{e}_{n,k,\eta}\|^2\,\mathrm{d}\mathbf{x}\,\mathrm{d}t+\sigma\iint_{]0,T[\times\mathbb{R}^3}\mathbf{f}\cdot\mathbf{e}_{n,k,\eta}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t\\
&\le\frac{\varepsilon_0}{2}\iint_{\mathbb{R}^3}\|\mathcal{P}_{W_N}(\mathbf{e}_0)\|^2\,\mathrm{d}\mathbf{x}.
\end{aligned}$$

Combining these three inequalities, we get an energy inequality

$$\begin{aligned}
&\mathrm{E}_{n,k,\eta}(T)+\frac{\alpha}{1+\alpha^2}\iint_{]0,T[\times\Omega}\|\frac{\partial \mathbf{m}_{n,k,\eta}}{\partial t}\|^2\,\mathrm{d}\mathbf{x}\,\mathrm{d}t+\frac{\sigma}{\mu_0}\iint_{]0,T[\times\mathbb{R}^3}\|\mathbf{e}_{n,k,\eta}\|^2\,\mathrm{d}\mathbf{x}\,\mathrm{d}t\\
&+\frac{\sigma}{\mu_0}\iint_{]0,T[\times\mathbb{R}^3}\mathbf{f}\cdot\mathbf{e}_{n,k,\eta}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t\\
&\le\frac{A}{2}\int_\Omega\|\nabla\mathcal{P}_{V_n}(\mathbf{m}_0)\|^2\,\mathrm{d}\mathbf{x}+\frac12\int_\Omega(\mathbf{K}(\mathbf{x})\mathcal{P}_{V_n}(\mathbf{m}_0))\cdot\mathcal{P}_{V_n}(\mathbf{m}_0)\,\mathrm{d}\mathbf{x}\\
&\quad+\frac{k}{4}\int_\Omega(\|\mathcal{P}_{V_n}(\mathbf{m}_0)\|^2-1)^2\,\mathrm{d}\mathbf{x}+\mathrm{E}_s^\eta(\mathcal{P}_{V_n}(\mathbf{m}_0))\\
&\quad+\frac{\varepsilon_0}{2\mu_0}\int_{\mathbb{R}^3}\|\mathcal{P}_{W_N}(\mathbf{e}_0)\|^2\,\mathrm{d}\mathbf{x}+\frac12\int_{\mathbb{R}^3}\|\mathcal{P}_{W_N}(\mathbf{h}_0)\|^2\,\mathrm{d}\mathbf{x}
\end{aligned}\tag{6.10}$$

with

$$\begin{aligned}
\mathrm{E}_{n,k,\eta}(T)&=\frac{A}{2}\int_\Omega\|\nabla\mathbf{m}_{n,k,\eta}(T,\cdot)\|^2\,\mathrm{d}\mathbf{x}+\frac12\int_\Omega(\mathbf{K}(\mathbf{x})\mathbf{m}_{n,k,\eta}(T,\mathbf{x}))\cdot\mathbf{m}_{n,k,\eta}(T,\mathbf{x})\,\mathrm{d}\mathbf{x}\\
&\quad+\frac{k}{4}\int_\Omega(\|\mathbf{m}_{n,k,\eta}(T,\mathbf{x}))\|^2-1)^2\,\mathrm{d}\mathbf{x}\\
&\quad+\frac{\varepsilon_0}{2\mu_0}\int_{\mathbb{R}^3}\|\mathbf{e}_{n,k,\eta}(T,\mathbf{x})\|^2\,\mathrm{d}\mathbf{x}+\frac12\int_{\mathbb{R}^3}\|\mathbf{h}_{n,k,\eta}(T,\mathbf{x})\|^2\,\mathrm{d}\mathbf{x}\\
&\quad+\mathrm{E}_s^\eta(\mathbf{m}_{n,k,\eta}(T,\cdot))
\end{aligned}$$

The projection $\mathcal{P}_n(\mathbf{m}_0)$ converges to $\mathbf{m}_0$ in $\mathbb{H}^1(\Omega)$ and in $\mathbb{L}^6(\Omega)$ by Sobolev imbedding. The terms on the right hand-side remain bounded independently of n. The last term on the left hand-side may be dealt with by Young inequality. Thus, $\mathbf{m}_{n,k,\eta}$, $\mathbf{h}_{n,k,\eta}$ and $\mathbf{e}_{n,k,\eta}$ cannot explode in finite time and exist globally.

6.3. **Final step of Galerkin's method.** We now have n tend to $+\infty$ By (6.10) and using Young inequality to deal with the term containing $\mathbf{f}$:

- $\mathbf{m}_{n,k,\eta}$ is bounded in $\mathrm{L}^\infty(\mathbb{R}^+;\mathbb{L}^4(\Omega))$ independently of n.
- $\nabla\mathbf{m}_{n,k,\eta}$ is bounded in $\mathrm{L}^\infty(\mathbb{R}^+;\mathbb{L}^2(\Omega))$ independently of n.
- $\frac{\partial \mathbf{m}_{n,k,\eta}}{\partial t}$ is bounded in $\mathrm{L}^2(\mathbb{R}^+;\mathbb{L}^2(\Omega))$ independently of n.
- $\mathbf{h}_{n,k,\eta}$ is bounded in $\mathrm{L}^\infty(\mathbb{R}^+;\mathbb{L}^2(\Omega))$ independently of n.
- $\mathbf{e}_{n,k,\eta}$ is bounded in $\mathrm{L}^\infty(\mathbb{R}^+;\mathbb{L}^2(\Omega))$ independently of n.

Thus, there exist $\mathbf{m}_{k,\eta}$ in the space $\mathrm{H}^1_{loc}([0,+\infty[;\mathbb{L}^2(\Omega))\cap\mathrm{L}^\infty(]0,+\infty[;\mathbb{H}^1(\Omega))$, $\mathbf{h}_{k,\eta}$ in the space $\mathrm{L}^\infty(\mathbb{R}^+;\mathbb{L}^2(\Omega))$, $\mathbf{e}_{k,\eta}$ in the space $\mathrm{L}^\infty(\mathbb{R}^+;\mathbb{L}^2(\Omega))$, such that up to a subsequence:

- $\mathbf{m}_{n,k,\eta}$ converges weakly to $\mathbf{m}_{k,\eta}$ in $\mathbb{H}^1(]0,T[\times\Omega)$.
- $\mathbf{m}_{n,k,\eta}$ converges strongly to $\mathbf{m}_{k,\eta}$ in $\mathbb{L}^2(]0,T[\times\Omega)$.
- $\mathbf{m}_{n,k,\eta}$ converges strongly to $\mathbf{m}_{k,\eta}$ in $\mathcal{C}([0,T];\mathbb{L}^2(\Omega))$ and in $\mathcal{C}([0,T];\mathbb{L}^p(\Omega))$ for all $1\le p<6$. See Lemma 5.1.
- $\nabla\mathbf{m}_{n,k,\eta}$ converges weakly to $\nabla\mathbf{m}_{k,\eta}$ in $\mathbb{L}^2(]0,T[\times\Omega)$.
- For all time T, $\nabla\mathbf{m}_{n,k,\eta}(T,\cdot)$ converges weakly to $\nabla\mathbf{m}_{k,\eta}(T,\cdot)$ in $\mathbb{L}^2(\Omega)$. The same subsequence can be used for all time $T\ge 0$, see Lemma 5.4.
- $\frac{\partial \mathbf{m}_{n,k,\eta}}{\partial t}$ converges weakly to $\frac{\partial \mathbf{m}_{k,\eta}}{\partial t}$ in $\mathbb{L}^2(\mathbb{R}^+\times\Omega)$.
- $\mathbf{h}_{n,k,\eta}$ converges star weakly to $\mathbf{h}_{k,\eta}$ in $\mathrm{L}^\infty(\mathbb{R}^+;\mathbb{L}^2(\Omega))$.
- $\mathbf{e}_{n,k,\eta}$ converges star weakly to $\mathbf{e}_{k,\eta}$ in $\mathrm{L}^\infty(\mathbb{R}^+;\mathbb{L}^2(\Omega))$.

Taking the limit in the energy inequality (6.10) as n tend to $+\infty$ is tricky: the terms involving the $\mathbb{L}^2(\Omega)$ norm of $\mathbf{e}_{n,k,\eta}(T,\cdot)$ and $\mathbf{h}_{n,k,\eta}(T,\cdot)$ are tricky. For all $T>0$, we can extract a subsequence of $\mathbf{e}_{n,k,\eta}(T,\cdot)$ that converges weakly to $\mathbf{e}^T_{k,\eta}$ in $\mathbb{L}^2(\Omega)$ as n tends to $+\infty$. The tricky part is that it is unproven that $\mathbf{e}^T_{k,\eta}$ is equal to $\mathbf{e}_{k,\eta}(T,\cdot)$. If we had strong convergence of $\mathbf{e}_{n,k,\eta}$ as a function defined on $\mathbb{R}^+\times\Omega$ or if we had the existence of a subsequence along which $\mathbf{e}_{n,k,\eta}(T,\cdot)$ converged weakly in $\mathbb{L}^2(\Omega)$ for almost all time T, then we could conclude directly. Unfortunately, while we have for all $T>0$, the existence of a subsequence of $\mathbf{e}_{n,k,\eta}(T,\cdot)$ that converges weakly in $\mathbb{L}^2(\Omega)$, the subsequence depends on T. We have the same problem for $\mathbf{h}_{n,k,\eta}$. There is no such problem with $\mathbf{m}(T,\cdot)$, see Lemma 5.4. To solve the problem, we first integrate (6.10) over $]T_1,T_2[$ where $0\le T_1<T_2<+\infty$ then we can take the limit as n tend to $+\infty$:

$$
\begin{aligned}
&\int_{T_1}^{T_2}\Big(\frac{A}{2}\int_\Omega\|\nabla\mathbf{m}_{k,\eta}(T,\cdot)\|^2\,\mathrm{d}\mathbf{x}+\frac12\int_\Omega(\mathbf{K}(\mathbf{x})\mathbf{m}_{k,\eta}(T,\mathbf{x}))\cdot\mathbf{m}_{k,\eta}(T,\mathbf{x})\,\mathrm{d}\mathbf{x}\\
&+\frac{k}{4}\int_\Omega(\|\mathbf{m}_{k,\eta}(T,\mathbf{x})\|^2-1)^2\,\mathrm{d}\mathbf{x}+\frac{\varepsilon_0}{2\mu_0}\int_{\mathbb{R}^3}\|\mathbf{e}_{k,\eta}(T,\mathbf{x})\|^2\,\mathrm{d}\mathbf{x}\\
&+\frac12\int_{\mathbb{R}^3}\|\mathbf{h}_{k,\eta}(T,\mathbf{x})\|^2\,\mathrm{d}\mathbf{x}+\mathrm{E}^\eta_s(\mathbf{m}_{k,\eta}(T,\cdot))+\frac{\alpha}{1+\alpha^2}\iint_{]0,T[\times\Omega}\|\frac{\partial\mathbf{m}_{k,\eta}}{\partial t}\|^2\,\mathrm{d}\mathbf{x}\,\mathrm{d}t\\
&+\frac{\sigma}{\mu_0}\iint_{]0,T[\times\mathbb{R}^3}\|\mathbf{e}_{k,\eta}\|^2\,\mathrm{d}\mathbf{x}\,\mathrm{d}t+\frac{\sigma}{\mu_0}\iint_{]0,T[\times\mathbb{R}^3}\mathbf{f}\cdot\mathbf{e}_{k,\eta}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t\Big)dT\\
&\le(T_2-T_1)\mathcal{E}^\eta_0,
\end{aligned}
$$

for all $0\le T_1<T_2<+\infty$, where

$$
\mathcal{E}^\eta_0=\frac{A}{2}\int_\Omega\|\nabla\mathbf{m}_0\|^2\,\mathrm{d}\mathbf{x}+\frac12\int_\Omega(\mathbf{K}(\mathbf{x})\mathbf{m}_0)\cdot\mathbf{m}_0\,\mathrm{d}\mathbf{x}+\mathrm{E}^\eta_s(\mathbf{m}_0)+\frac{\varepsilon_0}{2\mu_0}\int_{\mathbb{R}^3}\|\mathbf{e}_0\|^2\,\mathrm{d}\mathbf{x}
$$

$$+\frac{1}{2}\int_{\mathbb{R}^3}\|\mathbf{h}_0\|^2\,\mathrm{d}\mathbf{x}.$$

Since the equality holds for all T_1 and T_2, we have that for almost all $T>0$,

$$\begin{aligned}
&\frac{A}{2}\int_\Omega\|\nabla\mathbf{m}_{k,\eta}(T,\mathbf{x})\|^2\,\mathrm{d}\mathbf{x}+\frac{1}{2}\int_\Omega(\mathbf{K}(\mathbf{x})\mathbf{m}_{k,\eta}(T,\mathbf{x}))\cdot\mathbf{m}_{k,\eta}(T,\mathbf{x})\,\mathrm{d}\mathbf{x}\\
&+\frac{k}{4}\int_\Omega(\|\mathbf{m}_{k,\eta}(T,\mathbf{x})\|^2-1)^2\,\mathrm{d}\mathbf{x}+\frac{\varepsilon_0}{2\mu_0}\int_{\mathbb{R}^3}\|\mathbf{e}_{k,\eta}(T,\mathbf{x})\|^2\,\mathrm{d}\mathbf{x}\\
&+\frac{1}{2}\int_{\mathbb{R}^3}\|\mathbf{h}_{k,\eta}(T,\mathbf{x})\|^2\,\mathrm{d}\mathbf{x}+\mathrm{E}_s^\eta(\mathbf{m}_{k,\eta}(T,\cdot))+\frac{\alpha}{1+\alpha^2}\iint_{]0,T[\times\Omega}\|\frac{\partial\mathbf{m}_{k,\eta}}{\partial t}\|^2\,\mathrm{d}\mathbf{x}\,\mathrm{d}t\\
&+\frac{\sigma}{\mu_0}\iint_{]0,T[\times\mathbb{R}^3}\|\mathbf{e}_{k,\eta}\|^2\,\mathrm{d}\mathbf{x}\,\mathrm{d}t+\frac{\sigma}{\mu_0}\iint_{]0,T[\times\mathbb{R}^3}\mathbf{f}\cdot\mathbf{e}_{k,\eta}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t\le\mathcal{E}_0^\eta.
\end{aligned}\tag{6.11}$$

We take the limit in (6.9a) as n tends to $+\infty$:

$$\begin{aligned}
&\iint_{]0,T[\times\Omega}\alpha\frac{\partial\mathbf{m}_{k,\eta}}{\partial t}\cdot\boldsymbol{\phi}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t+\iint_{]0,T[\times\Omega}\Big(\mathbf{m}_{k,\eta}\wedge\frac{\partial\mathbf{m}_{k,\eta}}{\partial t}\Big)\cdot\boldsymbol{\phi}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t\\
&=-(1+\alpha^2)A\iint_{]0,T[\times\Omega}\sum_{i=1}^3\frac{\partial\mathbf{m}_{k,\eta}}{\partial x_i}\cdot\frac{\partial\boldsymbol{\phi}}{\partial x_i}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t\\
&\quad-(1+\alpha^2)\iint_{]0,T[\times\Omega}(\mathbf{K}(\mathbf{x})\mathbf{m}_{k,\eta}(t,\mathbf{x}))\cdot\boldsymbol{\phi}(t,\mathbf{x})\,\mathrm{d}\mathbf{x}\,\mathrm{d}t\\
&\quad+(1+\alpha^2)\iint_{]0,T[\times\Omega}\mathbf{h}_{k,\eta}\cdot\boldsymbol{\phi}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t\\
&\quad+(1+\alpha^2)\frac{K_s}{\eta}\iint_{]0,T[\times(B\times]-\eta,\eta[)}((\boldsymbol{\nu}\cdot\mathbf{m}_{k,\eta})\boldsymbol{\nu}-\mathbf{m}_{k,\eta})\cdot\boldsymbol{\phi}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t\\
&\quad+(1+\alpha^2)\frac{J_1}{\eta}\iint_{]0,T[\times(B\times]-\eta,\eta[)}(\mathbf{m}^*_{k,\eta}-\mathbf{m}_{k,\eta})\cdot\boldsymbol{\phi}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t\\
&\quad+2(1+\alpha^2)\frac{J_2}{\eta}\iint_{]0,T[\times(B\times]-\eta,\eta[)}\Big((\mathbf{m}_{k,\eta}\cdot\mathbf{m}^*_{k,\eta})\mathbf{m}^*_{n,k,\eta}\\
&\qquad-\|\mathbf{m}^*_{k,\eta}\|^2\mathbf{m}_{k,\eta}\Big)\cdot\boldsymbol{\phi}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t,
\end{aligned}\tag{6.12a}$$

for all $\boldsymbol{\phi}$ in $\bigcup_n\mathcal{C}^\infty([0,T[;V_n^3)$. By density, it also holds for all $\boldsymbol{\phi}$ in $\mathbb{H}^1(]0,T[\times\Omega)$.

We integrate (6.9b) by parts then take the limit as n tends to $+\infty$.

$$\begin{aligned}
&-\mu_0\iint_{\mathbb{R}^+\times\mathbb{R}^3}(\mathbf{h}_{k,\eta}+\mathbf{m}_{k,\eta}))\frac{\partial\boldsymbol{\psi}}{\partial t}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t+\iint_{\mathbb{R}^+\times\mathbb{R}^3}\mathbf{e}_{k,\eta}\cdot\operatorname{curl}\boldsymbol{\psi}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t\\
&=\mu_0\int_{\mathbb{R}^3}(\mathbf{h}_0+\mathbf{m}_0))\cdot\boldsymbol{\psi}(0,\cdot)\,\mathrm{d}\mathbf{x},
\end{aligned}\tag{6.12b}$$

for all $\boldsymbol{\psi}$ in $\bigcup_n\mathcal{C}_c^\infty([0,+\infty[;W_n)$. By density, it also holds for all $\boldsymbol{\psi}$ in $\mathrm{L}^1(\mathbb{R}^+;\mathbb{H}^1(\Omega))$ such that $\frac{\partial\boldsymbol{\psi}}{\partial t}$ belongs to $\mathrm{L}^1(\mathbb{R}^+;\mathbb{L}^2(\Omega))$.

We integrate (6.9c) by parts then take the limit as n tends to $+\infty$.

$$\begin{aligned}
&- \varepsilon_0 \iint_{\mathbb{R}^+\times\mathbb{R}^3} \mathbf{e}_{k,\eta} \cdot \frac{\partial \mathbf{\Theta}}{\partial t} \,\mathrm{d}\mathbf{x}\,\mathrm{d}t - \iint_{\mathbb{R}^+\times\mathbb{R}^3} \mathbf{h}_{k,\eta} \cdot \operatorname{curl} \mathbf{\Theta} \,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\
&+ \sigma \iint_{\mathbb{R}^+\times\Omega} (\mathbf{e}_{k,\eta} + \mathbf{f}) \cdot \mathbf{\Theta} \,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\
&= \varepsilon_0 \int_{\mathbb{R}^3} \mathbf{e}_0 \cdot \mathbf{\Theta}(0,\cdot) \,\mathrm{d}\mathbf{x},
\end{aligned} \tag{6.12c}$$

for all $\mathbf{\Theta}$ in $\bigcup_n \mathcal{C}_c^\infty([0,+\infty[;W_n)$. By density, it also holds for all $\mathbf{\Theta}$ in $\mathrm{L}^1(\mathbb{R}^+;\mathbb{H}^1(\Omega))$ such that $\frac{\partial \mathbf{\Theta}}{\partial t}$ belongs to $\mathrm{L}^1(\mathbb{R}^+;\mathbb{L}^2(\Omega))$.

6.4. **Limit as k tends to $+\infty$.** By (6.11) and using Young inequality to deal with the term containing $\mathbf{f}$:

- $\mathbf{m}_{k,\eta}$ is bounded in $\mathrm{L}^\infty(\mathbb{R}^+;\mathbb{L}^4(\Omega))$ independently of n.
- $\nabla\mathbf{m}_{k,\eta}$ is bounded in $\mathrm{L}^\infty(\mathbb{R}^+;\mathbb{L}^2(\Omega))$ independently of n.
- $\frac{\partial \mathbf{m}_{k,\eta}}{\partial t}$ is bounded in $\mathrm{L}^2(\mathbb{R}^+;\mathbb{L}^2(\Omega))$ independently of n.
- $\mathbf{h}_{k,\eta}$ is bounded in $\mathrm{L}^\infty(\mathbb{R}^+;\mathbb{L}^2(\Omega))$ independently of n.
- $\mathbf{e}_{k,\eta}$ is bounded in $\mathrm{L}^\infty(\mathbb{R}^+;\mathbb{L}^2(\Omega))$ independently of n.
- $k(\|\mathbf{m}_{k,\eta}\|^2 - 1)$ is bounded in $\mathrm{L}^\infty(\mathbb{R}^+;\mathbb{L}^2(\Omega))$ independently of n.

Thus, there exist $\mathbf{m}_\eta$, $\mathbf{h}_\eta$, $\mathbf{e}_\eta$, such that up to a subsequence:

- $\mathbf{m}_{k,\eta}$ converges weakly to $\mathbf{m}_\eta$ in $\mathbb{H}^1(]0,T[\times\Omega)$.
- $\mathbf{m}_{k,\eta}$ converges strongly to $\mathbf{m}_\eta$ in $\mathbb{L}^2(]0,T[\times\Omega)$.
- $\mathbf{m}_{k,\eta}$ converges strongly to $\mathbf{m}_\eta$ in $\mathcal{C}([0,T];\mathbb{L}^2(\Omega))$ and in $\mathcal{C}([0,T];\mathbb{L}^p(\Omega))$ for all $1 \le p < 6$. See Lemma 5.1.
- $\nabla\mathbf{m}_{k,\eta}$ converges weakly to $\nabla\mathbf{m}_\eta$ in $\mathbb{L}^2(]0,T[\times\Omega)$.
- For all time T, $\nabla\mathbf{m}_{k,\eta}(T,\cdot)$ converges weakly to $\nabla\mathbf{m}_\eta(T,\cdot)$ in $\mathbb{L}^2(\Omega)$.
- $\frac{\partial \mathbf{m}_{k,\eta}}{\partial t}$ converges weakly to $\frac{\partial \mathbf{m}_\eta}{\partial t}$ in $\mathbb{L}^2(\mathbb{R}^+\times\Omega)$.
- $\mathbf{h}_{k,\eta}$ converges star weakly to $\mathbf{h}_\eta$ in $\mathrm{L}^\infty(\mathbb{R}^+;\mathbb{L}^2(\Omega))$.
- $\mathbf{e}_{k,\eta}$ converges star weakly to $\mathbf{e}_\eta$ in $\mathrm{L}^\infty(\mathbb{R}^+;\mathbb{L}^2(\Omega))$.

Since $\|\mathbf{m}_{k,\eta}\|^2 - 1$ converges to 0, $\|\mathbf{m}_\eta\| = 1$ almost everywhere on $\mathbb{R}^+\times\Omega$.

For the reasons explained in §6.3, we integrate (6.11) over $[T_1,T_2]$, drop the term $k\|\|\mathbf{m}_\eta\|^2 - 1\|^2_{\mathrm{L}^2(\Omega)}/4$, and compute the limit as k tends to $+\infty$. After the limit is taken, we drop the integral over $[T_1,T_2]$ and obtain that for almost all $T > 0$:

$$\begin{aligned}
&\frac{A}{2}\int_\Omega \|\nabla\mathbf{m}_\eta(T,\cdot)\|^2 \,\mathrm{d}\mathbf{x} + \frac{1}{2}\int_\Omega (\mathbf{K}(\mathbf{x})\mathbf{m}_\eta(T,\mathbf{x}))\cdot\mathbf{m}_\eta(T,\mathbf{x}) \,\mathrm{d}\mathbf{x} \\
&+ \frac{\varepsilon_0}{2\mu_0}\int_{\mathbb{R}^3} \|\mathbf{e}_\eta(T,\mathbf{x})\|^2 \,\mathrm{d}\mathbf{x} + \frac{1}{2}\int_{\mathbb{R}^3} \|\mathbf{h}_\eta(T,\mathbf{x})\|^2 \,\mathrm{d}\mathbf{x} \\
&+ \mathrm{E}_s^\eta(\mathbf{m}_\eta(T,\cdot)) + \frac{\alpha}{1+\alpha^2}\iint_{]0,T[\times\Omega} \|\frac{\partial \mathbf{m}_\eta}{\partial t}\|^2 \,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\
&+ \frac{\sigma}{\mu_0}\iint_{]0,T[\times\mathbb{R}^3} \|\mathbf{e}_\eta\|^2 \,\mathrm{d}\mathbf{x}\,\mathrm{d}t + \frac{\sigma}{\mu_0}\iint_{]0,T[\times\mathbb{R}^3} \mathbf{f}\cdot\mathbf{e}_\eta \,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\
&\le \frac{A}{2}\int_\Omega \|\nabla\mathbf{m}_0\|^2 \,\mathrm{d}\mathbf{x} + \frac{1}{2}\int_\Omega (\mathbf{K}(\mathbf{x})\mathbf{m}_0)\cdot\mathbf{m}_0 \,\mathrm{d}\mathbf{x} \\
&\quad + \mathrm{E}_s^\eta(\mathbf{m}_0) + \frac{\varepsilon_0}{2\mu_0}\int_{\mathbb{R}^3} \|\mathbf{e}_0\|^2 \,\mathrm{d}\mathbf{x} + \frac{1}{2}\int_{\mathbb{R}^3} \|\mathbf{h}_0\|^2 \,\mathrm{d}\mathbf{x}.
\end{aligned} \tag{6.13}$$

We replace $\boldsymbol{\phi}$ in (6.12a) with $\mathbf{m}_{k,\eta}\wedge\boldsymbol{\varphi}$ where $\boldsymbol{\varphi}$ is $\mathcal{C}_c^\infty(\mathbb{R}^+\times\Omega;\mathbb{R}^3)$:

$$\begin{aligned}
&-\alpha\iint_{]0,T[\times\Omega}\Big(\mathbf{m}_{k,\eta}\wedge\frac{\partial\mathbf{m}_{k,\eta}}{\partial t}\Big)\cdot\boldsymbol{\varphi}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t+\iint_{]0,T[\times\Omega}\|\mathbf{m}_{k,\eta}\|^2\frac{\partial\mathbf{m}_{k,\eta}}{\partial t}\cdot\boldsymbol{\varphi}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t\\
&=\iint_{]0,T[\times\Omega}\Big(\mathbf{m}_{k,\eta}\cdot\frac{\partial\mathbf{m}_{k,\eta}}{\partial t}\Big)(\mathbf{m}_{k,\eta}\cdot\boldsymbol{\varphi})\,\mathrm{d}\mathbf{x}\,\mathrm{d}t\\
&\quad+(1+\alpha^2)A\iint_{]0,T[\times\Omega}\sum_{i=1}^3\Big(\mathbf{m}_{k,\eta}\wedge\frac{\partial\mathbf{m}_{k,\eta}}{\partial x_i}\Big)\cdot\frac{\partial\boldsymbol{\varphi}}{\partial x_i}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t\\
&\quad+(1+\alpha^2)\iint_{]0,T[\times\Omega}(\mathbf{m}_{k,\eta}(t,\mathbf{x})\wedge\mathbf{K}(\mathbf{x})\mathbf{m}_{k,\eta}(t,\mathbf{x}))\cdot\boldsymbol{\varphi}(t,\mathbf{x})\,\mathrm{d}\mathbf{x}\,\mathrm{d}t\\
&\quad-(1+\alpha^2)\iint_{]0,T[\times\Omega}(\mathbf{m}_{k,\eta}\wedge\mathbf{h}_{k,\eta})\cdot\boldsymbol{\varphi}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t\\
&\quad-(1+\alpha^2)\frac{K_s}{\eta}\iint_{]0,T[\times(B\times]-\eta,\eta[)}(\boldsymbol{\nu}\cdot\mathbf{m}_{k,\eta})(\mathbf{m}_{k,\eta}\wedge\boldsymbol{\nu})\cdot\boldsymbol{\varphi}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t\\
&\quad-(1+\alpha^2)\frac{J_1}{\eta}\iint_{]0,T[\times(B\times]-\eta,\eta[)}(\mathbf{m}_{k,\eta}\wedge\mathbf{m}^*_{k,\eta})\cdot\boldsymbol{\varphi}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t\\
&\quad-2(1+\alpha^2)\frac{J_2}{\eta}\iint_{]0,T[\times(B\times]-\eta,\eta[)}(\mathbf{m}_{k,\eta}\cdot\mathbf{m}^*_{k,\eta})(\mathbf{m}_{k,\eta}\wedge\mathbf{m}^*_{k,\eta})\cdot\boldsymbol{\varphi}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t,
\end{aligned}$$

We then take the limit as k tends to $+\infty$:

$$\begin{aligned}
&-\alpha\iint_{]0,T[\times\Omega}\Big(\mathbf{m}_{\eta}\wedge\frac{\partial\mathbf{m}_{\eta}}{\partial t}\Big)\cdot\boldsymbol{\varphi}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t+\iint_{]0,T[\times\Omega}\frac{\partial\mathbf{m}_{\eta}}{\partial t}\cdot\boldsymbol{\varphi}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t\\
&=+(1+\alpha^2)A\iint_{]0,T[\times\Omega}\sum_{i=1}^3\Big(\mathbf{m}_{\eta}\wedge\frac{\partial\mathbf{m}_{\eta}}{\partial x_i}\Big)\cdot\frac{\partial\boldsymbol{\varphi}}{\partial x_i}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t\\
&\quad+(1+\alpha^2)\iint_{]0,T[\times\Omega}(\mathbf{m}_{\eta}(t,\mathbf{x})\wedge\mathbf{K}(\mathbf{x})\mathbf{m}_{\eta}(t,\mathbf{x}))\cdot\boldsymbol{\varphi}(t,\mathbf{x})\,\mathrm{d}\mathbf{x}\,\mathrm{d}t\\
&\quad-(1+\alpha^2)\iint_{]0,T[\times\Omega}(\mathbf{m}_{\eta}\wedge\mathbf{h}_{\eta})\cdot\boldsymbol{\varphi}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t\\
&\quad-(1+\alpha^2)\frac{K_s}{\eta}\iint_{]0,T[\times(B\times]-\eta,\eta[)}(\boldsymbol{\nu}\cdot\mathbf{m}_{\eta})(\mathbf{m}_{\eta}\wedge\boldsymbol{\nu})\cdot\boldsymbol{\varphi}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t\\
&\quad-(1+\alpha^2)\frac{J_1}{\eta}\iint_{]0,T[\times(B\times]-\eta,\eta[)}(\mathbf{m}_{\eta}\wedge\mathbf{m}^*_{\eta})\cdot\boldsymbol{\varphi}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t\\
&\quad-2(1+\alpha^2)\frac{J_2}{\eta}\iint_{]0,T[\times(B\times]-\eta,\eta[)}(\mathbf{m}_{\eta}\cdot\mathbf{m}^*_{\eta})(\mathbf{m}_{\eta}\wedge\mathbf{m}^*_{\eta})\cdot\boldsymbol{\varphi}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t,
\end{aligned}\tag{6.14a}$$

We take the limit in (6.12b) as k tends to $+\infty$:

$$\begin{aligned}
&-\mu_0\iint_{\mathbb{R}^+\times\mathbb{R}^3}(\mathbf{h}_\eta+\mathbf{m}_\eta))\frac{\partial\boldsymbol{\psi}}{\partial t}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t+\iint_{\mathbb{R}^+\times\mathbb{R}^3}\mathbf{e}_\eta\operatorname{curl}\boldsymbol{\psi}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t\\
&=\mu_0\int_{\mathbb{R}^3}(\mathbf{h}_0+\mathbf{m}_0))\cdot\boldsymbol{\psi}(0,\cdot)\,\mathrm{d}\mathbf{x}
\end{aligned}\tag{6.14b}$$

for all $\boldsymbol{\psi}$ in $\mathrm{L}^1(\mathbb{R}^+;\mathbb{H}^1(\Omega))$ such that $\frac{\partial\boldsymbol{\psi}}{\partial t}$ belongs to $\mathrm{L}^1(\mathbb{R}^+;\mathbb{L}^2(\Omega))$.

We take the limit in (6.12c) as k tends to $+\infty$,

$$
\begin{aligned}
&-\varepsilon_0 \iint_{\mathbb{R}^+\times\mathbb{R}^3} \mathbf{e}_\eta \cdot \frac{\partial \boldsymbol{\Theta}}{\partial t}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t - \iint_{\mathbb{R}^+\times\mathbb{R}^3} \mathbf{h}_\eta \cdot \operatorname{curl}\boldsymbol{\Theta}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\
&+\sigma \iint_{\mathbb{R}^+\times\Omega} (\mathbf{e}_\eta + \mathbf{f}) \cdot \boldsymbol{\Theta}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\
&= \varepsilon_0 \int_{\mathbb{R}^3} \mathbf{e}_0 \cdot \boldsymbol{\Theta}(0,\cdot)\,\mathrm{d}\mathbf{x},
\end{aligned}
\tag{6.14c}
$$

for all $\boldsymbol{\Theta}$ in in $\mathrm{L}^1(\mathbb{R}^+;\mathbb{H}^1(\Omega))$ such that $\frac{\partial\boldsymbol{\Theta}}{\partial t}$ belongs to $\mathrm{L}^1(\mathbb{R}^+;\mathbb{L}^2(\Omega))$.

6.5. **Limit as η tends to 0.** Since $\mathbb{H}^1(\Omega)$ is continuously imbedded in $\mathcal{C}^0\big(]-L^-,L^+[\setminus\{0\};\mathbb{L}^4(B)\big)$, $\mathrm{E}_s^\eta(\mathbf{m}_0)$ remains bounded independently of η and converges to $\mathrm{E}_s(\mathbf{m}_0)$. Thus, using (6.13) and the constraint $\|\mathbf{m}_\eta\| = 1$ almost everywhere:

- $\mathbf{m}_\eta$ is bounded in $\mathbb{L}^\infty(\mathbb{R}^+\times\Omega)$ by 1.
- $\nabla\mathbf{m}_\eta$ is bounded in $\mathrm{L}^\infty(\mathbb{R}^+;\mathbb{L}^2(\Omega))$ independently of η.
- $\frac{\partial \mathbf{m}_{k,\eta}}{\partial t}$ is bounded in $\mathrm{L}^2(\mathbb{R}^+;\mathbb{L}^2(\Omega))$ independently of η.
- $\mathbf{h}_{k,\eta}$ is bounded in in $\mathrm{L}^\infty(\mathbb{R}^+;\mathbb{L}^2(\Omega))$ independently of η.
- $\mathbf{e}_{k,\eta}$ is bounded in in $\mathrm{L}^\infty(\mathbb{R}^+;\mathbb{L}^2(\Omega))$ independently of η.

Thus, there exists $\mathbf{m}$ in $\mathbb{L}^\infty(\mathbb{R}^+;\mathbb{H}^1(\Omega))$ and in $\mathrm{H}^1_{\mathrm{loc}}([0,+\infty[;\mathbb{L}^2(\Omega))$, $\mathbf{h}$ in $\mathrm{L}^\infty(\mathbb{R}^+;\mathbb{L}^2(\Omega))$ and $\mathbf{e}$ in $\mathbb{L}^\infty(\mathrm{R}^+;\mathbb{L}^2(\Omega))$ such that up to a subsequence

- $\mathbf{m}_\eta$ converges weakly to $\mathbf{m}$ in $\mathbb{H}^1(]0,T[\times\Omega)$.
- $\mathbf{m}_\eta$ converges strongly to $\mathbf{m}$ in $\mathbb{L}^2(]0,T[\times\Omega)$.
- $\mathbf{m}_\eta$ converges strongly to $\mathbf{m}$ in $\mathcal{C}([0,T];\mathbb{L}^2(\Omega))$ and thus in $\mathcal{C}([0,T];\mathbb{L}^p(\Omega))$ for all $1\le p<+\infty$.
- $\nabla\mathbf{m}_\eta$ converges weakly to $\nabla\mathbf{m}$ in $\mathbb{L}^2(]0,T[\times\Omega)$.
- For all time T, $\nabla\mathbf{m}_\eta(T,\cdot)$ converges weakly to $\nabla\mathbf{m}(T,\cdot)$ in $\mathbb{L}^2(\Omega)$.
- $\frac{\partial\mathbf{m}_\eta}{\partial t}$ converges weakly to $\frac{\partial\mathbf{m}}{\partial t}$ in $\mathrm{L}^2(\mathbb{R}^+\times\Omega)$.
- $\mathbf{h}_\eta$ converges star weakly to $\mathbf{h}$ in $\mathrm{L}^\infty(\mathbb{R}^+;\mathbb{L}^2(\Omega))$.
- $\mathbf{e}_\eta$ converges star weakly to $\mathbf{e}$ in $\mathrm{L}^\infty(\mathbb{R}^+;\mathbb{L}^2(\Omega))$.

As $\|\mathbf{m}^\eta\| = 1$ almost everywhere, $\|\mathbf{m}\| = 1$ almost everywhere. Moreover, as $\mathbf{m}_\eta(0,\cdot) = \mathbf{m}_0$, we have $\mathbf{m}(0,\cdot) = \mathbf{m}_0$.

For the reasons explained in §6.3, we integrate (6.13) over $[T_1,T_2]$, and compute the limit as η tends to 0. All the volume terms converge to their intuitive limit. Taking the limit in the surfacic terms requires more work. The space $\mathbb{H}^1(]0,T[\times\Omega)$ is compactly imbedded into

$$\mathcal{C}^0([-L^-,0];\mathbb{L}^2(]0,T[\times B)) \otimes \mathcal{C}^0([0,L^+];\mathbb{L}^2(]0,T[\times B)).$$

This is a direct application of Lemma 5.2 with $\mathcal{O} =]0,T[\times B$ and, thus a direct consequence of the extended Aubin's lemma 5.1. Therefore, $\mathbf{m}_\eta$ converges strongly to $\mathbf{m}$ in

$$\mathcal{C}^0([-L^-,0];\mathbb{L}^2(]0,T[\times B)) \otimes \mathcal{C}^0([0,L^+];\mathbb{L}^2(]0,T[\times B)).$$

Since $\|\mathbf{m}_\eta\| = 1$, the convergence is strong in

$$\mathcal{C}^0([-L^-,0];\mathbb{L}^p(]0,T[\times B)) \otimes \mathcal{C}^0([0,L^+];\mathbb{L}^p(]0,T[\times B)),$$

for all $p<+\infty$. Therefore,

$$\limsup_{\eta\to 0} \int_{T_1}^{T_2} \|\mathrm{E}_s^\eta(\mathbf{m}_\eta(t,\cdot)) - \mathrm{E}_s^\eta(\mathbf{m}(t,\cdot))\|\,\mathrm{d}t$$

$$
\begin{aligned}
&\le \limsup_{\eta\to 0} \frac{1}{2\eta}\int_{-\eta}^{\eta}\int_{T_1}^{T_2}\iint_B \|P(\mathbf{m}_\eta(t),\mathbf{m}_\eta^*(t)) - P(\mathbf{m}(t),\mathbf{m}^*(t))\|\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}t\\
&\le \limsup_{\eta\to 0}\sup_{z\in[-\eta,\eta]}\int_{T_1}^{T_2}\iint_B \|P(\mathbf{m}_\eta(t),\mathbf{m}_\eta^*(t)) - P(\mathbf{m}(t),\mathbf{m}^*(t))\|\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t\\
&\le 0,
\end{aligned}
$$

where P is some polynomial. Moreover, $\mathbf{m}(\cdot,\cdot)$ belongs to:

$$
\mathcal{C}^0\big([-L^-,0];\mathbb{L}^p(]0,T[\times B)\big)\otimes\mathcal{C}^0\big([0,L^+];\mathbb{L}^p(]0,T[\times B)\big).
$$

Therefore,

$$
\begin{aligned}
&\limsup_{\eta\to 0}\int_{T_1}^{T_2}\|\mathrm{E}_s^\eta(\mathbf{m}(t,\cdot)) - \mathrm{E}_s(\mathbf{m}(t,\cdot))\|\,\mathrm{d}t\\
&\le \limsup_{\eta\to 0}\frac{1}{2\eta}\int_{T_1}^{T_2}\int_{-\eta}^{\eta}\iint_B \|P(\mathbf{m}(t),\mathbf{m}^*(t))\\
&\quad - P(\mathbf{m}(x,y,0^+,t),\mathbf{m}(x,y,0^-,t))\|\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}t\\
&\le \limsup_{\eta\to 0}\sup_{z\in[-\eta,\eta]}\int_{T_1}^{T_2}\iint_B \|P(\mathbf{m}(t),\mathbf{m}^*(t))\\
&\quad - P(\mathbf{m}(x,y,0^+,t),\mathbf{m}(x,y,0^-,t))\|\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \le 0.
\end{aligned}
$$

Hence, the integral over $[T_1,T_2]$ of inequality (4.3d) holds for all $0<T_1<T_2$, therefore inequality (4.3d) is satisfied for almost all $T>0$.

We take the limit in (6.14a) as η tends to 0. All the volume terms converges to their intuitive limit. Moreover, because of the strong convergence, along a subsequence, of $\mathbf{m}_\eta$ to $\mathbf{m}$ in

$$
\mathcal{C}^0([-L^-,0];\mathbb{L}^p(]0,T[\times B))\otimes\mathcal{C}^0([0,L^+];\mathbb{L}^p(]0,T[\times B)),
$$

for all $p<+\infty$, we have

$$
\begin{aligned}
&\limsup_{\eta\to 0}\frac{1}{\eta}\Big|\iint_{]0,T[\times(B\times]-\eta,\eta[)}(\boldsymbol{\nu}\cdot\mathbf{m}_\eta)(\mathbf{m}_\eta\wedge\boldsymbol{\nu})\cdot\boldsymbol{\varphi}(t,\mathbf{x})\,\mathrm{d}\mathbf{x}\,\mathrm{d}t\\
&-\iint_{]0,T[\times(B\times]-\eta,\eta[)}(\boldsymbol{\nu}\cdot\mathbf{m})(\mathbf{m}\wedge\boldsymbol{\nu})\cdot\boldsymbol{\varphi}(t,\mathbf{x})\,\mathrm{d}\mathbf{x}\,\mathrm{d}t\Big| = 0,\\
&\limsup_{\eta\to 0}\frac{1}{\eta}\Big|\iint_{]0,T[\times(B\times]-\eta,\eta[)}(\mathbf{m}_\eta\wedge\mathbf{m}_\eta^*)\cdot\boldsymbol{\varphi}(t,\mathbf{x})\,\mathrm{d}\mathbf{x}\,\mathrm{d}t\\
&-\iint_{]0,T[\times(B\times]-\eta,\eta[)}(\mathbf{m}\wedge\mathbf{m}^*)\cdot\boldsymbol{\varphi}(t,\mathbf{x})\,\mathrm{d}\mathbf{x}\,\mathrm{d}t\Big| = 0,\\
&\limsup_{\eta\to 0}\frac{1}{\eta}\Big|\iint_{]0,T[\times(B\times]-\eta,\eta[)}(\mathbf{m}_\eta\cdot\mathbf{m}_\eta^*)(\mathbf{m}_\eta\wedge\mathbf{m}_{k,\eta}^*)\cdot\boldsymbol{\varphi}(t,\mathbf{x})\,\mathrm{d}\mathbf{x}\,\mathrm{d}t\\
&-\iint_{]0,T[\times(B\times]-\eta,\eta[)}(\mathbf{m}\cdot\mathbf{m}^*)(\mathbf{m}\wedge\mathbf{m}^*)\cdot\boldsymbol{\varphi}(t,\mathbf{x})\,\mathrm{d}\mathbf{x}\,\mathrm{d}t\Big| = 0.
\end{aligned}
$$

Since $\mathbf{m}$ belongs to

$$
\mathcal{C}^0([-L^-,0];\mathbb{L}^p(]0,T[\times B))\otimes\mathcal{C}^0([0,L^+];\mathbb{L}^p(]0,T[\times B)),
$$

each surface term also converges to its surface intuitive limit. Therefore, the weak formulation (4.3a) is also satisfied.

We take the limits as η tends to 0 in (6.14b) and (6.14b). All the volume terms converges to their intuitive limit. Hence, relations (4.3b) and (4.3c) are satisfied. This finishes our proof of Theorem 4.2.

7. Characterization of the ω-limit set

We consider $(\mathbf{m}, \mathbf{h}, \mathbf{e})$ a weak solution to the Landau-Lifshitz-Maxwell system given by Theorem 4.2.

We consider $\mathbf{u} \in \omega(m)$. There exists a non decreasing sequence $(t_n)_n$ such that $t_n \to +\infty$, and $\mathbf{m}(t_n, \cdot) \rightharpoonup \mathbf{u}$ in $\mathbb{H}^1(\Omega)$ weak. Since Ω is a smooth bounded domain, then $\mathbf{m}(t_n, \cdot)$ tends to $\mathbf{u}$ in $\mathbb{L}^p(\Omega)$ strongly for $p \in [1, 6[$, and extracting a subsequence, we assume that $\mathbf{m}(t_n, \cdot)$ tends to $\mathbf{u}$ almost everywhere, so that the saturation constraint $\|\mathbf{u}\| = 1$ is satisfied almost everywhere.

In addition, we remark that for all n, $\|\mathbf{m}(t_n, \cdot)\| = 1$ almost everywhere, so that $\|\mathbf{m}(t_n, \cdot)\|_{\mathbb{L}^\infty(\Omega)} = 1$. By interpolation inequalities in the L^p spaces, we obtain that for all $p < +\infty$, $\mathbf{m}(t_n, \cdot)$ tends to $\mathbf{u}$ in $\mathbb{L}^p(\Omega)$ strongly.

First Step. we fix a a non negative real number. for $s \in]-a, a[$ and $x \in \Omega$, for n large enough, we set

$$\mathbf{U}_n(s, \mathbf{x}) = \mathbf{m}(t_n + s, \mathbf{x}).$$

We have the following estimate:

$$\begin{aligned}
\frac{1}{2a} \int_{-a}^{a} \int_\Omega \|\mathbf{U}_n(s, \mathbf{x}) - \mathbf{m}(t_n, \mathbf{x})\|^2 \,\mathrm{d}\mathbf{x} \,\mathrm{d}s &= \frac{1}{2a} \int_{-a}^{a} \int_\Omega \| \int_0^s \frac{\partial \mathbf{m}}{\partial t}(t_n + \tau, \mathbf{x}) d\tau \|^2 \,\mathrm{d}\mathbf{x} \,\mathrm{d}s \\
&\leq \frac{1}{2a} \int_{-a}^{a} |s| \int_\Omega \int_{t_n - a}^{+\infty} \| \frac{\partial \mathbf{m}}{\partial t}(\tau, \mathbf{x}) \|^2 d\tau \,\mathrm{d}\mathbf{x} \,\mathrm{d}s \\
&\leq a \int_{t_n - a}^{+\infty} \int_\Omega \| \frac{\partial \mathbf{m}}{\partial t}(\tau, \mathbf{x}) \|^2 d\tau \,\mathrm{d}\mathbf{x}.
\end{aligned}$$

Since $\frac{\partial \mathbf{m}}{\partial t}$ is in $\mathbb{L}^2(\mathbb{R}^+ \times \Omega)$, we obtain that

$$\int_{-a}^{a} \int_\Omega \|\mathbf{U}_n(s, \mathbf{x}) - \mathbf{m}(t_n, \mathbf{x})\|^2 \,\mathrm{d}\mathbf{x} \,\mathrm{d}s \to 0 \quad \text{as } n \text{ tends to } +\infty.$$

Since $\mathbf{m}(t_n, \cdot)$ tends strongly to $\mathbf{u}$ in $\mathbb{L}^2(\Omega)$, then

$$\mathbf{U}_n \text{ tends strongly to } \mathbf{u} \text{ in } \mathrm{L}^2(]-a, a[; \mathbb{L}^2(\Omega)). \tag{7.1}$$

We remark now that the sequence $(\nabla \mathbf{U}_n)_n$ is bounded in $\mathrm{L}^\infty(]-a, a[; \mathbb{L}^2(\Omega))$. In addition, $(\frac{\partial \mathbf{U}_n}{\partial t})_n$ is bounded in $\mathrm{L}^2(]-a, a[; \mathbb{L}^2(\Omega))$. So, by applying Aubin's Lemma with $X = \mathbb{H}^1(\Omega)$, $B = \mathbb{H}^{\frac{3}{4}}(\Omega)$, $Y = \mathbb{L}^2(\Omega)$, $r = 2$ and $p = +\infty$, we obtain that $(\mathbf{U}_n)_n$ is compact in $\mathcal{C}^0([-a, a]; \mathbb{H}^{\frac{3}{4}}(\Omega))$, so that

$$\mathbf{U}_n \text{ tends strongly to } \mathbf{u} \text{ in } \mathcal{C}^0([-a, a]; \mathbb{H}^{\frac{3}{4}}(\Omega)). \tag{7.2}$$

By continuity of the trace operator, since $\mathbb{H}^{\frac{1}{4}}(\Gamma) \subset \mathbb{L}^2(\Gamma)$, we obtain that

$$\gamma(\mathbf{U}_n) \to \gamma(\mathbf{u}) \text{ strongly in } \mathcal{C}^0([-a, a]; \mathbb{L}^2(\Gamma)).$$

In addition, by classical properties of the trace operator, for all n, we have $\|\mathbf{U}_n\|_{\mathrm{L}^\infty(]-a,a[\times\Omega)} = 1$, so $\|\gamma(\mathbf{U}_n)\|_{\mathrm{L}^\infty([-a,a]\times\Gamma)} \leq 1$. We obtain then in particular that

$$\gamma(\mathbf{U}_n) \to \gamma(\mathbf{u}) \text{ strongly in } \mathbb{L}^p(]-a, a[\times\partial\Omega),\ p < +\infty.$$

Second step. We consider a smooth positive function ρ_a compactly supported in $[-a,a]$ such that

$$\begin{gathered}\rho_a(\tau)=1 \quad \text{for } \tau\in[-a+1,a-1],\\ 0\le \rho_a\le 1,\quad |\rho_a'|\le 2.\end{gathered}$$

For n large enough, we set

$$\mathbf{h}_a^n(\mathbf{x})=\frac{1}{2a}\int_{-a}^{a}\mathbf{h}(t_n+s,\mathbf{x})\rho_a(s)\,\mathrm{d}s,\quad \mathbf{e}_a^n(\mathbf{x})=\frac{1}{2a}\int_{-a}^{a}\mathbf{e}(t_n+s,\mathbf{x})\rho_a(s)\,\mathrm{d}s.$$

By construction of $(\mathbf{m},\mathbf{h},\mathbf{e})$, we know that $\mathbf{h}$ and $\mathbf{e}$ are in $\mathrm{L}^\infty(\mathbb{R}^+;\mathbb{L}^2(\mathbb{R}^3))$. We have the estimate

$$\begin{aligned}\|\mathbf{h}_a^n\|_{\mathbb{L}^2(\mathbb{R}^3)}^2&=\int_{\mathbb{R}^3}\|\frac{1}{2a}\int_{-a}^{a}\mathbf{h}(t_n+s,\mathbf{x})\rho_a(s)\,\mathrm{d}s\|^2\,\mathrm{d}\mathbf{x}\\ &\le\frac{1}{2a}\int_{-a}^{a}\rho_a^2(s)\,\mathrm{d}s\frac{1}{2a}\int_{\mathbb{R}^3}\int_{-a}^{a}\|\mathbf{h}(t_n+s,\mathbf{x})\|^2\,\mathrm{d}s\,\mathrm{d}\mathbf{x}\\ &\le\frac{2a}{2a}\|\mathbf{h}\|_{\mathrm{L}^\infty(\mathbb{R}^+;\mathbb{L}^2(\mathbb{R}^3))}.\end{aligned}$$

Therefore,

$$\forall a\ge 1,\ \forall n,\ \|\mathbf{h}_a^n\|_{\mathbb{L}^2(\mathbb{R}^3)}\le\|\mathbf{h}\|_{\mathrm{L}^\infty(\mathbb{R}^+;\mathbb{L}^2(\mathbb{R}^3))}. \tag{7.3}$$

In the same way, we prove that

$$\forall a\ge 1,\ \forall n,\ \|\mathbf{e}_a^n\|_{\mathbb{L}^2(\mathbb{R}^3)}\le\|\mathbf{e}\|_{\mathrm{L}^\infty(\mathbb{R}^+;\mathbb{L}^2(\mathbb{R}^3))}. \tag{7.4}$$

So for a fixed value of a we can assume by extracting a subsequence that $\mathbf{h}_a^n$ and $\mathbf{e}_a^n$ converge weakly in $\mathbb{L}^2(\mathbb{R}^3)$ when n tends to $+\infty$:

$$\mathbf{h}_a^n\rightharpoonup\mathbf{h}_a \text{ and } \mathbf{e}_a^n\rightharpoonup\mathbf{e}_a \text{ weakly in } \mathbb{L}^2(\mathbb{R}^3) \text{ when } n\to+\infty.$$

In the weak formulation (4.3a), we take $\boldsymbol{\phi}(t,\mathbf{x})=\frac{1}{2a}\rho_a(t-t_n)\boldsymbol{\psi}(\mathbf{x})$ where $\boldsymbol{\psi}\in\mathcal{D}(\overline{\Omega})$. We obtain after the change of variables $s=t-t_n$:

$$\frac{1}{2a}\int_{-a}^{a}\int_\Omega\Big(\frac{\partial\mathbf{U}_n}{\partial t}-\alpha\mathbf{U}_n\wedge\frac{\partial\mathbf{U}_n}{\partial t}\Big)\cdot\boldsymbol{\psi}(\mathbf{x})\rho_a(s)\,\mathrm{d}\mathbf{x}\,\mathrm{d}s=T_1+\ldots+T_6$$

with

$$\begin{gathered}T_1=(1+\alpha^2)A\frac{1}{2a}\int_{-a}^{a}\int_\Omega\sum_{i=1}^{3}\Big(\mathbf{U}_n(s,\mathbf{x})\wedge\frac{\partial\mathbf{U}_n}{\partial x_i}(t,\mathbf{x})\Big)\cdot\frac{\partial\boldsymbol{\psi}}{\partial x_i}(\mathbf{x})\,\mathrm{d}\mathbf{x}\,\mathrm{d}s,\\ T_2=(1+\alpha^2)\frac{1}{2a}\int_{-a}^{a}\int_\Omega(\mathbf{U}_n(s,\mathbf{x})\wedge\mathbf{K}(\mathbf{x})\mathbf{U}_n(s,\mathbf{x}))\cdot\boldsymbol{\psi}(\mathbf{x})\rho_a(s)\,\mathrm{d}\mathbf{x}\,\mathrm{d}s,\\ T_3=-(1+\alpha^2)\frac{1}{2a}\int_{-a}^{a}\int_\Omega(\mathbf{U}_n(s,\mathbf{x})\wedge\mathbf{h}(t_n+s,\mathbf{x}))\cdot\boldsymbol{\psi}(\mathbf{x})\rho_a(s)\,\mathrm{d}\mathbf{x}\,\mathrm{d}s,\\ T_4=-(1+\alpha^2)K_s\frac{1}{2a}\int_{-a}^{a}\int_{(\Gamma^\pm)}(\boldsymbol{\nu}\cdot\gamma\mathbf{U}_n)(\gamma\mathbf{U}_n\wedge\boldsymbol{\nu})\cdot\gamma\boldsymbol{\psi}(\hat{\mathbf{x}})\rho_a(s)\,\mathrm{d}S(\hat{\mathbf{x}})\,\mathrm{d}s,\\ T_5=-(1+\alpha^2)J_1\frac{1}{2a}\int_{-a}^{a}\int_{(\Gamma^\pm)}(\gamma\mathbf{U}_n\wedge\gamma^*\mathbf{U}_n)\cdot\gamma\boldsymbol{\psi}(\hat{\mathbf{x}})\rho_a(s)\,\mathrm{d}S(\hat{\mathbf{x}})\,\mathrm{d}s,\\ T_6=-2(1+\alpha^2)J_2\frac{1}{2a}\int_{-a}^{a}\int_{(\Gamma^\pm)}(\gamma\mathbf{U}_n\cdot\gamma^*\mathbf{U}_n)(\gamma\mathbf{U}_n\wedge\gamma^*\mathbf{U}_n)\cdot\gamma\boldsymbol{\psi}(\hat{\mathbf{x}})\rho_a(s)\,\mathrm{d}S(\hat{\mathbf{x}})\,\mathrm{d}s.\end{gathered}$$

Now for a fixed value of the parameter a, we take the limit of the previous equation when n tends to $+\infty$.

Left hand side term: we have the following estimates.

$$
\begin{aligned}
&\Big|\frac{1}{2a}\int_{-a}^{a}\int_{\Omega}\Big(\frac{\partial \mathbf{U}_n}{\partial t}-\alpha \mathbf{U}_n\wedge\frac{\partial \mathbf{U}_n}{\partial t}\Big)\cdot\boldsymbol{\psi}(\mathbf{x})\rho_a(s)\,\mathrm{d}\mathbf{x}\,\mathrm{d}s\Big|\\
&\le (1+\alpha)\frac{1}{2a}\int_{-a}^{a}\rho_a(s)\|\frac{\partial \mathbf{U}_n}{\partial t}(s,\cdot)\|_{\mathbb{L}^2(\Omega)}\|\boldsymbol{\psi}\|_{\mathbb{L}^2(\Omega)}\\
&\le \frac{1}{\sqrt{2a}}\|\boldsymbol{\psi}\|_{\mathbb{L}^2(\Omega)}(1+\alpha)\Big(\int_{-a}^{a}\int_{\Omega}\|\frac{\partial \mathbf{U}_n}{\partial t}\|^2\,\mathrm{d}\mathbf{x}\,\mathrm{d}s\Big)^{1/2}\\
&\le \frac{1}{\sqrt{2a}}\|\boldsymbol{\psi}\|_{\mathbb{L}^2(\Omega)}(1+\alpha)\Big(\int_{t_n-a}^{+\infty}\int_{\Omega}\|\frac{\partial \mathbf{m}}{\partial t}\|^2\,\mathrm{d}\mathbf{x}\,\mathrm{d}s\Big)^{1/2}
\end{aligned}
$$

Since $\frac{\partial \mathbf{m}}{\partial t}\in \mathrm{L}^2(\mathbb{R}^+;\mathbb{L}^2(\Omega))$, the last right hand side term tends to zero when n (and so t_n) tends to $+\infty$. Therefore

$$
\frac{1}{2a}\int_{-a}^{a}\int_{\Omega}\Big(\frac{\partial \mathbf{U}_n}{\partial t}-\alpha \mathbf{U}_n\wedge\frac{\partial \mathbf{U}_n}{\partial t}\Big)\cdot\boldsymbol{\psi}(\mathbf{x})\rho_a(s)\,\mathrm{d}\mathbf{x}\,\mathrm{d}s\to 0 \text{ when } n\to+\infty.
$$

Limit for T_1: since $\mathbf{U}_n\to\mathbf{u}$ strongly in $\mathbb{L}^2(]-a,a[\times\Omega)$, since $\frac{\partial \mathbf{U}_n}{\partial x_i}\rightharpoonup\frac{\partial \mathbf{u}}{\partial x_i}$ in $\mathbb{L}^2(]-a,a[\times\Omega)$ weak, we obtain that

$$
T_1\to(1+\alpha^2)A\frac{1}{2a}\int_{-a}^{a}\rho_a(s)\,\mathrm{d}s\int_{\Omega}\sum_{i=1}^{3}\Big(\mathbf{u}(\mathbf{x})\wedge\frac{\partial \mathbf{u}}{\partial x_i}(\mathbf{x})\Big)\cdot\frac{\partial \boldsymbol{\psi}}{\partial x_i}(\mathbf{x})\,\mathrm{d}\mathbf{x}.
$$

Limit for T_2: since $\mathbf{U}_n$ tends to $\mathbf{u}$ strongly in $\mathbb{L}^2(]-a,a[\times\Omega)$,

$$
T_2\to(1+\alpha^2)A\frac{1}{2a}\int_{-a}^{a}\rho_a(s)\,\mathrm{d}s\int_{\Omega}(\mathbf{u}(\mathbf{x})\wedge\mathbf{K}(\mathbf{x})\mathbf{u}(\mathbf{x}))\cdot\boldsymbol{\psi}(\mathbf{x})\,\mathrm{d}\mathbf{x}.
$$

Limit for T_3: we write

$$
\begin{aligned}
T_3&=-(1+\alpha^2)\int_{\Omega}(\mathbf{u}\wedge\mathbf{h}_a^n)\cdot\boldsymbol{\psi}\,\mathrm{d}\mathbf{x}\\
&\quad+(1+\alpha^2)\frac{1}{2a}\int_{-a}^{a}\int_{\Omega}((\mathbf{u}-\mathbf{U}_n)\wedge\mathbf{h}(t_n+s,\mathbf{x}))\cdot\boldsymbol{\psi}(\mathbf{x})\rho_a(s)\,\mathrm{d}\mathbf{x}\,\mathrm{d}s.
\end{aligned}
$$

We estimate the right hand side term as follows:

$$
\begin{aligned}
&\Big|\frac{1}{2a}\int_{-a}^{a}\int_{\Omega}((\mathbf{u}-\mathbf{U}_n)\wedge\mathbf{h}(t_n+s,x))\cdot\boldsymbol{\psi}(\mathbf{x})\rho_a(s)\,\mathrm{d}\mathbf{x}\,\mathrm{d}s\Big|\\
&\le\frac{1}{2a}\|\boldsymbol{\psi}\|_{\mathbb{L}^\infty(\Omega)}\|\mathbf{u}-\mathbf{U}_n\|_{\mathbb{L}^2(]-a,a[\times\Omega)}\|\mathbf{h}\|_{\mathbb{L}^2(]t_n-a,t_n+a[\times\Omega)}.
\end{aligned}
$$

So since $\mathbf{U}_n$ tends to $\mathbf{u}$ in $\mathbb{L}^2(]-a,a[\times\Omega)$, we obtain that

$$
T_3\to-(1+\alpha^2)\int_{\Omega}(\mathbf{u}\wedge\mathbf{h}_a)\cdot\boldsymbol{\psi}\,\mathrm{d}\mathbf{x}.
$$

Limit for T_4, T_5 and T_6: since $\gamma(\mathbf{U}_n)\to\gamma(\mathbf{u})$ strongly in $\mathbb{L}^p(]-a,a[\times\Gamma^\pm)$ for $p<+\infty$, the same occurs for $\gamma^*(\mathbf{U}_n)$ so that we obtain:

$$
T_4\to-(1+\alpha^2)K_s\frac{1}{2a}\int_{-a}^{a}\rho_a(s)\,\mathrm{d}s\int_{(\Gamma^\pm)}(\boldsymbol{\nu}\cdot\gamma\mathbf{u})(\gamma\mathbf{u}\wedge\boldsymbol{\nu})\cdot\gamma\boldsymbol{\psi}(\hat{\mathbf{x}})\,\mathrm{d}S(\hat{\mathbf{x}}),
$$

$$T_5 \to -(1+\alpha^2) J_1 \frac{1}{2a} \int_{-a}^{a} \rho_a(s)\,\mathrm{d}s \int_{(\Gamma^\pm)} (\gamma \mathbf{u} \wedge \gamma^* \mathbf{u}) \cdot \gamma \boldsymbol{\psi}(\hat{\mathbf{x}})\,\mathrm{d}S(\hat{\mathbf{x}}),$$
$$T_6 \to -2(1+\alpha^2) J_2 \frac{1}{2a} \int_{-a}^{a} \rho_a(s)\,\mathrm{d}s \int_{(\Gamma^\pm)} (\gamma \mathbf{u} \cdot \gamma^* \mathbf{u})(\gamma \mathbf{u} \wedge \gamma^* \mathbf{u}) \cdot \gamma \boldsymbol{\psi}(\hat{\mathbf{x}})\,\mathrm{d}S(\hat{\mathbf{x}}).$$

So we obtain that $\mathbf{u}$ satisfies for all $\boldsymbol{\psi} \in \mathcal{D}'(\overline{\Omega})$:

$$\begin{aligned}
&A \int_\Omega \sum_{i=1}^{3} \Big(\mathbf{u}(\mathbf{x}) \wedge \frac{\partial \mathbf{u}}{\partial x_i}(\mathbf{x})\Big) \cdot \frac{\partial \boldsymbol{\psi}}{\partial x_i}(\mathbf{x})\,\mathrm{d}\mathbf{x} + A \int_\Omega (\mathbf{u}(\mathbf{x}) \wedge \mathbf{K}(\mathbf{x})\mathbf{u}(\mathbf{x})) \cdot \boldsymbol{\psi}(\mathbf{x})\,\mathrm{d}\mathbf{x} \\
&- \frac{2a}{\int_{-a}^{a} \rho_a(s)\,\mathrm{d}s} (1+\alpha^2) \int_\Omega \mathbf{u} \wedge \mathbf{h}_a \boldsymbol{\psi}\,\mathrm{d}\mathbf{x} - K_s \int_{(\Gamma^\pm)} (\boldsymbol{\nu} \cdot \gamma \mathbf{u})(\gamma \mathbf{u} \wedge \boldsymbol{\nu}) \cdot \gamma \boldsymbol{\psi}(\hat{\mathbf{x}})\,\mathrm{d}S(\hat{\mathbf{x}}) \\
&- J_1 \int_{(\Gamma^\pm)} (\gamma \mathbf{u} \wedge \gamma^* \mathbf{u}) \cdot \gamma \boldsymbol{\psi}(\hat{\mathbf{x}})\,\mathrm{d}S(\hat{\mathbf{x}}) \\
&- 2J_2 \int_{(\Gamma^\pm)} (\gamma \mathbf{u} \cdot \gamma^* \mathbf{u})(\gamma \mathbf{u} \wedge \gamma^* \mathbf{u}) \cdot \gamma \boldsymbol{\psi}(\hat{\mathbf{x}})\,\mathrm{d}S(\hat{\mathbf{x}}) = 0.
\end{aligned}$$

We remark that by density, we can extend this equality for all $\boldsymbol{\psi} \in \mathbb{H}^1(\Omega)$. We take now the limit when a tends to $+\infty$. By definition of ρ_a we obtain that

$$\frac{2a}{\int_{-a}^{a} \rho_a(s)\,\mathrm{d}s} \to 1.$$

Concerning $\mathbf{h}_a$, by taking the weak limit in Estimate (7.3), we obtain that

$$\forall a \geq 1, \quad \|\mathbf{h}_a\|_{\mathbb{L}^2(\mathbb{R}^3)} \leq \|\mathbf{h}\|_{\mathrm{L}^\infty(\mathbb{R}^+;\mathbb{L}^2(\mathbb{R}^3))}. \tag{7.5}$$

So by extracting a subsequence, we can assume that

$$\mathbf{h}_a \rightharpoonup \mathbf{H} \text{ in } \mathbb{L}^2(\mathbb{R}^3) \text{ weak when } a \to +\infty.$$

In (4.3b), we take $\boldsymbol{\psi}(t, \mathbf{x}) = \theta_a(t - t_n)\nabla \xi(\mathbf{x})$ where $\xi \in \mathcal{D}(\mathbb{R}^3)$ and where

$$\theta_a(t) = \int_a^t \rho_a(s)\,\mathrm{d}s.$$

We obtain then that

$$\begin{aligned}
&- \mu_0 \int_{-a}^{a} \int_{\mathbb{R}^3} (\mathbf{h}(t_n + s, \mathbf{x}) + \overline{\mathbf{U}_n(s, \mathbf{x})}) \cdot \nabla \xi(\mathbf{x}) \rho_a(s)\,\mathrm{d}\mathbf{x}\,\mathrm{d}s \\
&= \mu_0 \int_{\mathbb{R}^3} (\mathbf{h}_0 + \overline{\mathbf{m}_0}) \cdot \nabla \xi(\mathbf{x}) \theta_a(0)\,\mathrm{d}\mathbf{x} = 0
\end{aligned}$$

since $\operatorname{div}(\mathbf{h}_0 + \overline{\mathbf{m}_0}) = 0$

So for all $\xi \in \mathcal{D}'(\mathbb{R}^3)$, for all $a \geq 1$ and all n great enough,

$$-\mu_0 \int_{\mathbb{R}^3} (\mathbf{h}_a^n(\mathbf{x}) + \frac{1}{2a} \int_{-a}^{a} \overline{\mathbf{U}_n(s, \mathbf{x})} \rho_a(s)\,\mathrm{d}s) \cdot \nabla \xi(\mathbf{x})\,\mathrm{d}\mathbf{x} = 0.$$

We take the limit of this equality when n tends to $+\infty$ for a fixed a:

$$-\mu_0 \int_{\mathbb{R}^3} (\mathbf{h}_a(\mathbf{x}) + \frac{1}{2a} \int_{-a}^{a} \rho_a(s)\,\mathrm{d}s \overline{\mathbf{u}(\mathbf{x})}) \cdot \nabla \xi(\mathbf{x})\,\mathrm{d}\mathbf{x} = 0,$$

and taking the limit when a tends to $+\infty$, we obtain

$$-\mu_0 \int_{\mathbb{R}^3} (\mathbf{H}(\mathbf{x}) + \overline{\mathbf{u}(\mathbf{x})}) \cdot \nabla \xi(\mathbf{x})\,\mathrm{d}\mathbf{x} = 0;$$

that is,

$$\operatorname{div}(\mathbf{H}+\overline{\mathbf{u}})=0 \quad \text{in } \mathcal{D}'(\mathbb{R}^3).$$

In (4.3c), we take $\mathbf{\Theta}(t,\mathbf{x})=\frac{1}{2a}\rho_a(t-t_n)\xi(\mathbf{x})$, where $\xi\in\mathcal{D}(\mathbb{R}^3;\mathbb{R}^3)$. We obtain:

$$\begin{aligned}
&-\varepsilon_0\frac{1}{2a}\int_{-a}^{a}\int_{\mathbb{R}^3}\mathbf{e}(t_n+s,\mathbf{x})\cdot\rho_a'(s)\xi(\mathbf{x})\,\mathrm{d}\mathbf{x}\,\mathrm{d}s-\int_{\mathbb{R}^3}\mathbf{h}_a^n\cdot\operatorname{curl}\xi\,\mathrm{d}\mathbf{x}\\
&+\sigma\int_\Omega\mathbf{e}_a^n\cdot\xi(\mathbf{x})\,\mathrm{d}\mathbf{x}+\sigma\int_\Omega\frac{1}{2a}\int_{-a}^{a}\mathbf{f}(t_n+s,\mathbf{x})\cdot\rho_a(s)\xi(\mathbf{x})\,\mathrm{d}\mathbf{x}\,\mathrm{d}s \qquad (7.6)\\
&=\varepsilon_0\int_{\mathbb{R}^3}\mathbf{e}_0\cdot\xi(\mathbf{x})\rho_a(-t_n)\,\mathrm{d}\mathbf{x}.
\end{aligned}$$

For n large enough, the right hand side term vanishes. We denote by γ_a^n the term

$$\gamma_a^n=-\varepsilon_0\frac{1}{2a}\int_{-a}^{a}\int_{\mathbb{R}^3}\mathbf{e}(t_n+s,\mathbf{x})\cdot\rho_a'(s)\xi(\mathbf{x})\,\mathrm{d}\mathbf{x}\,\mathrm{d}s.$$

We have

$$|\gamma_a^n|\le\frac{2\varepsilon_0}{a}\|\xi\|_{\mathbb{L}^2(\mathbb{R}^3)}\|\mathbf{e}\|_{\mathrm{L}^\infty(\mathbb{R}^+;\mathbb{L}^2(\mathbb{R}^3))}.$$

So for a fixed a, we can extract a subsequence till denoted γ_a^n which converges to a limit γ_a such that

$$|\gamma_a|\le\frac{2\varepsilon_0}{a}\|\xi\|_{\mathbb{L}^2(\mathbb{R}^3)}\|\mathbf{e}\|_{\mathrm{L}^\infty(\mathbb{R}^+;\mathbb{L}^2(\mathbb{R}^3))}.$$

Moreover,

$$\begin{aligned}
&\Big\|\frac{1}{2a}\int_{-a}^{a}\int_\Omega\mathbf{f}(t_n+s,\mathbf{x})\rho_a(s)\xi(\mathbf{x})\,\mathrm{d}\mathbf{x}\,\mathrm{d}s\Big\|\\
&\le\frac{1}{2a}\Big(\int_{t_n-a}^{t_n+a}\|\mathbf{f}(s,\cdot)\|^2_{\mathbb{L}^2(\Omega)}\,\mathrm{d}s\Big)^{1/2}\Big(\int_{-a}^{a}(\rho_a(s))^2\,\mathrm{d}s\Big)^{1/2}\|\xi\|_{\mathbb{L}^2(\Omega)}.
\end{aligned}$$

So

$$\Big\|\frac{1}{2a}\int_{-a}^{a}\int_\Omega\mathbf{f}(t_n+s,\mathbf{x})\rho_a(s)\xi(\mathbf{x})\,\mathrm{d}\mathbf{x}\,\mathrm{d}s\Big\|\le\frac{1}{\sqrt{2a}}\|\xi\|_{\mathbb{L}^2(\Omega)}\Big(\int_{t_n-a}^{+\infty}\|\mathbf{f}(s,\cdot)\|^2_{\mathbb{L}^2(\Omega)}\,\mathrm{d}s\Big)^{1/2}$$

thus for a fixed a, since $\mathbf{f}\in\mathbb{L}^2(\mathbb{R}^+\times\Omega)$, this term tends to zero as n tends to $+\infty$.

Therefore, taking the limit when n tends to $+\infty$ in (7.6) we obtain

$$\gamma_a-\int_{\mathbb{R}^3}\mathbf{h}_a\cdot\operatorname{curl}\xi\,\mathrm{d}\mathbf{x}+\sigma\int_\Omega\mathbf{e}_a\cdot\xi(\mathbf{x})\,\mathrm{d}\mathbf{x}=0.$$

Taking now the limit when a tends to $+\infty$ yields

$$-\int_{\mathbb{R}^3}\mathbf{H}\cdot\operatorname{curl}\xi\,\mathrm{d}\mathbf{x}+\sigma\int_\Omega\mathbf{E}\cdot\xi(\mathbf{x})\,\mathrm{d}\mathbf{x}=0, \qquad (7.7)$$

where $\mathbf{E}$ is a weak limit of a subsequence of $(\mathbf{e}_a)_a$.

In the same way, in (4.3b), we take $\boldsymbol{\psi}(t,\mathbf{x})=\rho_a(t-t_n)\xi(\mathbf{x})$. By the same arguments, we obtain that

$$\int_{\mathbb{R}^3}\mathbf{E}\cdot\operatorname{curl}\xi=0;$$

that is, $\operatorname{curl}E=0$ in $\mathcal{D}'(\mathbb{R}^3)$.

So we remark the $\mathbf{E}$ is in $\mathbb{H}_{\rm curl}(\mathbb{R}^3)$ and by density of $\mathcal{D}(\mathbb{R}^3;\mathbb{R}^3)$ in this space, we can take $\xi = \mathbf{E}$ in (7.7). We obtain then that

$$\sigma \int_\Omega \|\mathbf{E}\|^2 = 0.$$

Therefore from (7.7) we obtain that for all $\xi \in \mathcal{D}(\mathbb{R}^3;\mathbb{R}^3)$,

$$\int_{\mathbb{R}^3} \mathbf{H} \cdot \operatorname{curl} \xi \, \mathrm{d}\mathbf{x} = 0;$$

that is, $\operatorname{curl} \mathbf{H} = 0$ in $\mathcal{D}'(\mathbb{R}^3)$. So $\mathbf{H}$ satisfies:

$$\operatorname{div}(\mathbf{H} + \overline{\mathbf{u}}) = 0, \quad \operatorname{curl} \mathbf{H} = 0.$$

This concludes the proof of Theorem 4.4.

Conclusion. In this article, we have proven the existence of solutions to the Landau-Lifshitz-Maxwell system with nonlinear Neumann boundary conditions arising from surface energies. We have also characterized the ω-limit set of those weak solutions.

Further improvements should be possible. On the one hand, we expect that extending these results to curved spacers should be possible. No fundamental new idea should be necessary to carry out such an extension of our results as long as the spacer fully separates the domain in two. However, even in that case, the technicalities would lengthen the proof and the statement of the theorem as it would be necessary to write down geometric conditions on the spacers (the spacer cannot share a tangent plane with the domain boundary as it would create cusps).

On the other hand, the construction of more regular solutions for this model remains open.

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