Electronic Journal of Differential Equations, Vol. 2015 (2015), No. 56, pp. 1–6.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

A NEW PROOF OF BOREL'S LEMMA IN TWO DIMENSIONS

DENING LI

Abstract. Existence of solutions for nonlinear problems can often be established by a Newton's scheme, near an approximate solution, combined with a regularization procedure. This article provides a new method of constructing an infinite order C^∞ approximate solution for proving Borel's Lemma, without using the usual C^∞ cut-off functions.

1. Approximate solutions of infinite order

In the study of nonlinear problems, the linear iteration method is widely used to obtain the existence of solutions. Depending upon the nature of the energy estimate for the linearized problem, various iteration methods are carried out, usually near an approximate solution for the nonlinear problem. The existence of such an approximate solution depends upon the compatibility of the initial data and boundary or free boundary conditions. This compatibility is a necessary condition for the existence of the solution with some given regularity.

Consider the $n \times n$ system of nonlinear partial differential equations (the 3-dimensional Euler system is a special example with $n = 5$):

$$\mathscr{L}(u)u = A_0(u)\partial_t u + A_1(u)\partial_x u + A_2(u)\partial_y u + A_3(u)\partial_z u + C(u)u = f. \tag{1.1}$$

Here, the matrix $A_0(u)$ is assumed to be positively definite in the range of u.

The piece-wise smooth solutions (such as shock waves, rarefaction waves, or contact discontinuity) for the system (1.1) are usually formulated as initial-free-boundary problems. After a change of coordinates (depending upon the free boundary), the problem can be further transformed into an initial-boundary value problem. The approximate solutions for the unknown functions describing free boundaries can be constructed separately. Hence for simplicity, we will omit that part and consider only the following initial-boundary conditions

$$u(0, x, y, z) = u_0(x, y, z), \quad x \geq \phi(y, z),\ (y, z) \in \mathbb{R}^2, \tag{1.2}$$

$$B(u)u(t, x, y, z) = g(t, y, z), \quad t \geq 0,\ x = \psi(t, y, z),\ (y, z) \in \mathbb{R}^2. \tag{1.3}$$

Here $B(u)$ is in general an $m \times n$ matrix of nonlinear zero-order operators, and $\psi(0, y, z) = \phi(y, z)$.

For the nonlinear problem (1.1)-(1.3) to be solvable, at least locally in time, the compatibility is a standard requirement. Such requirement is necessary so that

2000 *Mathematics Subject Classification.* 35L50, 40C05.
Key words and phrases. Borel's Lemma; infinite order approximate solution.

Submitted March 18, 2014. Published February 27, 2015.

the value $u_0(x,y,z)$ in (1.2) and the value $u(t,x,y,z)$ required by the boundary condition (1.3) do not conflict with the value determined from the partial differential equations (1.1) at the intersection curve $x=\phi(y,z)$, $t=0$.

The 0-order compatibility comes from the fact the solution is continuous at the intersection curve. The values of the u-components obtained from (1.3) must be identical to the values prescribed in (1.2). From this 0-order compatibility of u_0 and g, one obtains that all the derivatives $\partial_y^{j_2}\partial_z^{j_3}u$ can be uniquely determined at $x=\phi(y,z)$ and $t=0$ by (1.2) and (1.3).

The first-order compatibility condition is derived from the fact that the solution is continuously differentiable at the intersection curve $x=\phi(y,z)$, $t=0$. From (1.1) and (1.2), the values of u_t at $t=0$ for a classical solution $u(t,x,y,z)$ can be uniquely determined. On the other hand, for m components of u, the derivative u_t can also be determined by (1.3) at $x=\phi(y,z)$. Therefore, in order that problem (1.1)-(1.3) have a classical solution $u(t,x,y,z)$, these two values must coincide at the intersection of $x=\phi(y,z)$ and $t=0$. This implies that the values $u_0(x,y,z)$ and $g(y,z,t)$ must satisfy certain constraints at the intersection of $x=\phi(y,z)$ and $t=0$. These constraints consist of the first order compatibility for the initial and boundary data (u_0,g). In other words, the data (u_0,g) are first-order compatible if and only if one can uniquely determine the values of u_t,u_x at the intersections of $x=\phi(y,z)$ and $t=0$.

Once the values of u_t and u_x are obtained at the intersection curve $x=\phi(y,z)$ and $t=0$, all the derivatives $\partial_y^{j_2}\partial_z^{j_3}u_t$ and $\partial_y^{k_2}\partial_z^{k_3}u_x$ are also known at $x=\phi(y,z)$ and $t=0$.

In general, the k-th order compatibility of the data (u_0,g) can be defined similarly from the continuity of k-th order derivatives of the solution. With k-th order compatible data (u_0,g), all the derivatives $\partial^\alpha u$, $(|\alpha|\le k)$ at the intersection of $x=\phi(y,z)$ and $t=0$ are uniquely determined. Here, we use the multi-index convention that

$$\partial^\alpha=\partial_t^{\alpha_0}\partial_x^{\alpha_1}\partial_y^{\alpha_2}\partial_z^{\alpha_3},\quad |\alpha|=\alpha_0+\alpha_1+\alpha_2+\alpha_3. \tag{1.4}$$

A k-th order approximate solution is closely related to the k-th order compatibility of the data. In particular, let $u\in C^k$ be a solution for (1.1)-(1.3) (which implies k-order compatibility of the data (u_0,g)), a k-th order approximate solution w for t problem (1.1)-(1.3) is a function $w(t,x,y,z)\in C^k$ near the intersections of $x=\phi(y,z)$ and $t=0$ such that

$$\partial^\alpha w(0,\phi(y,z),y,z)=\partial^\alpha u(0,\phi(y,z),y,z),\quad \forall|\alpha|\le k. \tag{1.5}$$

Equivalently, a function $w(t,x,y,z)\in C^k$ is a k-th order approximate solution if $w(0,x,y,z)=u_0(x,y,z)$, and both the interior equation (1.1) and the boundary conditions (1.3) are satisfied up to the order of $O(t^k)$; i.e.,

$$\mathscr{L}(u)u-f=O(t^k),\quad B(u)u-g=O(t^k). \tag{1.6}$$

To find a solution by a linear iteration for a nonlinear initial-boundary value problem such as (1.1)-(1.3), the iteration are proceeded near an approximate solution, see e.g., [8]. For various free boundary problems (essentially nonlinear); see e.g., [1, 4, 6], the linear iterations needs also to be carried out around an approximate solution.

The order requirement of the approximate solution varies, depending upon the nature of different iteration schemes. When an appropriate a priori estimate is available for the solution of linearized problem which would support the iteration indefinitely, the standard Picard's linear iteration is used, and the order for the approximate solution is usually the same as the smoothness order of required solution. However, when the linearized problem admits only a weaker estimate, the Nash-Moser iteration is a powerful tool which requires only a family of so-called tame estimates [5] for the linearized problems. In such cases, the order for the approximate solution could be much higher than the smoothness order of the required solution.

Given the initial-boundary data which are compatible up to any order, then an approximate solution of infinite order can be obtained by Borel's Lemma. In the following section, a new construction of such an approximate solution will be presented.

2. A new proof of Borel's lemma

From the k-th order compatibility condition, all the derivatives $u^{(\alpha)}(|\alpha| \le k)$ of the solution for (1.1)-(1.3) are uniquely determined. Then the k-th order approximate solution can be constructed immediately by using the Taylor polynomials.

Specifically, let $y \in \mathbb{R}^m$, given a family of C^k functions $\{c_\alpha(y) \in C^k,\ \alpha = (\alpha_0, \alpha_1),\ |\alpha| \le k\}$, the corresponding k-th order approximate solution $w(t,x,y)$ can be obtained by

$$w(t,x,y) = \sum_{|\alpha|=0}^{k} \frac{c_\alpha(y)}{\alpha!} t^{\alpha_0} x^{\alpha_1}. \tag{2.1}$$

However, the construction in (2.1) cannot be directly generalized to the case of infinite order approximate solution, because the corresponding Taylor series may not have a non-zero radius of convergence. This difficulty is usually overcome by using Borel's technique; i.e., introducing a sequence of C_0^∞ cut-off functions in the coefficients of (2.1). Indeed, we have the following result.

Theorem 2.1 (Borel's Lemma). *Let $\{c_\alpha(y) \in C^\infty, \alpha = (\alpha_0, \alpha_1), |\alpha| \ge 0\}$ be a given sequence of smooth functions, and $x = \phi(y)$ with $\phi(0) = 0$ be a C^∞ surface in (x,y) space near $(0,0)$. Then there is a C^∞ function $w(t,x,y)$ near $(t,x,y) = (0,0,0)$ satisfying*

$$w^{(\alpha)}(0, \phi(y), y) = c_\alpha(y), \quad |\alpha| = 0, 1, 2, \cdots . \tag{2.2}$$

Here $w^{(\alpha)}(t,x,y) = \partial_t^{\alpha_0} \partial_x^{\alpha_1} w(t,x,y)$.

The one-dimensional result of Theorem 2.1 was first proved by Borel in [2]. More generalized versions are also available, see e.g.[8]. The two-dimensional version of Theorem 2.1 is proved in [3] in a somewhat simplified form, using a modified Taylor series of (2.1), with added C_0^∞ coefficients ϕ_α with rapidly shrinking support as $|\alpha| \to \infty$. In the following, we present a completely different proof without introducing any C_0^∞ functions. Instead, we will use a more elementary construction for the infinite order approximate solution in Theorem 2.1. The method might be of interest because of its explicit expression.

Proof. First we define a sequence of functions $\gamma_\alpha(r)$ $(|\alpha| \ge 1)$ as follows.

For $|\alpha| = 1$,

$$\gamma_\alpha(r, y) \equiv \sin(b_\alpha(y) r);$$

for $|\alpha| = 2$,

$$\gamma_\alpha(r, y) \equiv \int_0^r \sin(b_\alpha(y)s)ds;$$

and for $|\alpha| \geq 3$,

$$\gamma_\alpha(r) \equiv \int_0^r \int_0^{s_{|\alpha|-1}} \cdots \int_0^{s_3} \int_0^{s_2} \sin(b_\alpha(y)s_1)ds_1 ds_2 \cdots ds_{|\alpha|-2} ds_{|\alpha|-1}, \tag{2.3}$$

where $b_\alpha = b_\alpha(y)$ depends only upon the parameter $y \in \mathbb{R}^m$ and will be chosen later.

$\gamma_\alpha(r)$ is a scalar function of the variable r, depending upon the parameter $y \in \mathbb{R}^m$ through $b_\alpha(y)$. Let $\gamma_\alpha^{(j)}$ denote the j-th order derivative with respect to r. It is readily checked that we have the following statement.

Lemma 2.2. *The functions $\gamma_\alpha(r)$ defined in* (2.3) *have the following properties:*

(1) $\gamma_\alpha \in C^\infty(\mathbb{R})$;
(2) $|\gamma_\alpha^{(j)}(r)| \leq 1$ *for all* $j < |\alpha|$, $r \in (-1, 1)$;
(3) $\gamma_\alpha^{(j)}(0) = 0$ *for all* $j < |\alpha|$;
(4) $\gamma_\alpha^{(|\alpha|)}(0) = b_\alpha$.

Now we define the function

$$w(t, x, y) = c_0(y) + \sum_{|\alpha|\geq 1} \frac{1}{|\alpha|!} \gamma_\alpha\Big(t + \frac{x - \phi(y)}{\alpha_1 + 1}\Big). \tag{2.4}$$

Remark 2.3. The choice of the factor $(\alpha_1 + 1)^{-1}$ in (2.4) serves to distinguish the different α with the same $|\alpha|$. Its specific form is only for convenience and can obviously be made differently, e.g., $(\alpha_1 + 1)$ or 2^{α_1}, etc. However, α_1 cannot be replaced by, say $|\alpha|$ or $|\alpha|!$, as it will be seen later in (2.7) and (2.8).

From the property 2 in Lemma 2.2, the function $w(t, x, y)$ in (2.4) is well-defined and C^∞ in the region: $\{(t, x, y) : |t| + |x - \phi(y)| < 1\}$. From the property 3 in Lemma 2.2, it is obvious that $w^{(0)}(0, \phi(y), y) = c_0(y)$. To show that it is the required function in Theorem 2.1, it remains to choose $b_\alpha(y)$ such that $w^{(\alpha)}(0, \phi(y), y) = c_\alpha(y)$ for all α. This is achieved by induction on $k = |\alpha|$ as follows.

- For $|\alpha| = k = 1$, let $b_\alpha(y) = c_\alpha(y)$.
- Assume that $b_\alpha(y)$ be already chosen for all $|\alpha| < k$. This means that all the functions γ_α, together with all the derivatives $\gamma_\alpha^{(j)}$ are known for $|\alpha| < k$.

We proceed to choose the vector $b_\alpha(y)$ for all $|\alpha| = k$ simultaneously such that for any $\beta = (\beta_0, \beta_1)$ with $|\beta| = k$,

$$\begin{aligned} w^{(\beta)}(t, x, y)\big|_{t=0,x=\phi(y)} &= \sum_{|\alpha|\geq 1} \frac{1}{|\alpha|!} \partial^\beta \gamma_\alpha\Big(t + \frac{x - \phi(y)}{\alpha_1 + 1}\Big)\big|_{t=0,x=\phi(y)} \\ &= c_\beta(y). \end{aligned} \tag{2.5}$$

Since

$$\partial^\beta \gamma_\alpha\Big(t + \frac{x - \phi(y)}{\alpha_1 + 1}\Big) = \gamma_\alpha^{(|\beta|)}\Big(t + \frac{x - \phi(y)}{\alpha_1 + 1}\Big)\Big(\frac{1}{\alpha_1 + 1}\Big)^{\beta_1},$$

by Property 3 in Lemma 2.2, all the terms in the summation of (2.5) with $k < |\alpha|$ vanish, i.e., for all $|\alpha| > |\beta| = k$,

$$\partial^\beta \gamma_\alpha\Big(t + \frac{x-\phi(y)}{\alpha_1+1}\Big)\Big|_{t=0,x=\phi(y)} = 0.$$

Then (2.5) becomes

$$w^{(\beta)}(t,x,y)\big|_{t=0,x=\phi(y)} = \sum_{|\alpha|=1}^{k} \frac{1}{|\alpha|!}\Big(\frac{1}{\alpha_1+1}\Big)^{\beta_1} \gamma_\alpha^{(|\beta|)}(0) = c_\beta(y),$$

or equivalently

$$\frac{1}{k!}\sum_{|\alpha|=k}\Big(\frac{1}{\alpha_1+1}\Big)^{\beta_1} b_\alpha(y) = c_\beta(y) - \sum_{|\alpha|=1}^{k-1}\frac{1}{|\alpha|!}\Big(\frac{1}{\alpha_1+1}\Big)^{\beta_1}\gamma_\alpha^{(|\beta|)}(0). \tag{2.6}$$

For all multi-index β with $|\beta| = k$, (2.6) consists of $k+1$ linear equations for $k+1$ variables $b_\alpha(y)$ with $|\alpha| = k$. (2.6) admits a unique vector solution $b_\alpha(y)$ if (omitting the non-zero factor $1/k!$) the following coefficient $(k+1)\times(k+1)$ matrix is nonsingular

$$\mathscr{A} = \big\{\big(\frac{1}{\alpha_1+1}\big)^{\beta_1},\ \alpha_1,\beta_1 = 0,1,\dots,k.\big\} \tag{2.7}$$

Computed explicitly, (2.7) becomes

$$\mathscr{A} = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1/2 & \cdots & 1/(k+1) \\ \cdots & \cdots & \cdots & \cdots \\ 1 & 1/2^k & \cdots & 1/(k+1)^k \end{pmatrix} \tag{2.8}$$

This matrix is the well-known Vandermonde matrix with the following non-zero determinant

$$\det\mathscr{A} = \prod_{1\le i<j\le k+1}(1/i - 1/j) \neq 0. \tag{2.9}$$

This completes the proof. □

Remark 2.4. As mentioned in section 1, the existence of an approximate solution of infinite order with explicit structure in (2.4) can be a useful tool in the study of some nonlinear problems, especially when Nash-Moser iteration is required to obtain the existence of the solution. This was first successfully used in the context of multi-dimensional rarefaction waves [1] to establish the existence of solution, see also [6]. Later on, it was also used in studying the general initial-boundary value problems in [8], and the 2-dimensional contact discontinuity problems [4] for the Euler system in gas-dynamics, etc.

References

[1] S. Alinhac; *Existence d'ondes de rarefaction pour des systèmes quasi-linéaires hyperboliques multidimensionnels*, Comm. Partial Differential Equations, 14(1989),173-230.

[2] E. Borel, *Sur quelques points de la theorie des fonctions*, Ann. Sci. Ecole Norm. Sup. 12 (1895), 9-55.

[3] S. Chen, D. Li; *Cauchy problem with general discontinuous initial data along a smooth curve for 2-D Euler system*, J. Differential Equations 257 (2014), no. 6, 1939–1988.

[4] J-F. Coulombel, P. Secchi; *Nonlinear compressible vortex sheets in two space dimensions*, Ann. Sci. Ec. Norm. Super.,41(2008),85-139.

[5] R. S. Hamilton; *The inverse function theorem of Nash and Moser*, Bull. Amer. Math. Soc., 7(1982), 65-222.
[6] D. Li; *Rarefaction and shock waves for multi-dimensional hyperbolic conservation laws*, Comm. in PDEs, 16(1991), 425-450.
[7] A. Majda; *The stability of multi-dimensional shock front, The existence of multi-dimensional shock front*, Memoirs Amer. Math. Soc., 275, 281 (1983).
[8] M. Poppenberg, *Nash-Moser techniques for nonlinear boundary-value problems,* Electronic Journal of Differential Euqations, 2003 (2003), no. 54, pp. 1-33.

Dening Li
Department of Mathematics, West Virginia University, Morgantown, WV 26506, USA
E-mail address: li@math.wvu.edu