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EXISTENCE AND ASYMPTOTIC BEHAVIOR OF A UNIQUE SOLUTION TO A SINGULAR DIRICHLET BOUNDARY-VALUE PROBLEM WITH A CONVECTION TERM

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ABSTRACT. In this article, we consider the problem

$$-\Delta u = b(x)g(u) + \lambda a(x)|\nabla u|^q + \sigma(x),\ u > 0,\ x \in \Omega, \quad u|_{\partial\Omega} = 0$$

with $\lambda \in \mathbb{R}$, $q \in [0, 2]$ in a smooth bounded domain Ω of $\mathbb{R}^N$. The weight functions b, a, σ belong to $C^\alpha_{\rm loc}(\Omega)$ satisfying $b(x), a(x) > 0$, $\sigma(x) \ge 0$, $x \in \Omega$, which may vanish or be singular on the boundary. $g \in C^1((0,\infty),(0,\infty))$ satisfies $\lim_{t\to 0^+} g(t) = \infty$. Our results include the existence, uniqueness and the exact boundary asymptotic behavior and global asymptotic behavior of the solution.

1. Introduction and main results

In this article we study the existence and asymptotic behavior of the unique classical solution to the problem

$$-\Delta u = b(x)g(u) + \lambda a(x)|\nabla u|^q + \sigma(x),\ u > 0,\ x \in \Omega, \quad u|_{\partial\Omega} = 0, \tag{1.1}$$

where Ω is a bounded domain with smooth boundary in $\mathbb{R}^N$, $\lambda \in \mathbb{R}$, $q \in [0, 2]$, b, a, σ satisfy

(H1) $b, a, \sigma \in C^\alpha_{\rm loc}(\Omega)$ for some $\alpha \in (0, 1)$, and $b(x), a(x) > 0, \sigma(x) \ge 0, x \in \Omega$,

and g satisfies the following hypotheses, not necessary simultaneously:

(G1) $g \in C^1((0,\infty),(0,\infty))$, $\lim_{t\to 0^+} g(t) = \infty$;
(G2) there exists $t_0 > 0$ such that $g'(t) < 0$, for all $t \in (0, t_0)$;
(G3) g is decreasing on $(0,\infty)$;
(G4) there exists $D_g \ge 0$ such that

$$\lim_{t\to 0^+} g'(t) \int_0^t \frac{ds}{g(s)} = -D_g.$$

When $\lambda = 0$ and $\sigma \equiv 0$ in Ω, problem (1.1) becomes

$$-\Delta u = b(x)g(u),\ u > 0,\ x \in \Omega, \quad u|_{\partial\Omega} = 0. \tag{1.2}$$

This problem arises in the study of non-Newtonian fluids, boundary layer phenomena for viscous fluids, chemical heterogeneous catalysts, as well as in the theory

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of heat conduction in electrical materials, and has been studied and extended by many authors, for instance [2]-[8], [13], [17, 18], [21], [25, 26], [29]-[32], [34, 35], [38], [44]-[48], [51], [53]-[56] and the references therein.

Next, we review works about the existence, uniqueness and asymptotic behavior of classical solutions to (1.1), which are summarized as the following two parts.

Part I: Existence and boundary behavior. For $b \equiv 1$ in Ω, when g satisfies (G1) and (G3), Crandall, Rabinowitz and Tartar [13], Fulks and Maybee [17], Stuart [45] showed that (1.2) has a unique solution $u \in C^{2,\alpha}_{\rm loc}(\Omega) \cap C(\bar\Omega)$, and the authors in [13] established the asymptotic behavior of the unique solution. Moreover, Anedda [2], Berhanu, Gladiali and Porru [5], Berhanu, Cuccu and Porru [6], Ghergu and Rădulescu [18], Ghergu and Rădulescu [21], McKenna and Reichel [34], Mi and Liu [35], Zhang [54] analyzed the first or second estimate of the solution near the boundary to (1.2). In particular, when $b \in C^\alpha(\bar\Omega)$ satisfies the following assumptions: there exist a constant $\delta > 0$ and a positive non-decreasing function $k_1 \in C((0,\delta))$ such that

- (B01) $\lim_{d(x)\to 0} \frac{b(x)}{k_1(d(x))} = b_0 \in (0,\infty)$, where $d(x) := \operatorname{dist}(x, \partial\Omega)$;
- (B02) $\lim_{t\to 0^+} k_1(t)g(t) = \infty$;

and g satisfies (G1), (G3) and the conditions

- (G01) there exist positive c_0, η_0 and $\gamma \in (0,1)$ such that $g(t) \le c_0 t^{-\gamma}$, for all $t \in (0,\eta_0)$;
- (G02) there exist $\theta > 0$ and $t_0 \ge 1$ such that $g(\xi t) \ge \xi^{-\theta} g(t)$ for all $\xi \in (0,1)$ and $0 < t \le t_0\xi$;
- (G03) the mapping $\xi \in (0,\infty) \to T(\xi) = \lim_{t\to 0^+} \frac{g(\xi t)}{\xi g(t)}$ is a continuous function.

Ghergu and Rădulescu [18] showed that the unique solution u of (1.2) satisfies $u \in C^{1,1-\alpha}(\bar\Omega) \cap C^2(\Omega)$ and

$$\lim_{d(x)\to 0} \frac{u(x)}{\phi(d(x))} = \bar\xi,$$

where $T(\bar\xi) = b_0^{-1}$, and $\phi \in C^1([0,c]) \cap C^2((0,c])$ $(c \in (0,\delta))$ is the local solution of the problem

$$-\phi''(t) = k_1(t)g(\phi(t)),\ \phi(t) > 0,\ t \in (0,c),\ \phi(0) = 0.$$

Zhang [51] extended the above result to the case where g is normalized regularly varying at zero with index $-\gamma$ ($\gamma > 0$) and k_1 in (B01) is normalized regularly varying at zero with index $-\beta$ ($\beta \in (0,2)$).

Later, Ben Othman et al [3], Gontara et al [25] extended the results in [18, 51] to a large class of functions b which belongs to the Kato class $K(\Omega)$ and g is normalized regularly varying at zero with index $-\gamma$ ($\gamma \ge 0$). In particular, they established an exact boundary behavior of the unique solution to the problem

$$-\Delta v = b(x),\ v > 0,\ x \in \Omega,\ v|_{\partial\Omega} = 0, \tag{1.3}$$

when b satisfies (H1) and the condition

(B03)

$$0 < \tilde b_2 := \liminf_{d(x)\to 0} \frac{b(x)}{k_1(d(x))} \le \tilde b_1 := \limsup_{d(x)\to 0} \frac{b(x)}{k_1 d(x)} < \infty$$

with $k_1(t) = t^{-2}\prod_{i=1}^m (\ln_i(t^{-1}))^{-\mu_i}$, $t \in (0,\delta)$, for some $\delta > 0$, where $\ln_i(t^{-1}) = \ln\circ\ln\circ\ln\circ\cdots\circ\ln(t^{-1})$ (i times) and $\mu_1 = \mu_2 = \cdots = \mu_{j-1} = 1$, $\mu_j > 1$ and $\mu_i \in \mathbb{R}$ for $j+1 \le i \le 1$.

For the convenience of discussions, we introduce two classes of Karamata functions as follows.

(i) Denote by Λ the set of all positive functions in $C^1((0,\delta_0]) \cap L^1((0,\delta_0])$ which satisfy

$$\lim_{t\to 0^+} \frac{d}{dt}\Big(\frac{K(t)}{k(t)}\Big) \in (0,\infty), \quad K(t) = \int_0^t k(s)ds \tag{1.4}$$

and for each $k \in \Lambda$ there exists $\delta_k \in (0,\delta_0]$ such that k is monotonic on $(0,\delta_k]$.

(ii) Denote by $\mathcal{K}$ the set of all positive functions k defined on $(0,\delta_0]$ by

$$\begin{gathered} k(t) := c\exp\Big(\int_t^{\delta_0} \frac{y(s)}{s} ds\Big), \quad c > 0 \text{ and} \\ y \in C((0,\delta_0]) \text{ with } \lim_{t\to 0^+} y(t) = 0. \end{gathered} \tag{1.5}$$

Define

$$D_k := \lim_{t\to 0^+} \frac{d}{dt}\Big(\frac{K(t)}{k(t)}\Big) \quad \text{for each } k \in \Lambda\cup\mathcal{K}.$$

Indeed, if $k \in \mathcal{K}$, then it follows by Proposition 2.8(i) and a direct calculation that $D_k = 1$.

The set Λ was first introduced by Cîrstea and Rădulescu [9]-[12] for non-decreasing functions and by Mohammed [37] for non-increasing functions to study the boundary behavior and uniqueness of solutions for boundary blow-up elliptic problems, which enables us to obtain significant information about the qualitative behavior of the large solution in a general framework. Later, Based on their ideas, Huang et al. [27]-[28], Mi and Liu [36], Zhang [55] and Repovš [41] further studied the asymptotic behavior of boundary blow-up solutions.

Recently, Zhang and Li [56] obtained the following results.

(i) Let b satisfy (H1), (B03), g satisfy (G1), (G3)-(G4) with $D_g > 0$, then for the unique classical solution u of (1.2),

$$\Big(\frac{\tilde{b}_2}{\mu_j - 1}\Big)^{1-D_g} \le \liminf_{d(x)\to 0} \frac{u(x)}{\psi(h(d(x)))} \le \limsup_{d(x)\to 0} \frac{u(x)}{\psi(h(d(x)))} \le \Big(\frac{\tilde{b}_1}{\mu_j - 1}\Big)^{1-D_g},$$

where

$$h(t) = (\ln_j(t^{-1}))^{1-\mu_j} \prod_{i=j+1}^{m} (\ln_i(t^{-1}))^{-\mu_i},\ t \in (0,\delta),$$

and the function ψ is uniquely determined by

$$\int_0^{\psi(t)} \frac{ds}{g(s)} = t, \quad t > 0. \tag{1.6}$$

(ii) Let b satisfy (H1) and the condition that there exists $k \in \Lambda$ such that

$$0 < b_2 := \liminf_{d(x)\to 0} \frac{b(x)}{k^2(d(x))} \le b_1 := \limsup_{d(x)\to 0} \frac{b(x)}{k^2(d(x))} < \infty, \tag{1.7}$$

and g satisfy (G1), (G3)-(G4) with $D_g > 0$. If

$$D_k + 2D_g > 2, \tag{1.8}$$

then for the unique classical solution u of (1.2),

$$\xi_2^{1-D_g} \le \liminf_{d(x)\to 0} \frac{u(x)}{\psi(K^2(d(x)))} \le \limsup_{d(x)\to 0} \frac{u(x)}{\psi(K^2(d(x)))} \le \xi_1^{1-D_g},$$

where

$$\xi_i = \frac{b_i}{2(D_k + 2D_g - 2)},\ i = 1, 2. \tag{1.9}$$

Later, Zeddini, Alsaedi and Mâagli [48] extended the above results so that they cover the case $b(t) = t^{-2}k(t)$, where k belongs to $\mathcal{K}$ and satisfies

$$\int_0^{\delta_0} \frac{k(s)}{s} ds < \infty. \tag{1.10}$$

They obtained the following theorem.

Theorem 1.1. *Let b satisfy* (H1) *and there exist $k \in \mathcal{K}$ such that*

$$0 < \tilde{b}_2 := \liminf_{d(x)\to 0} \frac{b(x)}{(d(x))^{\gamma-1}k(d(x))} \le \tilde{b}_1 := \limsup_{d(x)\to 0} \frac{b(x)}{(d(x))^{\gamma-1}k(d(x))} < \infty,$$

where $\gamma \ge 0$ and

$$\int_0^{\delta_0} \frac{k(s)}{s} ds = \infty,$$

then the unique classical solution u of (1.2) *in the case of $g(u) = u^{-\gamma}$ satisfies*

$$\begin{aligned} a_2^{1/(1+\gamma)} &\le \liminf_{d(x)\to 0} \frac{u(x)}{d(x)\Big(\int_{d(x)}^{\delta_0} \frac{k(s)}{s} ds\Big)^{1/(1+\gamma)}} \\ &\le \limsup_{d(x)\to 0} \frac{u(x)}{d(x)\Big(\int_{d(x)}^{\delta_0} \frac{k(s)}{s} ds\Big)^{1/(1+\gamma)}} \le a_1^{1/(1+\gamma)}, \end{aligned}$$

where $a_i = \tilde{b}_i(1+\gamma)$, $i = 1, 2$.

This improves the result of Lazer and Mckenna [30]. Recently, Alsaedi, Mâagli and Zeddini [1] extended the results in [48] to the case where Ω is an exterior domain in $\mathbb{R}^N$ with $N \ge 3$.

When $\lambda > 0$, $q = 2$, $b, a \equiv 1$, $\sigma \equiv 0$ in Ω and $g(u) = u^{-\gamma}$, $\gamma > 0$, by using the change of variable $v = e^{\lambda u} - 1$, Zhang and Yu [49] proved that (1.1) possesses a unique classical solution for each $\lambda \in (0, \infty)$. This was then used to deduce the existence and nonexistence of classical solutions to (1.1) in the case $q \in (0, 2)$.

When $\lambda = \pm 1$, $q \in (0, 2)$, $b, a \equiv 1$, $\sigma \equiv 0$ in Ω and the function $g : (0, \infty) \to (0, \infty)$ is locally Lipschitz continuous and decreasing, Giarrusso and Porru [22] showed that if g satisfies the following conditions:

(i) $\int_0^1 g(s)ds = \infty$, $\int_1^\infty g(s)ds < \infty$;
(ii) there exist positive constants δ and M with $M > 1$ such that

$$G_1(t) < MG_1(2t), \quad \forall t \in (0, \delta), \quad G_1(t) = \int_t^\infty g(s)ds, \quad t > 0,$$

then the unique solution u to (1.1) has the properties:

(i) $|u(x) - \Psi(d(x))| < c_0 d(x)$, $\forall x \in \Omega$ for $q \in (0, 1]$;

(ii) $|u(x) - \Psi(d(x))| < c_0 d(x) \left(G_1(\Psi(d(x)))\right)^{(q-1)/2}$, for all $x \in \Omega$ for $q \in (1,2)$, where c_0 is a suitable positive constant and $\Psi \in C([0,\infty)) \cap C^2((0,\infty))$ is uniquely determined by
$$\int_0^{\Psi(t)} \frac{ds}{\sqrt{2G_1(s)}} = t, \quad t > 0. \tag{1.11}$$

This implies
$$\lim_{d(x)\to 0} \frac{u(x)}{\Psi(d(x))} = 1.$$

When $\lambda \in \mathbb{R}$ and g satisfies (G1) with $\lim_{t\to\infty} g(t) = 0$, (G3), Zhang [52] showed that

(i) if $q = 2$, b satisfies (H1) and (1.3) possesses a unique solution which belongs to $C^{2,\alpha}_{\rm loc}(\Omega) \cap C(\bar\Omega)$, then (1.1) has a unique solution $u_\lambda \in C^{2,\alpha}_{\rm loc}(\Omega) \cap C(\bar\Omega)$ for every $\lambda \geq 0$;

(iii) if $b \equiv 1$ in Ω, then (1.1) has a unique solution $u_\lambda \in C^{2,\alpha}_{\rm loc}(\Omega) \cap C(\bar\Omega)$ in one of the following three cases: (i) $q \in [0,2]$, $\lambda \leq 0$; (ii) $q \in [0,1)$, $\lambda \geq 0$; (iii) $q = 1$, $0 \leq \lambda < \lambda_1^{1/2}$, where λ_1 is the first eigenvalue of Laplace operator $(-\Delta)$ with the Dirichlet boundary condition.

When $\lambda > 0$, $a \equiv 1$, $\sigma \equiv 0$ in Ω, and g satisfies (G1), (G3), (G4), Zhang et al [57] studied the boundary asymptotic behavior of the unique solution to (1.1) in the following two cases: (i) $q = 2$ and $b \in C^{\alpha}_{\rm loc}(\Omega)$; (ii) $q \in (0,2)$ and $b \equiv 1$ in Ω.

For other works, we refer the reader to [14]-[16], [19]-[20], [23], [39], [31] and the references therein.

Part II: Existence and global behavior. In this part, we review these works about the existence and global asymptotic behavior of classical solutions to (1.2) in the case that $g \in C^1((0,\infty))$ is a nonnegative function. For the convenience , we introduce the notation below.

For two nonnegative functions f and g defined on a set Ω,
$$f(x) \approx g(x), \quad x \in \Omega,$$
means that there exists some constant $c > 0$ such that
$$\frac{f(x)}{c} \leq g(x) \leq cf(x), \quad \text{for all } x \in \Omega.$$
Let φ_1 denote the positive normalized (i.e, $\max_{x\in\Omega} \varphi_1(x) = 1$) eigenfunction corresponding to the first positive eigenvalue λ_1 of the Laplace operator $(-\Delta)$. It is well known (please refer to [40]) that $\varphi_1 \in C^2(\bar\Omega)$ is a positive function, and we have for $x \in \Omega$,
$$\varphi_1(x) \approx d(x). \tag{1.12}$$
When $g(u) = u^{-\gamma}$, $\gamma > 1$ and b satisfies the condition that $b(x) \approx (d(x))^{-\mu}$, $x \in \Omega$, where $\mu \in (0,2)$. Lazer and Mckenna [30] showed that (1.2) has a unique solution u satisfying
$$c_2 (d(x))^{2/(1+\gamma)} \leq u(x) \leq c_1 (d(x))^{(2-\mu)/(1+\gamma)}, \quad \text{for } x \in \Omega,$$
where c_1, c_2 are two positive constants.

When g satisfies (G1), (G3) and the conditions

(G04) there exist $\gamma > 1$ and $c > 0$ such that $\lim_{t\to 0^+} t^\gamma g(t) = c$;

(G05) $\int_1^\infty g(t)dt < \infty$,

and the weight function b satisfies

(B04) there exists $\beta \in (0,2)$ such that $b(x) \approx (\varphi_1(x))^{-\beta}$, $x \in \Omega$,

Zhang and Cheng [50] obtained the following results:

(i) problem (1.2) has a unique solution $u \in C^{2,\alpha}_{\rm loc}(\Omega) \cap C(\bar{\Omega})$ satisfying
$$u(x) \approx \Psi((\varphi_1(x))^{\eta}), \quad x \in \Omega,$$
where Ψ is uniquely determined by (1.11) and $\eta = (2-\beta)/2$;

(ii) $u \in H^1_0(\Omega)$ if and only if
$$\int_\Omega \varphi_1^{\beta} g(\Psi(\varphi_1^{\eta}))\Psi(\varphi_1^{\eta})dx < \infty.$$

Moreover, when $g(u) = u^{-\gamma}$, $\gamma > 0$, they also obtained some more precise results. specifically, for $b \equiv 1$ in Ω, i.e., $\beta = 0$, they proved that the above results still hold as the condition (G04) is omitted.

Later, applying Karamata regular variation theory, many authors further studied the global estimate of solutions to (1.2) (please refer to [4], [7], [25], [32]). In particular, Mâagli [32] proved the following theorem.

Theorem 1.2. *If b satisfies* (H1) *and for all $x \in \Omega$,*
$$b(x) \approx (d(x))^{-\mu}\tilde{k}(d(x)), \quad \mu \le 2$$
and $\int_0^l \frac{\tilde{k}(s)}{s}ds < \infty$, where $\tilde{k} \in C^1((0,l))$ ($l > \max\{\delta_0, \operatorname{diam}(\Omega)\}$) is a positive extension of $k \in \mathcal{K}$, i.e.,
$$\tilde{k} := \begin{cases} k(t), & 0 < t \le \delta_0, \\ \tilde{k}(t), & \delta_0 < t < l, \end{cases}$$
then (1.2) *in the case of $g(u) = u^{-\gamma}$, $\gamma > -1$ has a unique classical solution u satisfying, for $x \in \Omega$,*
$$u(x) \approx (d(x))^{\min\{1,(2-\mu)/(1+\gamma)\}}\Psi_{\tilde{k},\mu,\gamma}(d(x)),$$
where
$$\Psi_{\tilde{k},\mu,\gamma}(t) := \begin{cases} \big(\int_0^t \frac{\tilde{k}(s)}{s}ds\big)^{1/(1+\gamma)}, & \text{if } \mu = 2, \quad \text{(i)} \\ (\tilde{k}(t))^{1/(1+\gamma)}, & \text{if } 1-\gamma < \mu < 2, \quad \text{(ii)} \\ \big(\int_t^l \frac{\tilde{k}(s)}{s}ds\big)^{1/(1+\gamma)}, & \text{if } \mu = 1-\gamma, \quad \text{(iii)} \\ 1, & \text{if } \mu < 1-\gamma. \quad \text{(iv)} \end{cases} \tag{1.13}$$

Recently, Ben Othman and Khamessi [4] improved and generalized the above result as follows.

Let $\tilde{k}_1, \tilde{k}_2 \in C^1((0,l))$ ($l > \max\{\delta_0, \operatorname{diam}(\Omega)\}$) be, respectively, the extensions of k_1, $k_2 \in \mathcal{K}$ and
$$\int_0^{\delta_0} \frac{k_i(s)}{s}ds < \infty, \quad i = 1,2.$$
Assume that b satisfies (H1) and the condition
$$(d(x))^{-\mu_2}\tilde{k}_2(d(x)) \le b(x) \le (d(x))^{-\mu_1}\tilde{k}_1(d(x)), \quad x \in \Omega$$
with $\mu_2 \le \mu_1 \le 2$, and $g \in C^1((0,\infty))$ is a nonnegative function satisfying
$$c_2 u^{-\gamma_2} \le g(u) \text{ for } 0 < u \le 1 \text{ and } g(u) \le c_1 u^{-\gamma_1} \text{ for } u > 0,$$

where $\gamma_1 \geq \gamma_2 > -1$ and $c_1 > c_2 > 0$. Then (1.2) has a unique classical solution u satisfying for each $x \in \Omega$,

$$\begin{aligned} &c^{-1}(d(x))^{\min\{1,\,(2-\mu_2)/(1-\gamma_2)\}}\Psi_{\tilde{k}_2,\,\mu_2,\,\gamma_2}(d(x)) \\ &\leq u(x) \leq c(d(x))^{\min\{1,\,(2-\mu_1)/(1-\gamma_1)\}}\Psi_{\tilde{k}_1,\,\mu_1,\,\gamma_1}(d(x)), \end{aligned}$$

for some constant $c > 0$.

Inspired by the above works, in this paper we continue to study the existence and asymptotic behavior of the unique classical solution to (1.1). For $q \in [0,1]$, we first establish a local comparison principle of the unique solution to (1.1), where we omit the usual condition that g is decreasing on $(0,\infty)$ as in [57]. Then we consider the exact asymptotic behavior of the unique solution near the boundary to (1.1) and reveal that the nonlinear term $\lambda a(x)|\nabla u|^q + \sigma(x)$ does not affect the asymptotic behavior for several kinds of functions b, a and σ. For $q \in [0,2]$, in view of the ideas of boundary estimate we investigate the existence and global asymptotic behavior of the unique solution to (1.1), and our approach is very different from that one in [32].

In particular, when $\lambda = 0$ and $\sigma \equiv 0$ in Ω, we improve and extend the results in [32] and [48] as follows:

- (I1) By Theorem 1.4, we extend the result of Theorem 1.1 from the nonlinearity $g(u) = u^{-\gamma}$ with $\gamma \geq 0$ to the case where g is normalized regularly varying at zero with index $-\gamma$, $\gamma = D_g/(1-D_g) \geq 0$, $D_g < 1$;
- (I2) By Theorems 1.8-1.10, we extend partial results of Theorem 1.2, i.e., the nonlinearity $g(u) = u^{-\gamma}$, $\gamma \geq 0$ is extended to the case where g is normalized regularly varying at zero with index $-\gamma$, $\gamma = D_g/(1-D_g) \geq 0$, $D_g < 1$. Exactly, we extend expressions (i), (ii) and (iii)-(iv) in (1.13) by Theorems 1.10, 1.8 and 1.9, respectively. It is worthwhile to point out that in our results, if b satisfies (B3) and (1.8) holds or b satisfies (B4), then g is admitted to be rapidly varying at zero.

Moreover, when $\lambda > 0$, $a \equiv 1$, $\sigma \equiv 0$ in Ω and $q \in [0,1)$, we extend the result of [57, Theorem 1.4] from the case where $b \equiv 1$ in Ω to $b \in C^{\alpha}_{\rm loc}(\Omega)$ for some $\alpha \in (0,1)$, i.e., we extend the range of D_g in (G4) from $D_g > 1/2$ to $D_g \geq 0$.

To our aim, we assume that b satisfies one of the following conditions:

(B1) there exists $k \in \Lambda \cup \mathcal{K}$ such that

$$0 < b_2 := \liminf_{d(x)\to 0} \frac{b(x)}{k^2(d(x))} \leq b_1 := \limsup_{d(x)\to 0} \frac{b(x)}{k^2(d(x))} < \infty; \tag{1.14}$$

(B2) there exists $k \in \mathcal{K}$ such that

$$0 < b_4 := \liminf_{d(x)\to 0} \frac{b(x)}{(d(x))^{-2}k(d(x))} \leq b_3 := \limsup_{d(x)\to 0} \frac{b(x)}{(d(x))^{-2}k(d(x))} < \infty,$$

where k satisfies (1.10);

(B3) there exist $k \in \Lambda \cup \mathcal{K}$ and $a_i(c) > 0$, $i = 1,2$ for each $0 < c < \delta_0$ such that

$$a_2(c) \leq \inf_{x\in\Omega} \frac{b(x)}{k^2(c\varphi_1(x))} \leq \sup_{x\in\Omega} \frac{b(x)}{k^2(c\varphi_1(x))} \leq a_1(c); \tag{1.15}$$

(B4) there exist $k \in \mathcal{K}$ and $a_i(c) > 0$, $i = 1,2$ for each $0 < c < \delta_0$ such that

$$a_2(c) \leq \inf_{x\in\Omega} \frac{b(x)}{(c\varphi_1(x))^{-2}k(c\varphi_1(x))} \leq \sup_{x\in\Omega} \frac{b(x)}{(c\varphi_1(x))^{-2}k(c\varphi_1(x))} \leq a_1(c), \tag{1.16}$$

where k satisfies (1.10),

and a, σ satisfy one of the following conditions:

(H2) there exist constants ρ_i, $i = 1, 2$ satisfying
$$\rho_1 < \frac{(q-1)(2-2D_g)+D_k(2-q)}{D_k}, \quad \rho_2 < \frac{2D_k+2D_g-2}{D_k}$$
and functions $\hat{k}_i \in \mathcal{K}$, $i = 1, 2$ such that
$$\limsup_{d(x)\to 0} \frac{a(x)}{(d(x))^{-\rho_1}\hat{k}_1(d(x))} < \infty, \quad \limsup_{d(x)\to 0} \frac{\sigma(x)}{(d(x))^{-\rho_2}\hat{k}_2(d(x))} < \infty; \tag{1.17}$$

(H3) there exist constants ρ_i, $i = 1, 2$ satisfying
$$\rho_1,\, \rho_2 < \frac{D_k(\gamma+2)-2}{D_k}$$
and functions $\hat{k}_i \in \mathcal{K}$, $i = 1, 2$ such that (1.17) holds;

(H4) there exist constants ρ_i, $i = 1, 2$ satisfying
$$\begin{cases} \rho_1 \le 2-q, & q \in (1,2], \\ \rho_1 < 2-q, & q \in [0,1], \\ \rho_2 < 2, & q \in [0,2] \end{cases}$$
and functions $\hat{k}_i \in \mathcal{K}$, $i = 1, 2$ such that (1.17) holds here, and if $\rho_1 = 2-q$, then
$$\limsup_{d(x)\to 0} \hat{k}_1(d(x)) < \infty. \tag{1.18}$$

Our results are summarized as the following two parts and the key of our estimates is the solution ψ of (1.6).

Part 1: Boundary asymptotic behavior.

Theorem 1.3. *Let b, a, σ satisfy* (H1)–(H2), (B1), *g satisfy* (G1)–(G2), (G4) *with $D_g + q < 3$ and* (1.8) *hold.*

(i) *If $q \in [0, 1)$, then the unique solution u_λ of* (1.1) *for each $\lambda \in \mathbb{R}$ satisfies*
$$\xi_2^{1-D_g} \le \liminf_{d(x)\to 0} \frac{u_\lambda(x)}{\psi(K^2(d(x)))} \le \limsup_{d(x)\to 0} \frac{u_\lambda(x)}{\psi(K^2(d(x)))} \le \xi_1^{1-D_g}, \tag{1.19}$$
where ξ_i, $i = 1, 2$ are as defined in (1.9).

(ii) *If $q = 1$, then there exists $\lambda_0 > 0$ such that the unique solution u_λ of* (1.1) *for each $\lambda \in (-\lambda_0, \lambda_0)$ satisfies* (1.19).

Theorem 1.4. *Let b, a, σ satisfy* (H1), (H3), (B1), *and g satisfy* (G1)–(G2) *with $\liminf_{t\to 0^+} t^\gamma g(t) > 0$. Also let* (G4) *and $D_k + 2D_g = 2$ hold, where $\gamma = D_g/(1-D_g)$, $D_g < 1$. Further assume that*

(H5) $\int_0^{\delta_0} k^2(s)s^{-\gamma}ds = \infty$;

(H6)
$$\lim_{t\to 0^+} \Big(g'(t)\int_0^t \frac{1}{g(s)}ds + D_g\Big)\frac{\int_t^{\delta_0} k^2(s)s^{-\gamma}ds}{k^2(t)t^{1-\gamma}} = E \in (-\infty,\, (1-D_g)^2).$$

Then the following hold:

(i) *when* $q \in [0, 1)$, *the unique solution* u_λ *of* (1.1) *for each* $\lambda \in \mathbb{R}$ *satisfies*

$$\begin{aligned} \xi_2^{1-D_g} &\le \liminf_{d(x)\to 0} \frac{u_\lambda(x)}{\psi\Big((d(x))^{1+\gamma}\int_{d(x)}^{\delta_1} k^2(s)s^{-\gamma}ds\Big)} \\ &\le \limsup_{d(x)\to 0} \frac{u_\lambda(x)}{\psi\Big((d(x))^{1+\gamma}\int_{d(x)}^{\delta_1} k^2(s)s^{-\gamma}ds\Big)} \le \xi_1^{1-D_g}, \end{aligned} \tag{1.20}$$

for some $\delta_1 > 0$, *where*

$$\xi_i = \frac{b_i}{1-(1-D_g)^{-2}E}, \quad i = 1, 2;$$

(ii) *when* $q = 1$, *there exists* $\lambda_0 > 0$ *such that the unique solution* u_λ *of* (1.1) *for each* $\lambda \in (-\lambda_0, \lambda_0)$ *satisfies* (1.20).

Theorem 1.5. *Let* b, a, σ *satisfy*(H1), (H4), (B2), g *satisfy* (G1)–(G2), (G4). *Also,* $D_g < 1$ *in* (G4) *if* $\rho_1 = 2 - q$ *in* (H4). *Then the following hold:*

(i) *when* $q \in [0, 1)$, *the unique solution* u_λ *of 1.1 for each* $\lambda \in \mathbb{R}$ *satisfies*

$$b_4^{1-D_g} \le \liminf_{d(x)\to 0} \frac{u_\lambda(x)}{\psi\Big(\int_0^{d(x)}\frac{k(s)}{s}ds\Big)} \le \limsup_{d(x)\to 0} \frac{u_\lambda(x)}{\psi\Big(\int_0^{d(x)}\frac{k(s)}{s}ds\Big)} \le b_3^{1-D_g}; \tag{1.21}$$

(ii) *when* $q = 1$, *there exists* $\lambda_0 > 0$ *such that the unique solution* u_λ *of* (1.1) *for each* $\lambda \in (-\lambda_0, \lambda_0)$ *satisfies* (1.21).

Remark 1.6. Let $k \in \Lambda \cup \mathcal{K}$ and $D_k + 2D_g = 2$ hold. Combining with Lemma 3.1 (iv) and Proposition 2.6, we know that there exists $k_1 \in \mathcal{K}$ such that $k^2(t) = t^{2(1-D_k)/D_k} k_1$, $t \in (0, \delta_0]$. Hence, it follows by Lemma 3.3 that

$$\lim_{t\to 0^+} \frac{\int_t^{\delta_0} k^2(s)s^{-\gamma}ds}{k^2(t)t^{1-\gamma}} = \lim_{t\to 0^+} \frac{\int_t^{\delta_0} \frac{k_1(s)}{s}ds}{k_1(t)} = \infty,$$

where $\gamma = D_g/(1 - D_g)$, $D_g < 1$.

Remark 1.7. In Theorem 1.4, let $\delta_0 = 1$, $k^2(t) = t^{\gamma-1}(-\ln t)^\beta$ and

$$\begin{aligned} g(t) &= ct^{-\gamma} \exp\Big(\int_t^1 \frac{-E(1+\gamma)^2(1+\beta)}{s(-\ln s)}ds\Big) \\ &= ct^{-\gamma}(-\ln t)^{-E(1+\gamma)^2(1+\beta)}, \end{aligned}$$

$c > 0$, $E \le 0$, $\beta \ge 0$, $t \in (0, 1)$. By [35, Lemma 3 (iii)], we know that (H6) holds.

Part 2: Existence and global asymptotic behavior.

Theorem 1.8. *Let* b, a, σ *satisfy* (H1)–(H2), (B3), g *satisfy* (G1), (G3)–(G4) *with* $D_g + q < 3$ *and* (1.8) *hold.*

(i) *If* $q \in (0, 1)$, *then for each* $\lambda \in \mathbb{R}$ *problem* (1.1) *has a unique classical solution* u_λ *satisfying*

$$u_\lambda(x) \approx \psi(\tilde{K}^2(d(x))), \quad x \in \Omega \tag{1.22}$$

with $\tilde{K}(t) = \int_0^t \tilde{k}(s)ds$, $t \in (0, \infty)$, *where* $\tilde{k} \in C^1((0, \infty))$ *is a positive extension of* $k \in C^1((0, \delta_0])$.

(ii) *If* $q \in [1, 2]$, *then there exists* $\lambda_0 > 0$ *such that for each* $\lambda \in (-\infty, \lambda_0)$ *problem* (1.1) *has a unique classical solution* u_λ *satisfying* (1.22).

Theorem 1.9. *Let b, a, σ satisfy* (H1), (H3), (B3), *g satisfy* (G1), (G3)–(G4) *and $2/(2+\gamma) < D_k \leq 2/(1+\gamma)$ hold, where $\gamma = D_g/(1-D_g)$, $D_g < 1$. If further assume that* (H6) *holds, then the following hold:*

- (i) *when $q \in (0,1)$, for each $\lambda \in \mathbb{R}$ problem* (1.1) *has a unique classical solution u_λ satisfying*
$$u_\lambda(x) \approx \psi\Big((d(x))^{1+\gamma} \int_{d(x)}^{l} \tilde{k}^2(s)s^{-\gamma}ds\Big), \quad x \in \Omega, \tag{1.23}$$
where $l > \max\{\delta_0, \operatorname{diam}(\Omega)\}$ and $\tilde{k} \in C^1((0,l))$ is a positive extension of $k \in C^1((0,\delta_0])$;
- (ii) *when $q \in [1,2]$, there exists $\lambda_0 > 0$ such that for each $\lambda \in (-\infty, \lambda_0)$ problem* (1.1) *has a unique classical solution u_λ satisfying* (1.23).

Theorem 1.10. *Let b, a, σ satisfy* (H1), (H4), (B4), *g satisfy* (G1), (G3)-(G4). *Moreover, $D_g < 1$ in* (G4) *if $\rho_1 = 2 - p$ in (H4).*

- (i) *If $q \in (0,1)$, then for each $\lambda \in \mathbb{R}$, problem* (1.1) *has a unique classical solution u_λ satisfying*
$$u_\lambda(x) \approx \psi\Big(\int_0^{d(x)} \frac{\tilde{k}(s)}{s} ds\Big), \quad x \in \Omega, \tag{1.24}$$
where $\tilde{k} \in C^1((0,l))$ is a positive extension of $k \in C^1((0,\delta_0])$ and $l >$ $\max\{\delta_0, \operatorname{diam}(\Omega)\}$.
- (ii) *If $q \in [1,2]$, then there exists $\lambda_0 > 0$ such that for each $\lambda \in (-\infty, \lambda_0)$, problem* (1.1) *has a unique classical solution satisfying* (1.24).

Remark 1.11. Assume that $b(x) \approx (d(x))^{-\alpha} k_1(d(x))$, $x \in \Omega$, $\alpha < 2$, where
$$k_1 = \exp\Big(\int_t^l \frac{y(s)}{s}\Big)ds, \quad y \in C((0,l]), \quad \lim_{t \to 0^+} y(t) = 0, \quad l > \max\{\operatorname{diam}(\Omega),\ \delta_0\}.$$
Then we can take $k \in \Lambda \cup \mathcal{K}$ such that
$$k^2(t) = t^{-\alpha} \exp\Big(\int_t^{\delta_0} \frac{y(s)}{s} ds\Big), \quad t \in (0, \delta_0]$$
such that (1.15) holds for each $c \in (0, \min\{\delta_0, 1/c_1\})$ and
$$a_1(c) = c_0 c^\alpha c_1^{|\alpha|}(c_1/c)^\beta M_0, \quad a_2(c) = c_0^{-1} c^\alpha c_1^{-|\alpha|}(c/c_1)^\beta M_0,$$
where $\beta = \max_{t \in (0,l]} |y(t)|$, $M_0 = \exp\Big(\int_{\delta_0}^l \frac{y(s)}{s} ds\Big)$, and c_0, c_1 are two large enough constants.

Remark 1.12. As in Remark 1.11, If $b(x) \approx (d(x))^{-2} k_1(d(x))$, then we can choose $k \in \mathcal{K}$ such that (1.16) holds.

Remark 1.13. For each $k \in C^1((0,\delta_0])$, there exists a positive function $\tilde{k} \in C^1((0,\infty))$ such that $\tilde{k} \equiv k$ on $(0,\delta_0]$, for instance, define
$$\tilde{k}(t) := \begin{cases} k(t), & 0 < t \leq \delta_0, \\ k(\delta_0)\exp\big(\frac{k'(\delta_0)(t-\delta_0)}{k(\delta_0)}\big), & \delta_0 < t, \end{cases}$$
which is our desired function.

Remark 1.14. In Theorems 1.8 and 1.10, g is admitted to be rapidly varying at zero.

Remark 1.15. If $q = 0$, then Theorems 1.8-1.10 still hold for each $\lambda \in [0, \infty)$.

Remark 1.16. In Theorem 1.9, if $\lambda = 0$, then the lower bound of D_k can be reduced to $2/(\gamma + 3)$.

We close this section with an outline of the paper. In Section 2, we give preliminary considerations. In Section 3, we collect some auxiliary results. Section 4 is devoted to prove Theorems 1.3-1.5. The proofs of Theorems 1.8-1.10 are given in Section 5.

2. Preliminary results

In this section, we present some bases of Karamata regular variation theory which come from Introductions and Appendix in Maric [33], Preliminaries in Resnick [42], Seneta [43].

Definition 2.1. A positive measurable function g defined on $(0, a_1)$, for some $a_1 > 0$, is called *regularly varying at zero* with index ρ, written as $g \in \mathrm{RVZ}_\rho$, if for each $\xi > 0$ and some $\rho \in \mathbb{R}$,
$$\lim_{t \to 0^+} \frac{g(\xi t)}{g(t)} = \xi^\rho. \tag{2.1}$$
In particular, when $\rho = 0$, g is called *slowly varying at zero.*

Clearly, if $g \in \mathrm{RVZ}_\rho$, then $t \mapsto g(t)t^{-\rho}$ is slowly varying at zero. Some basic examples of slowly varying functions at zero are

(i) every measurable function on $(0, a_1)$ which has a positive limit at zero;
(ii) $(-\ln t)^p$, $(\ln(-\ln t))^p$, $t \in (0, 1)$, $p \in \mathbb{R}$;
(iii) $\exp\big((-\ln t)^p\big)$, $t > 0$, $0 < p < 1$.

Definition 2.2. A positive measurable function g defined on $(0, a_1)$, for some $a_1 > 0$, is called *rapidly varying at zero* if for each $p > 1$
$$\lim_{t \to 0^+} g(t)t^p = \infty.$$

Proposition 2.3 (Uniform convergence theorem)**.** *If $g \in RVZ_\rho$, then* (2.1) *holds uniformly for $\xi \in [c_1, c_2]$ with $0 < c_1 < c_2 < a_1$.*

Proposition 2.4 (Representation theorem)**.** *A function k is slowly varying at zero if and only if it may be written in the form*
$$k(t) = \varphi(t) \exp\Big(\int_t^{a_0} \frac{y(\tau)}{\tau} d\tau\Big), \quad t \in (0, a_0],$$
for some $a_0 \in (0, a_1)$, where the functions φ and y are measurable and for $t \to 0^+, y(t) \to 0$ and $\varphi(t) \to c$, with $c > 0$.

We call that
$$k(t) = c \exp\Big(\int_t^{a_0} \frac{y(\tau)}{\tau} d\tau\Big), \quad t \in (0, a_0], \tag{2.2}$$
is *normalized* slowly varying at zero and $g(t) = t^\rho k(t)$, $t \in (0, a_0]$ is *normalized* regularly varying at zero with index ρ and written $g \in NRVZ_\rho$. By the above definition, we know that $\mathcal{K} \subseteq NRVZ_0$. On the other hand, if $k \in NRVZ_0 \cap C^1((0, \delta_0])$, then $k \in \mathcal{K}$.

Assume that g belongs to $C^1((0,a_0])$ for some $a_0>0$ and is positive on $(0,a_0]$. Then, $g\in NRVZ_\rho$ if and only if

$$\lim_{t\to 0^+}\frac{tg^{'}(t)}{g(t)}=\rho.$$

Proposition 2.5. *If functions k, k_1 are slowly varying at zero, then*

(i) *k^ρ for every $\rho\in\mathbb{R}$, $c_1k+c_2k_1$ ($c_1\geq 0, c_2\geq 0$ with $c_1+c_2>0$), $k\circ k_1$ if $k_1(t)\to 0$ as $t\to 0^+$), are also slowly varying at zero;*

(ii) *for every $\rho>0$ and $t\to 0^+$, $t^\rho k(t)\to 0$, $t^{-\rho}k(t)\to\infty$;*

(iii) *for $\rho\in\mathbb{R}$ and $t\to 0^+$, $\ln k(t)/\ln t\to 0$ and $\ln(t^\rho k(t))/\ln t\to\rho$.*

Proposition 2.6. *If $g_1\in RVZ_{\rho_1}$, $g_2\in RVZ_{\rho_2}$ with $\lim_{t\to 0^+}g_2(t)=0$, then $g_1\circ g_2\in RVZ_{\rho_1\rho_2}$.*

Proposition 2.7. *If $g_1\in RVZ_{\rho_1}$, $g_2\in RVZ_{\rho_2}$, then $g_1\cdot g_2\in RVZ_{\rho_1+\rho_2}$.*

Proposition 2.8 (Asymptotic Behavior)**.** *If a function k is slowly varying at zero, then for $a>0$ and $t\to 0^+$,*

(i) *$\int_0^t s^\rho k(s)ds\cong(1+\rho)^{-1}t^{1+\rho}k(t)$ for $\rho>-1$;*

(ii) *$\int_t^a s^\rho k(s)ds\cong-(1+\rho)^{-1}t^{1+\rho}k(t)$ for $\rho<-1$.*

3. Auxiliary results

In this section, we collect some useful results.

Lemma 3.1. *Let $k\in\Lambda\cup\mathcal{K}$. Then*

(i) $\lim_{t\to 0^+}\frac{K(t)}{k(t)}=0$;

(ii) $\lim_{t\to 0^+}\frac{tk(t)}{K(t)}=D_k^{-1}$, *i.e.,* $K\in NRVZ_{D_k^{-1}}$;

(iii) $\lim_{t\to 0^+}\frac{K(t)k'(t)}{k^2(t)}=1-D_k$;

(iv) $\lim_{t\to 0^+}\frac{tk'(t)}{k(t)}=\frac{1-D_k}{D_k}$.

Proof. Here, we only prove the results in the case of $k\in\mathcal{K}$ because the ones have been given by Lemma 2.1 in [54] when $k\in\Lambda$.

(i)-(iii) By Proposition 2.8(i), we obtain that (i)-(iii) hold. (iv) (iv) follows by (ii)-(iii). □

Lemma 3.2 ([56, Lemma 2.2])**.** *Let g satisfy* (G1)-(G2),

(i) *if g satisfies (G4), then $\lim_{t\to 0^+}\frac{g(t)}{t}\int_0^t\frac{ds}{g(s)}=1-D_g$ and $D_g\leq 1$;*

(ii) *(G4) holds with $D_g\in[0,1)$ if and only if $g\in NRVZ_{-D_g/(1-D_g)}$;*

(iii) *if (G4) holds with $D_g=1$, then g is rapidly varying at zero.*

Lemma 3.3 ([48, lemma 2.3])**.** *Let $k\in\mathcal{K}$, then*

$$\lim_{t\to 0^+}\frac{k(t)}{\int_t^{\delta_0}\frac{k(s)}{s}ds}=0.$$

If further $\int_0^{\delta_0}\frac{k(s)}{s}ds$ converges, then we have

$$\lim_{t\to 0^+}\frac{k(t)}{\int_0^t\frac{k(s)}{s}ds}=0.$$

Lemma 3.4. *Suppose g satisfies* (G1)–(G2), (G4) *and let ψ be the solution of* (1.6). *Then*

(i) $\psi'(t) = g(\psi(t))$, $\psi(t) > 0$, $\psi(0) = 0$ *and* $\psi''(t) = g(\psi(t))g'(\psi(t))$, $t > 0$;

(ii) $\lim_{t\to 0^+} \frac{t\psi'(t)}{\psi(t)} = 1 - D_g$;

(iii) $\lim_{t\to 0^+} \frac{t\psi''(t)}{\psi'(t)} = -D_g$;

(iv) *if* $k \in \Lambda \cup \mathcal{K}$ *and* $D_k(1+\gamma) \le 2$, $\gamma = D_g/(1-D_g)$, $D_g < 1$, *then*
$$\lim_{t\to 0^+} \frac{t^{1-\gamma}k^2(t)}{\int_t^{\delta_0} \frac{k(s)}{s} ds} = 0;$$

(v) *if* $k \in \Lambda \cup \mathcal{K}$ *and* (1.8) *holds, then*
$$\lim_{t\to 0^+} t^{-\rho_1}\hat{k}(t)(\psi'(K^2(t)))^{q-1}K^q(t)k^{q-2}(t) = 0;$$
$$\lim_{t\to 0^+} t^{-\rho_2}\hat{k}(t)\big(\psi'(K^2(t))k^2(t)\big)^{-1} = 0,$$
where $q \in [0,2]$, $D_g + q < 3$, $\hat{k} \in \mathcal{K}$ *and* ρ_1, ρ_2 *are as defined in* (H2);

(vi) *if* $k \in \Lambda \cup \mathcal{K}$ *and* $2/(2+\gamma) < D_k \le 2/(1+\gamma)$ *with* $\gamma = D_g/(1-D_g)$, $D_g < 1$, *then*
$$\lim_{t\to 0^+} t^{\gamma}(k^2(t))^{-1} \int_t^{\delta_0} k^2(s)s^{-\gamma} ds = 0;$$
$$\lim_{t\to 0^+} t^{-\rho_1}\hat{k}(t)(k^2(t))^{-1}t^{q\gamma}\Big[\psi'\Big(t^{1+\gamma}\int_t^{\delta_0} k^2(s)s^{-\gamma}ds\Big)\Big]^{q-1}\Big(\int_t^{\delta_0} k^2(s)s^{-\gamma}ds\Big)^q = 0;$$
$$\lim_{t\to 0^+} t^{-\rho_1}\hat{k}(t)(k^2(t))^{q-1}t^q\Big[\psi'\Big(t^{1+\gamma}\int_t^{\delta_0} k^2(s)s^{-\gamma}ds\Big)\Big]^{q-1} = 0;$$
$$\lim_{t\to 0^+} t^{-\rho_2}\hat{k}(t)\Big[\psi'\Big(t^{1+\gamma}\int_t^{\delta_0} k^2(s)s^{-\gamma}ds\Big)k^2(t)\Big]^{-1} = 0,$$
where $q \in [0,2]$, $\hat{k} \in \mathcal{K}$, *and* ρ_1, ρ_2 *are as defined in* ($\mathbf{H_3}$);

(vii) *if* $\hat{k}_1$, $\hat{k}_2$, $k \in \mathcal{K}$ *and* (1.10) *holds, then*
$$\lim_{t\to 0^+} t^{-\rho_1}\hat{k}_1(t)t^{2-q}(k(t))^{q-1}\Big[\psi'\Big(\int_0^t \frac{k(s)}{s}ds\Big)\Big]^{q-1} = 0;$$
$$\lim_{t\to 0^+} t^{2-\rho_2}\hat{k}_2(t)\Big[k(t)\psi'\Big(\int_0^t \frac{k(s)}{s}ds\Big)\Big]^{-1} = 0,$$
where $q \in [0,2]$ *and* ρ_1, ρ_2 *are as defined in (H4), moreover,* (1.18) *and* $D_g < 1$ *hold if* $\rho_1 = 2 - p$.

Proof. (i)-(iii) By the definition of ψ and a direct calculation, we get (i). By (i), Lemma 3.2 (i), we obtain that
$$\lim_{t\to 0^+} \frac{t\psi'(t)}{\psi(t)} = \lim_{t\to 0^+} \frac{g(\psi(t))}{\psi(t)} \int_0^{\psi(t)} \frac{ds}{g(s)} = 1 - D_g,$$
i.e. (ii) holds.
$$\lim_{t\to 0^+} \frac{t\psi''(t)}{\psi'(t)} = \lim_{t\to 0^+} tg'(\psi(t)) = \lim_{t\to 0^+} g'(\psi(t)) \int_0^{\psi(t)} \frac{ds}{g(s)} = -D_g,$$
i.e. (iii) holds.

(iv) By Lemma 3.1 (iv) and Proposition 2.6, we know that

$$k^2 \in NRVZ_{2(1-D_k)/D_k}. \tag{3.1}$$

Hence, there exists $k_0 \in \mathcal{K}$ such that

$$k^2(t)t^{-\gamma} = t^{(2(1-D_k)/D_k)-\gamma} k_0(t), \quad t \in (0, \delta_0]. \tag{3.2}$$

When $D_k(1+\gamma) = 2$, a simple calculation shows that $(2(1-D_k)/D_k) - \gamma = -1$. So, we conclude by Lemma 3.3 that

$$\lim_{t\to 0^+} \frac{t^{1-\gamma}k^2(t)}{\int_t^{\delta_0} k^2(s)s^{-\gamma}ds} = \frac{k_0(t)}{\int_t^{\delta_0} \frac{k_0(s)}{s}ds} = 0.$$

When $D_k(1+\gamma) < 2$, a simple calculation shows that $-1 < (2(1-D_k)/D_k) - \gamma$. So, we have

$$\lim_{t\to 0^+} \int_t^{\delta_0} k^2(s)s^{-\gamma}ds < \infty, \quad \text{i.e.,} \int_t^{\delta_0} k^2(s)s^{-\gamma}ds \in \mathcal{K}. \tag{3.3}$$

By (3.1) and Proposition 2.7, we know that there exists $k_0 \in \mathcal{K}$ such that

$$t^{1-\gamma}k^2(t) = t^{1-\gamma-2(1-D_k)/D_k} k_0(t), \quad t \in (0, \delta_0].$$

(iv) follows by Proposition 2.5 (ii).

(v) By Lemma 3.1, Proposition 2.6 and (iii), we obtain

$$K \in NRVZ_{D_k^{-1}}, \quad k \in NRVZ_{(1-D_k)/D_k} \text{ and } \psi' \circ K^2 \in NRVZ_{-2D_g/D_k}.$$

By Propositions 2.6 and 2.7, we arrive at

$$(\psi' \circ K^2)^{q-1} \cdot K^q \cdot k^{q-2} \in NRVZ_{\tau_1} \text{ and } (\psi' \circ K^2)^{-1}k^{-2} \in NRVZ_{\tau_2},$$

where

$$\tau_1 = \frac{(q-1)(2-2D_g) + D_k(2-q)}{D_k} > 0, \quad \tau_2 = \frac{2D_k + 2D_g - 2}{D_k} > 0.$$

Thus, there exist $k_1, k_2 \in \mathcal{K}$ such that

$$(\psi' \circ K^2(t))^{q-1} \cdot K^q(t) \cdot k^{q-2}(t) = t^{\tau_1}k_1(t),\ t \in (0, \delta_0],$$
$$(\psi' \circ K^2(t))^{-1}k^{-2}(t) = t^{\tau_2}k_2(t), \quad t \in (0, \delta_0].$$

It follows by Proposition 2.5 (ii) that

$$\lim_{t\to 0^+} t^{-\rho_1}\hat{k}(t)(\psi' \circ K^2(t))^{q-1} \cdot K^q(t) \cdot k^{q-2}(t) = \lim_{t\to 0^+} t^{\tau_1-\rho_1}\hat{k}(t)k_1(t) = 0;$$
$$\lim_{t\to 0^+} t^{-\rho_2}\hat{k}(t)(\psi'(K^2(t))k^2(t))^{-1} = \lim_{t\to 0^+} t^{\tau_2-\rho_2}\hat{k}(t)k_2(t) = 0.$$

(vi) By Lemma 3.1 (iv) and Proposition 2.6, we know that (3.1) holds. Hence, there exists $k_0 \in \mathcal{K}$ such that (3.2) holds here. As before, when $D_k = 2/(1+\gamma)$, we have $(2(1-D_k)/D_k) - \gamma = -1$. So, it follows by Lemma 3.3 that

$$\lim_{t\to 0^+} \frac{t\big(\int_t^{\delta_0} \frac{k_0(s)}{s}ds\big)'}{\int_t^{\delta_0} \frac{k_0(s)}{s}ds} = -\lim_{t\to 0^+} \frac{k_0(t)}{\int_t^{\delta_0} \frac{k_0(s)}{s}ds} = 0, \tag{3.4}$$

This implies

$$\int_t^{\delta_0} k^2(s)s^{-\gamma}ds \in \mathcal{K}.$$

Moreover, when $2/(2+\gamma) < D_k < 2/(1+\gamma)$, by the proof of (iv), we know that (3.3) holds here. Combining (iii) with Propositions 2.6 and 2.7, we see that there exist $k_i \in \mathcal{K}$, $i = 1, 2, 3, 4$ such that for any $t \in (0, \delta_0]$,

$$t^{\gamma}(k^2(t))^{-1}\int_t^{\delta_0} k^2(s)s^{-\gamma}ds = t^{((\gamma+2)D_k-2)/D_k}k_1(t);$$

$$\begin{aligned} &t^{-\rho_1}\hat{k}(t)(k^2(t))^{-1}t^{q\gamma}\Big[\psi'\Big(t^{1+\gamma}\int_t^{\delta_0} k^2(s)s^{-\gamma}ds\Big)\Big]^{q-1}\Big(\int_t^{\delta_0} k^2(s)s^{-\gamma}ds\Big)^{q}\\ &= t^{\tau_1-\rho_1}\hat{k}(t)k_2(t);\\ &t^{-\rho_1}\hat{k}(t)(k^2(t))^{q-1}t^{q}\Big[\psi'\Big(t^{1+\gamma}\int_t^{\delta_0} k^2(s)s^{-\gamma}ds\Big)\Big]^{q-1} = t^{\tau_2-\rho_1}\hat{k}(t)k_3(t);\end{aligned}$$

$$t^{-\rho_2}\hat{k}(t)\Big[\psi'\Big(t^{1+\gamma}\int_t^{\delta_0} k^2(s)s^{-\gamma}ds\Big)k^2(t)\Big]^{-1} = t^{\tau_3-\rho_2}\hat{k}(t)k_4(t),$$

where

$$\tau_1 = \tau_3 = \frac{D_k(\gamma+2)-2}{D_k} > 0, \quad \tau_2 = \frac{q(2-D_k(1+\gamma)) + D_k(\gamma+2)-2}{D_k} > 0.$$

Hence, (vi) follows by Proposition 2.5 (ii).

(vii) By (iii), we see that there exists $k_1 \in \mathcal{K}$ such that

$$k(t)\psi'\Big(\int_0^t \frac{k(s)}{s}ds\Big) = \frac{k(t)}{\int_0^t \frac{k(s)}{s}ds}\Big(\int_0^t \frac{k(s)}{s}ds\Big)^{1-D_g}k_1\Big(\int_0^t \frac{k(s)}{s}ds\Big),$$

where

$$\int_0^t \frac{k(s)}{s}ds \in \mathcal{K}, \tag{3.5}$$

which can be obtained by a simple calculation as for (3.4).

If $\rho_1 < 2-q$, then by Proposition 2.5 (ii) we have

$$\lim_{t\to 0^+} t^{-\rho_1}\hat{k}_1(t)t^{2-q}k^{q-1}(t)\Big[\psi'\Big(\int_0^t \frac{k(s)}{s}ds\Big)\Big]^{q-1} = 0. \tag{3.6}$$

If $\rho_1 = 2-q$, then it follows by Lemma 3.3 and Proposition 2.5 (ii) that (3.6) holds.

On the other hand, we conclude by Proposition 2.5 (ii) that

$$\lim_{t\to 0^+} t^{2-\rho_2}\hat{k}_2(t)\Big[k(t)\psi'\Big(\int_0^t \frac{k(s)}{s}ds\Big)\Big]^{-1} = 0.$$

□

4. Boundary asymptotic behavior

In this section, we prove Theorems 1.3-1.5. First, we introduce some notations and two significant lemmas, which are necessary for the proofs.

For $\delta > 0$, we define $\Omega_\delta = \{x \in \Omega : d(x) < \delta\}$. Since Ω is a C^2 - smooth domain, we take $\delta_1 \in (0, \delta_0]$ such that

$$d \in C^2(\Omega_{\delta_1}),\ |\nabla d(x)| = 1, \quad \Delta d(x) = -(N-1)H(\bar{x}) + o(1), \quad x \in \Omega_{\delta_1}, \tag{4.1}$$

where, for all $x \in \Omega$ near the boundary of Ω, $\bar{x} \in \partial\Omega$ is the nearest point to x, and $H(\bar{x})$ denotes the mean curvature of $\partial\Omega$ at $\bar{x}$ (please refer to [24, Lemmas 14.6 and 14.7]).

For a satisfies (H1), let $Va \in C^{2,\alpha}_{\rm loc}(\Omega) \cap C(\bar{\Omega})$ be the unique solution to the following problem

$$-\Delta v = a(x),\ v > 0,\ v|_{\partial\Omega} = 0.$$

specifically, if $a \equiv 1$ in Ω, then $V1 \in C^{2,\alpha}_{\rm loc}(\Omega) \cap C^1(\bar{\Omega})$. It follows by Höpf's maximum principle in [24] that

$$\nabla V1(x) \neq 0,\ \forall x \in \partial\Omega \text{ and } V1(x) \approx d(x), \forall x \in \Omega.$$

Lemma 4.1. *For fixed $\lambda \in \mathbb{R}$, let g satisfy* (G1)-(G2)*, b, a and σ satisfy (H1) and $q \in [0, 1]$. Let $u_\lambda \in C^2(\Omega_\delta) \cap C(\bar{\Omega}_\delta)$ be a unique solution to* (1.1)*, $\overline{u}_\lambda \in C^2(\Omega_\delta) \cap C(\bar{\Omega}_\delta)$ satisfy*

$$-\Delta\overline{u}_\lambda \geq b(x)g(\overline{u}_\lambda) + \lambda a(x)|\nabla\overline{u}_\lambda|^q + \sigma(x),\ \overline{u}_\lambda(x) > 0,\ x \in \Omega_\delta,\ \overline{u}_\lambda|_{\partial\Omega} = 0,$$

and $\underline{u}_\lambda \in C^2(\Omega_\delta) \cap C(\bar{\Omega}_\delta)$ satisfy

$$-\Delta\underline{u}_\lambda \leq b(x)g(\underline{u}_\lambda) + \lambda a(x)|\nabla\underline{u}_\lambda|^q + \sigma(x),\ \underline{u}_\lambda(x) > 0,\ x \in \Omega_\delta,\ \underline{u}_\lambda|_{\partial\Omega} = 0,$$

where δ sufficiently small such that $\overline{u}_\lambda(x), \underline{u}_\lambda(x), u_\lambda(x) \in (0, t_1)$, $x \in \Omega_\delta$. The constant $t_1 < t_0$ and t_0 is in (G2).

(I1) *When $q \in [0, 1)$, there exists a positive constant M such that*

$$\underline{u}_\lambda(x) - MVa(x) \leq u_\lambda(x) \leq \overline{u}_\lambda(x) + MVa(x),\ x \in \Omega_\delta. \tag{4.2}$$

(I2) *When $q = 1$, there exists two positive constants M and λ_0 such that if $\lambda \in (-\lambda_0, \lambda_0)$, then* (4.2) *still holds.*

Proof. (I1) When $q \in [0, 1)$, there exists a sufficiently large constant

$$M > \Big(|\lambda| \sup_{x\in\Omega} |\nabla Va(x)|\Big)^{1/(1-q)}$$

such that

$$\underline{u}_\lambda - MVa \leq u_\lambda(x) \leq \overline{u}_\lambda + MVa \quad \text{on } \{x \in \Omega : d(x) = \delta\}. \tag{4.3}$$

We assert that for all $x \in \Omega_\delta$

$$u_\lambda(x) \leq \overline{u}_\lambda(x) + MVa(x), \tag{4.4}$$
$$u_\lambda(x) \geq \underline{u}_\lambda(x) - MVa(x). \tag{4.5}$$

Assume the contrary, there exists $x_0 \in \Omega_\delta$ such that the following hold,

$$u_\lambda(x_0) - (\overline{u}_\lambda(x_0) + MVa(x_0)) > 0.$$

By the continuity of u_λ and $\overline{u}_\lambda$ on Ω_δ and $u_\lambda(x) = \overline{u}_\lambda(x) + MVa(x) = 0$, $x \in \partial\Omega$, we see that there exists $x_1 \in \Omega_\delta$ such that

$$u_\lambda(x_1) - (\overline{u}_\lambda(x_1) + MVa(x_1)) = \max_{x\in\Omega_\delta} u_\lambda(x) - (\overline{u}_\lambda(x) + MVa(x)) > 0.$$

At the point x_1, by using [24, Theorem 2.2] we have

$$\nabla\overline{u}_\lambda - \nabla u_\lambda = M\nabla Va \ \text{ and } \ -\Delta(u_\lambda - (\overline{u}_\lambda + MVa)) \geq 0. \tag{4.6}$$

By using the backward Minkowski inequality, we obtain

$$||\nabla u_\lambda|^q - |\nabla\overline{u}_\lambda|^q| \leq ||\nabla u_\lambda| - |\nabla\overline{u}_\lambda||^q. \tag{4.7}$$

Moreover, combining with (4.6), (4.7) and the basic fact

$$||\nabla u_\lambda| - |\nabla\overline{u}_\lambda||^q \leq |\nabla u_\lambda - \nabla\overline{u}_\lambda|^q,$$

we have
$$||\nabla u_\lambda|^q - |\nabla \overline{u}_\lambda|^q| \le M^q |\nabla Va|^q. \tag{4.8}$$
Thus, it follows by (H1), (G2) and (4.8) that
$$\begin{aligned}
&-\Delta(u_\lambda - (\overline{u}_\lambda + MVa))(x_1)\\
&\le b(x_1)(g(u_\lambda(x_1)) - g(\overline{u}_\lambda(x_1))) - Ma(x) + |\lambda| a(x)\big||\nabla u_\lambda|^q - |\nabla \overline{u}_\lambda|^q\big|\\
&\le b(x_1)(g(u_\lambda(x_1)) - g(\overline{u}_\lambda(x_1))) - Ma(x) + M^q|\lambda| a(x)|\nabla Va|^q < 0,
\end{aligned}$$
which is a contradiction. Hence, (4.4) holds. In the same way, we can show that (4.5) holds.

(I2) When $q = 1$, we can still choose a large $M > 0$ such that (4.3) holds. By the same proof as the above, we obtain that (4.4) and (4.5) hold in the case of
$$|\lambda| < \lambda_0 = \big(\sup_{x\in\Omega} |\nabla Va(x)|\big)^{-1}.$$
□

For the next lemma we assume that a satisfies
$$a(x) \approx (d(x))^{-\rho}\tilde{k}(d(x)), \quad x \in \Omega, \tag{4.9}$$
where $\rho \le 2$ and $\tilde{k} \in C^1((0, l])$ ($l > \max\{\delta_0, \operatorname{diam}(\Omega)\}$) is a positive extension of $k \in \mathcal{K}$, moreover, if $\rho = 2$, then (1.10) holds.

Lemma 4.2 ([32, Proposition 1]). *Assume that a satisfies* (H1) *and* (4.9). *Then $Va(x) \approx \varphi(d(x))$, $x \in \Omega$, where*
$$\varphi(t) := \begin{cases} \int_0^t \frac{\tilde{k}(s)}{s} ds, & \rho = 2;\\ t^{2-\rho}\tilde{k}(t), & 1 < \rho < 2;\\ t\int_t^l \frac{\tilde{k}(s)}{s} ds, & \rho = 1;\\ t, & \rho < 1. \end{cases} \tag{4.10}$$

Proof of Theorem 1.3. Let $\varepsilon \in (0, b_2/2)$ and put
$$\tau_1 = \xi_1 + \varepsilon\xi_1/b_1, \quad \tau_2 = \xi_2 - \varepsilon\xi_2/b_2.$$
We see that
$$\xi_2/2 < \tau_2 < \tau_1 < 3\xi_1/2.$$
Let
$$\overline{u}_\varepsilon = \psi(\tau_1 K^2(d(x))), \quad \underline{u}_\varepsilon = \psi(\tau_2 K^2(d(x))).$$
A straightforward calculation shows that
$$\begin{aligned}
&\Delta\overline{u}_\varepsilon + b(x)g(\overline{u}_\varepsilon) + \lambda a(x)|\nabla\overline{u}_\varepsilon|^q + \sigma(x)\\
&= \psi'(\tau_1K^2(d(x)))k^2(d(x))\Big[4\tau_1\Big(\tau_1K^2(d(x))g'(\psi(\tau_1K^2(d(x)))) + D_g\Big)\\
&\quad + 2\tau_1\Big(\frac{K(d(x))k'(d(x))}{k^2(d(x))} - (1-D_k)\Big) + 2\tau_1\Big(\frac{K(d(x))}{k(d(x))}\Big)\Delta d(x)\\
&\quad + \Big(\frac{b(x)}{k^2(d(x))} - b_1\Big) - 4\tau_1 D_g + 2\tau_1 + 2\tau_1(1-D_k) + b_1\\
&\quad + \lambda a(x)(2\tau_1)^q(\psi'(\tau_1K^2(d(x))))^{q-1}K^q(d(x))k^{q-2}(d(x))\\
&\quad + \sigma(x)\big(\psi'(\tau_1K^2(d(x)))k^2(d(x))\big)^{-1}\Big].
\end{aligned}$$

Combining Lemma 3.1, Lemma 3.4 (v) with the hypotheses (B1), (H2) and (G4), we obtain that for fixed $\varepsilon > 0$, there exists $\delta_\varepsilon \in (0, \delta_1)$ such that for $x \in \Omega_{\delta_\varepsilon}$,

$$\begin{aligned}
&\Big|4\tau_1\Big(\tau_1 K^2(d(x))g'(\psi(\tau_1 K^2(d(x)))) + D_g\Big) \\
&+ 2\tau_1\Big(\frac{K(d(x))k'(d(x))}{k^2(d(x))} - (1-D_k)\Big) + 2\tau_1\Big(\frac{K(d(x))}{k(d(x))}\Big)\Delta d(x) \\
&+ \Big(\frac{\lambda a(x)}{(d(x))^{-\rho_1}\hat{k}_1(d(x))}\Big)(d(x))^{-\rho_1}\hat{k}_1(d(x))(2\tau_1)^q(\psi'(\tau_1 K^2(d(x))))^{q-1}K^q(d(x)) \\
&\times k^{q-2}(d(x)) + \Big(\frac{\sigma(x)}{(d(x))^{-\rho_2}\hat{k}_2(d(x))}\Big)(d(x))^{-\rho_2}\hat{k}_2(d(x)) \\
&\times \big(\psi'(\tau_1 K^2(d(x)))k^2(d(x))\big)^{-1}\Big| \\
&< \varepsilon/2
\end{aligned}$$

and

$$k^2(d(x))(b_2 - \varepsilon/2) < b(x) < k^2(d(x))(b_1 + \varepsilon/2), \quad x \in \Omega_{\delta_\varepsilon}.$$

This implies that for $x \in \Omega_{\delta_\varepsilon}$, we have

$$\Delta\overline{u}_\varepsilon + b(x)g(\overline{u}_\varepsilon) + \lambda a(x)|\nabla\overline{u}_\varepsilon|^q + \sigma(x) \le 0,$$

i.e., $\overline{u}_\varepsilon$ is a supersolution of (1.1) in $\Omega_{\delta_\varepsilon}$.

In a similar way, we can show that $\underline{u}_\varepsilon$ is a subsolution of (1.1) in $\Omega_{\delta_\varepsilon}$.

Let $u_\lambda \in C^{2,\alpha}_{\rm loc}(\Omega) \cap C(\bar{\Omega})$ be the unique solution of (1.1). We choose $\delta < \delta_\varepsilon$ such that $\overline{u}_\varepsilon, \underline{u}_\varepsilon, u_\lambda \in (0, t_1)$, where t_1 is in Lemma 4.1. Now, we consider the following two cases.

Case 1: $q \in [0, 1)$**.** By Lemma 4.1 (I1), we know that there exists $M > 0$ such that

$$\underline{u}_\varepsilon(x) - MVa(x) \le u_\lambda(x) \le \overline{u}_\varepsilon(x) + MVa(x), \quad x \in \Omega_\delta, \tag{4.11}$$

i.e., for any $x \in \Omega_\delta$

$$\begin{gathered}
1 + \frac{MVa(x)}{\psi(\tau_1 K^2(d(x)))} \ge \frac{u_\lambda(x)}{\psi(\tau_1 K^2(d(x)))} \\
1 - \frac{MVa(x)}{\psi(\tau_2 K^2(d(x)))} \le \frac{u_\lambda(x)}{\psi(\tau_2 K^2(d(x)))}.
\end{gathered} \tag{4.12}$$

Subsequently, we prove

$$\lim_{d(x)\to 0} \frac{MVa(x)}{\psi(\tau_i K^2(d(x)))} = 0, \quad i = 1, 2. \tag{4.13}$$

In fact, by (H2) we can take a constant $c_1 > 0$ such that

$$a(x) < w(x),\ x \in \Omega,\ \text{ where } w(x) = c_1(d(x))^{-\rho_1}\tilde{\hat{k}}_1(d(x)) \tag{4.14}$$

with

$$2 - \rho_1 > \frac{q(D_k + 2D_g - 2) + 2(1 - D_g)}{D_k}, \tag{4.15}$$

where $\tilde{\hat{k}}_1 \in C^1((0, l))$ $(l > \max\{\delta_0, \operatorname{diam}(\Omega)\})$ is a positive extension of $\hat{k}_1$.

A basic fact, [24, Theorem 3.1], shows that $Va(x) \le Vw(x)$, $x \in \Omega$. We conclude by Lemma 4.2 that there exists a constant $c_2 > 0$ such that

$$Va(x) \le c_2\varphi(d(x)), \quad x \in \Omega, \tag{4.16}$$

where φ is defined by (4.10). Combining Lemma 3.1 (ii), Lemma 3.4 (ii) with Propositions 2.6 and 2.7, we obtain

$$\varphi \cdot (\psi \circ \tau_i K^2)^{-1} \in NRVZ_\rho,\ i=1,2 \text{ with } \rho = \min\{2-\rho_1,\, 1\} - 2(1-D_g)/D_k.$$

It follows by (1.8) and (4.15) that $\rho > 0$. Hence, we have

$$\lim_{d(x)\to 0} \frac{c_2\varphi(x)}{\psi(\tau_i K^2(d(x)))} = 0.$$

This fact, combined with (4.16), shows that (4.13) holds. It follows by (4.12) that

$$\limsup_{d(x)\to 0} \frac{u_\lambda(x)}{\psi(\tau_1 K^2(d(x)))} \le 1 \quad \text{and} \quad \liminf_{d(x)\to 0} \frac{u_\lambda(x)}{\psi(\tau_2 K^2(d(x)))} \ge 1. \tag{4.17}$$

Consequently, by Lemma 3.4 (ii), we deduce that

$$\begin{aligned} \tau_1^{1-D_g} &= \lim_{d(x)\to 0} \frac{\psi(\tau_1(K^2(d(x))))}{\psi((K^2(d(x))))} \ge \limsup_{d(x)\to 0} \frac{u_\lambda(x)}{\psi(K^2(d(x)))};\\ \tau_2^{1-D_g} &= \lim_{d(x)\to 0} \frac{\psi(\tau_2(K^2(d(x))))}{\psi((K^2(d(x))))} \le \liminf_{d(x)\to 0} \frac{u_\lambda(x)}{\psi(K^2(d(x)))}. \end{aligned} \tag{4.18}$$

Case 2: $p = 1$**.** By Lemma 4.1 (I2), we know that there exist positive constants M, λ_0 such that (4.11) holds here if $\lambda \in (-\lambda_0, \lambda_0)$. As in the proof of the above, we obtain (4.18) still holds.

The proof is complete when passing to the limit as $\varepsilon \to 0$. □

Proof of Theorem 1.4. Let $\varepsilon \in (0, b_2/2)$ and put

$$\tau_1 = \xi_1 + \varepsilon\xi_1/b_1, \quad \tau_2 = \xi_2 - \varepsilon\xi_2/b_2.$$

Clearly, $\xi_2/2 < \tau_2 < \tau_1 < 3\xi_1/2$. Let

$$\overline{u}_\varepsilon(x) = \psi\left(\tau_1(d(x))^{1+\gamma}\theta(x)\right), \quad \underline{u}_\varepsilon(x) = \psi\left(\tau_2(d(x))^{1+\gamma}\theta(x)\right),$$

where $\gamma = D_g/(1-D_g)$ and $\theta(x) = \int_{d(x)}^{\delta} k^2(s)s^{-\gamma}ds$. Denote

$$\begin{aligned} &I(x)\\ &= \tau_1(1+\gamma)^2\Big(\frac{\tau_1(d(x))^{1+\gamma}\theta(x)\psi''\big(\tau_1(d(x))^{1+\gamma}\theta(x)\big)}{\psi'\left(\tau_1(d(x))^{1+\gamma}\theta(x)\right)} + D_g\Big)\\ &\quad\times \frac{(d(x))^{\gamma-1}\theta(x)}{k^2(d(x))} - \tau_1(1+\gamma)^2 E\\ &\quad+ \frac{\tau_1(d(x))^{1+\gamma}\theta(x)\psi''\left(\tau_1(d(x))^{1+\gamma}\theta(x)\right)}{\psi'\left(\tau_1(d(x))^{1+\gamma}\theta(x)\right)} \frac{\tau_1(d(x))^{1-\gamma}k^2(d(x))}{\theta(x)}\\ &\quad- \Big(\frac{2(1+\gamma)\tau_1^2(d(x))^{1+\gamma}\theta(x)\psi''\left(\tau_1(d(x))^{1+\gamma}\theta(x)\right)}{\psi'\left(\tau_1(d(x))^{1+\gamma}\theta(x)\right)} + 2\tau_1\gamma\Big)\\ &\quad- \Big(\frac{2\tau_1 d(x)k'(d(x))}{k(d(x))} - \frac{2\tau_1(1-D_k)}{D_k}\Big) + \Big(\frac{\tau_1(1+\gamma)(d(x))^{\gamma}\theta(x)}{k^2(d(x))} - \tau_1 d(x)\Big)\Delta d(x)\\ &\quad+ \Big(\frac{\lambda\tau_1^q a(x)}{(d(x))^{-\rho_1}\hat{k}_1(d(x))}\Big)(d(x))^{-\rho_1}\hat{k}_1(d(x))(k^2(d(x)))^{q-1}\\ &\quad\times \left|\psi'\left(\tau_1(d(x))^{1+\gamma}\theta(x)\right)\right|^{q-1} \left|\frac{(1+\gamma)(d(x))^{\gamma}\theta(x)}{k^2(d(x))} - d(x)\right|^q \end{aligned}$$

$$+\Big(\frac{\sigma(x)}{(d(x))^{-\rho_2}\hat{k}_2(d(x))}\Big)(d(x))^{-\rho_2}\hat{k}_2(d(x))\big(\psi'\left(\tau_1(d(x))^{1+\gamma}\theta(x)\right)k^2(d(x))\big)^{-1}.$$

Combining Lemma 3.1 (iv) and Lemma 3.4 (iii)-(iv) (vi) with the hypotheses (B1), (H3), (H6) and (G4), we obtain that for fixed $\varepsilon > 0$, there exists $\delta_\varepsilon \in (0, \delta_1)$ such that $|I(x)| < \varepsilon/2$ for $x \in \Omega_{\delta_\varepsilon}$, and

$$k^2(d(x))(b_2 - \varepsilon/2) < b(x) < k^2(d(x))(b_1 + \varepsilon/2), \quad x \in \Omega_{\delta_\varepsilon}.$$

Hence, a straightforward calculation shows that

$$\begin{aligned}
&\Delta\overline{u}_\varepsilon + b(x)g(\overline{u}_\varepsilon) + \lambda a(x)|\nabla\overline{u}_\varepsilon|^q + \sigma(x)\\
&= \psi'\left(\tau_1(d(x))^{1+\gamma}\theta(x)\right)k^2(d(x))\big(I(x) + \tau_1(1+\gamma)^2E\\
&\quad - 2\tau_1(1-D_k)/D_k + 2\tau_1\gamma - \tau_1(1+\gamma) - \tau_1 + b(x)/k^2(d(x))\big)\\
&\le \psi'\left(\tau_1(d(x))^{1+\gamma}\theta(x)\right)k^2(d(x))\big(\varepsilon + \tau_1(1+\gamma)^2E - 2\tau_1(1-D_k)/D_k\\
&\quad + 2\tau_1\gamma - \tau_1(1+\gamma) - \tau_1 + b_1\big) \le 0;
\end{aligned}$$

i.e., $\overline{u}_\varepsilon$ is a supersolution of (1.1) in $\Omega_{\delta_\varepsilon}$.

In a similar way, we can show that $\underline{u}_\varepsilon$ is a subsolution of (1.1) in $\Omega_{\delta_\varepsilon}$.

Let $u_\lambda \in C^{2,\alpha}_{\rm loc}(\Omega) \cap C(\bar{\Omega})$ be the unique solution of (1.1). As before, we choose $\delta < \delta_\varepsilon$ such that $\overline{u}_\varepsilon$, $\underline{u}_\varepsilon$, $u_\lambda \in (0, t_1)$, where t_1 is in Lemma 4.1. Now, we consider the following two cases.

Case 1: $p \in [0, 1)$**.** By Lemma 4.1(I1), we know that there exists $M > 0$ such that (4.11) holds here, i.e., for any $x \in \Omega_\delta$

$$\begin{gathered}
1 + \frac{MVa(x)}{\psi(\tau_1(d(x))^{1+\gamma}\theta(x))} \ge \frac{u_\lambda(x)}{\psi(\tau_1(d(x))^{1+\gamma}\theta(x))};\\
1 - \frac{MVa(x)}{\psi(\tau_2(d(x))^{1+\gamma}\theta(x))} \le \frac{u_\lambda(x)}{\psi(\tau_2(d(x))^{1+\gamma}\theta(x))}.
\end{gathered} \tag{4.19}$$

Subsequently, we prove

$$\lim_{d(x)\to 0}\frac{MVa(x)}{\psi\left(\tau_i(d(x))^{1+\gamma}\theta(x)\right)} = 0, \quad i = 1, 2. \tag{4.20}$$

As before, by (H3), we can take a constant $c_1 > 0$ such that (4.14) holds here with $\rho_1 < 1$. On the other hand, we can also take a constant $c_2 > 0$ such that (4.16) holds here.

By Lemma 3.2 (ii) and the hypotheses on g, we know that there exists $k_1 \in \mathcal{K}$ such that

$$g(t) = t^{-\gamma}k_1(t),\ t \in (0, \delta_0] \text{ and } \liminf_{t\to 0^+} k_1(t) > 0.$$

Therefore, by (1.6) and Proposition 2.8 (i), as $t \to 0^+$, we obtain

$$\psi(t) \cong ((1+\gamma)tk_1(\psi(t)))^{1/(1+\gamma)}.$$

This fact, combined with Lemma 4.2 and (4.16), shows that (4.20) holds. Combining with (4.19), we have

$$\limsup_{d(x)\to 0}\frac{u_\lambda(x)}{\psi(\tau_1(d(x))^{1+\gamma}\theta(x))} \le 1 \quad \text{and} \quad \liminf_{d(x)\to 0}\frac{u_\lambda(x)}{\psi(\tau_2(d(x))^{1+\gamma}\theta(x))} \ge 1.$$

Consequently, by Lemma 3.4 (ii), we deduce that

$$\begin{aligned} \tau_1^{1-D_g} &= \lim_{d(x)\to 0} \frac{\psi(\tau_1(d(x))^{1+\gamma}\theta(x))}{\psi((d(x))^{1+\gamma}\theta(x))} \geq \limsup_{d(x)\to 0} \frac{u_\lambda(x)}{\psi((d(x))^{1+\gamma}\theta(x))}; \\ \tau_2^{1-D_g} &= \lim_{d(x)\to 0} \frac{\psi(\tau_2(d(x))^{1+\gamma}\theta(x))}{\psi((d(x))^{1+\gamma}\theta(x))} \leq \liminf_{d(x)\to 0} \frac{u_\lambda(x)}{\psi((d(x))^{1+\gamma}\theta(x))}. \end{aligned} \tag{4.21}$$

Case 2: $p = 1$**.** By Lemma 4.1 (I2), we know that there exist positive constants M, λ_0 such that (4.11) holds here if $\lambda \in (-\lambda_0, \lambda_0)$. As in the proof of the above, we obtain that (4.21) still holds.

The proof is complete, when passing to the limit $\varepsilon \to 0$. □

Proof of Theorem 1.5. Let $\varepsilon \in (0, b_4/2)$ and put

$$\tau_1 = b_3 + \varepsilon,\ \tau_2 = b_4 - \varepsilon.$$

We see that

$$b_4/2 < \tau_2 < \tau_1 < 3b_3/2.$$

Let

$$\overline{u}_\varepsilon(x) = \psi\Big(\tau_1 \int_0^{d(x)} \frac{k(s)}{s} ds\Big), \quad \underline{u}_\varepsilon(x) = \psi\Big(\tau_2 \int_0^{d(x)} \frac{k(s)}{s} ds\Big).$$

By using Lemma 3.1 (iv), Lemma 3.3 and Lemma 3.4 (vii), combining with the hypotheses (B2), (H4) and (G4) we obtain that for fixed $\varepsilon > 0$, there exists $\delta_\varepsilon \in (0, \delta)$ such that for $x \in \Omega_{\delta_\varepsilon}$

$$\begin{aligned} &\Big|\tau_1^2 k(d(x)) g'(\overline{u}_\varepsilon) + \tau_1\Big(\frac{d(x)k'(d(x))}{k(d(x))} + d(x)\Delta d(x)\Big) + \Big(\frac{\lambda a(x)}{(d(x))^{-\rho_1}\hat{k}_1(d(x))}\Big) \\ &\times (d(x))^{-\rho_1}\hat{k}_1(d(x))\tau_1^q (d(x))^{2-q}(k(d(x)))^{q-1}\Big[\psi'\Big(\tau_1 \int_0^{d(x)} \frac{k(s)}{s} ds\Big)\Big]^{q-1} |\nabla d(x)|^q \\ &+ \Big(\frac{\sigma(x)}{(d(x))^{-\rho_2}\hat{k}_2(d(x))}\Big)(d(x))^{2-\rho_2}\hat{k}_2(d(x))\Big[k(d(x))\psi'\Big(\tau_1 \int_0^{d(x)} \frac{k(s)}{s}\Big)\Big]^{-1}\Big| \\ &< \varepsilon/2 \end{aligned}$$

and

$$(d(x))^{-2}k(d(x))(b_4 - \varepsilon/2) < b(x) < (d(x))^{-2}k(d(x))(b_3 + \varepsilon/2),\ x \in \Omega_{\delta_\varepsilon}$$

Hence, we see that for $x \in \Omega_{\delta_\varepsilon}$

$$\begin{aligned} &\Delta\overline{u}_\varepsilon + b(x)g(\overline{u}_\varepsilon) + \lambda a(x)|\nabla\overline{u}_\varepsilon|^q + \sigma(x) \\ &= (d(x))^{-2}k(d(x))g(\overline{u}_\varepsilon)\Big[\tau_1^2 k(d(x))g'(\overline{u}_\varepsilon) + \tau_1\Big(\frac{d(x)k'(d(x))}{k(d(x))} + d(x)\Delta d(x)\Big) \\ &\quad + \frac{b(x)}{(d(x))^{-2}k(d(x))} - \tau_1 + \lambda a(x)\tau_1^q (d(x))^{2-q}(k(d(x)))^{q-1} \\ &\quad \times \Big[\psi'\Big(\tau_1 \int_0^{d(x)} \frac{k(s)}{s} ds\Big)\Big]^{q-1} |\nabla d(x)|^q + \sigma(x)(d(x))^2\Big[k(d(x))\psi'\Big(\tau_1 \int_0^t \frac{k(s)}{s}\Big)\Big]^{-1}\Big] \\ &\leq (d(x))^{-2}k(d(x))g(\overline{u}_\varepsilon)\big(\varepsilon + b_3 - \tau_1\big) \leq 0, \end{aligned}$$

i.e., $\overline{u}_\varepsilon$ is a supersolution of Eq. (1.1) in $\Omega_{\delta_\varepsilon}$.

In a similar way, we can show that $\underline{u}_\varepsilon$ is a subsolution of Eq. (1.1) in $\Omega_{\delta_\varepsilon}$.

Let $u_\lambda \in C^{2,\alpha}_{\rm loc}(\Omega)\cap C(\bar\Omega)$ be the unique solution of (1.1). We can still choose $\delta < \delta_\varepsilon$ such that $\overline{u}_\varepsilon$, $\underline{u}_\varepsilon$, $u_\lambda \in (0, t_1)$, where t_1 is in Lemma 4.1. As before, we consider the following two cases.

Case 1: $p \in [0,1)$**.** By Lemma 4.1 (I1), we know that there exists $M > 0$ such that (4.11) holds here, i.e., for any $x \in \Omega_\delta$,

$$\begin{gathered} 1+\frac{MVa(x)}{\psi\big(\tau_1\int_0^{d(x)}\frac{k(s)}{s}ds\big)} \ge \frac{u_\lambda(x)}{\psi\big(\tau_1\int_0^{d(x)}\frac{k(s)}{s}ds\big)},\\ 1-\frac{MVa(x)}{\psi\big(\tau_2\int_0^{d(x)}\frac{k(s)}{s}ds\big)} \le \frac{u_\lambda(x)}{\psi\big(\tau_2\int_0^{d(x)}\frac{k(s)}{s}ds\big)}. \end{gathered}$$

Subsequently, we prove

$$\lim_{d(x)\to 0}\frac{MVa(x)}{\psi\big(\tau_i\int_0^{d(x)}\frac{k(s)}{s}ds\big)} = 0, \quad i=1,2. \tag{4.22}$$

By (3.5), Lemma 3.4 (ii) and Proposition 2.7, we can see that

$$\psi\Big(\tau_i\int_0^{d(x)}\frac{k(s)}{s}ds\Big) \in \mathcal{K}, \quad i=1,2.$$

It follows by (H4) and Lemma 4.2 that (4.22) holds. Hence, we have

$$\limsup_{d(x)\to 0}\frac{u_\lambda(x)}{\psi\big(\tau_1\int_0^{d(x)}\frac{k(s)}{s}ds\big)} \le 1 \quad \liminf_{d(x)\to 0}\frac{u_\lambda(x)}{\psi\big(\tau_2\int_0^{d(x)}\frac{k(s)}{s}ds\big)} \ge 1.$$

Consequently, by Lemma 3.4 (ii), we deduce that

$$\begin{gathered} \tau_1^{1-D_g} = \lim_{d(x)\to 0}\frac{\psi\big(\tau_1\int_0^{d(x)}\frac{k(s)}{s}ds\big)}{\psi\big(\int_0^{d(x)}\frac{k(s)}{s}ds\big)} \ge \limsup_{d(x)\to 0}\frac{u_\lambda(x)}{\psi\big(\int_0^{d(x)}\frac{k(s)}{s}ds\big)};\\ \tau_2^{1-D_g} = \lim_{d(x)\to 0}\frac{\psi\big(\tau_2\int_0^{d(x)}\frac{k(s)}{s}ds\big)}{\psi\big(\int_0^{d(x)}\frac{k(s)}{s}ds\big)} \le \liminf_{d(x)\to 0}\frac{u_\lambda(x)}{\psi\big(\int_0^{d(x)}\frac{k(s)}{s}ds\big)}. \end{gathered} \tag{4.23}$$

Case 2: $p = 1$**.** As in the proofs of Theorems 1.3-1.4, there exists a positive constant λ_0 such that if $\lambda \in (-\lambda_0, \lambda_0)$, then (4.23) still holds. The proof is complete when passing to the limit $\varepsilon \to 0$.

5. Existence and global asymptotic behavior

In this section, we prove Theorems 1.8-1.10.

Proof of Theorem 1.8. Our proof is done in the following two steps.
Step 1 (Existence and global behavior) For $0 < c < \delta_0$, we define

$$\begin{gathered} Q_1(c,x) = -4\Big(K^2(c\,\varphi_1(x))g'((K^2(c\,\varphi_1(x)))) + D_g\Big) + 2(D_k + 2D_g - 2)\\ -2\Big(\frac{K(c\,\varphi_1(x))k'(c\,\varphi_1(x))}{k^2(c\varphi_1(x))} - (1-D_k)\Big),\ x\in\Omega;\\ Q_2(c,x) = c\psi'(K^2(c\varphi_1(x)))K(c\,\varphi_1(x))k(c\varphi_1(x)), \quad x\in\Omega. \end{gathered}$$

By (G4) and Lemma 3.1 (iii), we can take a small enough $0 < c_0 < \delta_0$ such that for $x \in \Omega$,

$$D_k + 2D_g - 2 \le Q_1(c_0, x) \le 4(D_k + 2D_g - 2).$$

Let
$$\overline{u}_\lambda := Ma_1(c_0)\psi(K^2(c_0\varphi_1)), \text{ in } \Omega,$$
where M is a positive constant to be determined.

First, by choosing a suitable M, we prove that $\overline{u}_\lambda$ is a supersolution of (1.1). Indeed, a straightforward calculation shows that
$$\begin{aligned}-\Delta\overline{u}_\lambda &= Ma_1(c_0)c_0^2 g(\psi((K^2(c_0\varphi_1))))k^2(c_0\varphi_1)Q_1(c_0,\cdot)|\nabla\varphi_1|^2\\ &\quad + 2M\lambda_1\varphi_1 a_1(c_0)Q_2(c_0,\cdot)\\ &\geq MI + 2M\lambda_1\varphi_1 a_1(c_0)Q_2(c_0,\cdot),\end{aligned}$$
where
$$I = a_1(c_0)c_0^2(D_k+2D_g-2)\, g(\psi(K^2(c_0\varphi_1)))k^2(c_0\varphi_1)|\nabla\varphi_1|^2.$$
By Hopf's maximum principle, there exist $\omega \Subset \Omega$ and a constant $\delta_1 > 0$ such that
$$|\nabla\varphi_1|^2 \geq \delta_1, \quad \text{in } \Omega\setminus\omega.$$
Put
$$M \geq \max\big\{2/(c_0^2\delta_1(D_k+2D_g-2)),\, 1/a_1(c_0)\big\}.$$
Combining with (B3) and (G3), we derive that for $x \in \Omega\setminus\omega$,
$$MI(x)/2 \geq b(x)g(Ma_1(c_0)\psi(K^2(c_0\varphi_1(x)))). \tag{5.1}$$
On the other hand, by (H2), (1.12), Proposition 2.3 and Lemma 3.4 (v) we see that
$$\begin{aligned}\lim_{d(x)\to 0}\Big[&\Big(\frac{a(x)}{(d(x))^{-\rho_1}\hat{k}_1(d(x))}\Big)(d(x))^{-\rho_1}\hat{k}_1(d(x))(\psi'(K^2(c_0\varphi_1(x))))^{q-1}\\ &\times K^q(c_0\varphi_1(x))k^{q-2}(c_0\varphi_1(x)) + \Big(\frac{\sigma(x)}{(d(x))^{-\rho_2}\hat{k}_2(d(x))}\Big)(d(x))^{-\rho_2}\\ &\times \hat{k}_2(d(x))\big(\psi'(K^2(c_0\varphi_1(x)))k^2(c_0\varphi_1(x))\big)^{-1}\Big] = 0.\end{aligned}\tag{5.2}$$
Hence, there exists $\omega' \Subset \Omega$ satisfying $\omega \Subset \omega'$ and $\mathrm{dist}(\omega',\partial\Omega) < \delta_0$ such that for $x \in \Omega\setminus\omega'$,
$$\begin{aligned}&MI(x)/2\\ &= k^2(c_0\varphi_1(x))\psi'(K^2(c_0\varphi_1(x)))\big(a_1(c_0)c_0^2(D_k+2D_g-2)|\nabla\varphi_1|^2/2\big)\\ &\geq k^2(c_0\varphi_1(x))\psi'(K^2(c_0\varphi_1(x)))\Big[\lambda(2Ma_1(c_0)c_0)^q|\nabla\varphi_1(x)|^q\Big(\frac{a(x)}{(d(x))^{-\rho_1}\hat{k}_1(d(x))}\Big)\\ &\quad\times (d(x))^{-\rho_1}\hat{k}_1(d(x))(\psi'(K^2(c_0\varphi_1(x))))^{q-1}K^q(c_0\varphi_1(x))k^{q-2}(c_0\varphi_1(x))\\ &\quad + \Big(\frac{\sigma(x)}{(d(x))^{-\rho_2}\hat{k}_2(d(x))}\Big)(d(x))^{-\rho_2}\hat{k}_2(d(x))\big(k^2(c_0\varphi_1(x))\psi'(K^2(c_0\varphi_1(x)))\big)^{-1}\Big]\\ &= \lambda a(x)(2Ma_1(c_0))^q Q_2^q(c_0,x)|\nabla\varphi_1(x)|^q + \sigma(x).\end{aligned}\tag{5.3}$$
This together with (5.1) implies that $\overline{u}_\lambda$ is a supersolution of (1.1) in $\Omega\setminus\omega'$.

Now, by taking a suitable constant $M > 0$, we prove $\overline{u}_\lambda$ is a supersolution of Eq. (1.1) in ω'. Define
$$m_1 := \sup_{x\in\omega'} b(x)g(\psi(K^2(c_0\varphi_1(x)))); \quad m_2 := \inf_{x\in\omega'} a_1(c_0)\varphi_1(x)Q_2(c_0,x);$$
$$m_3 := \sup_{x\in\omega'} a(x)(a_1(c_0))^q Q_2^q(c_0,x)|\nabla\varphi_1(x)|^q; \quad m_4 := \sup_{x\in\omega'}\sigma(x).$$

Let

$$M \geq \max\big\{m_1/m_2\lambda_1,\ 1/a_1(c_0)\big\}.$$

It follows from the monotonicity of g that for any $x \in \omega'$

$$M\lambda_1 a_1(c_0)\varphi_1(x)Q_2(c_0,x) \geq M\lambda_1 m_2 \geq m_1 \geq b(x)g(Ma_1(c_0)\psi(K^2(c_0\varphi_1(x)))). \tag{5.4}$$

Here, we distinguish the following two cases.

Case 1: $q \in [0,1)$**.** Let

$$M > \max\big\{\big(2^{q+1}m_3\max\{0,\lambda\}/m_2\lambda_1\big)^{1/(1-q)},\, 2m_4/\lambda_1 m_2\big\}.$$

By a direct calculation, we have for any $x \in \omega'$,

$$\begin{aligned} &M\lambda_1 a_1(c_0)\varphi_1(x)Q_2(c_0,x)/2\\ &\geq M\lambda_1 m_2/2 \geq m_3(2M)^q\max\{0,\lambda\}\\ &\geq \lambda a(x)(2Ma_1(c_0))^q Q_2^q(c_0,x)|\nabla\varphi_1(x)|^q \end{aligned} \tag{5.5}$$

and

$$M\lambda_1 a_1(c_0)\varphi_1(x)Q_2(c_0,x)/2 \geq M\lambda_1 m_2/2 \geq m_4 \geq \sigma(x). \tag{5.6}$$

So, we see that $\overline{u}_\lambda$ is a supersolution of (1.1) in ω'. Finally, combining with (5.1), (5.3)-(5.6), we conclude by choosing

$$\begin{aligned} M \geq \max\big\{&2/(c_0^2\delta_1(D_k+2D_g-2)),\, 1/a_1(c_0),\, m_1/(m_2\lambda_1),\\ &\big(2^{q+1}m_3\max\{0,\lambda\})/m_2\lambda_1\big)^{1/(1-q)},\, 2m_4/\lambda_1 m_2\big\} \end{aligned}$$

that $\overline{u}_\lambda$ is a supersolution of (1.1).

Case 2: $q \in [1,2]$**.** In this case, let

$$\begin{gathered} M \geq \max\big\{2/(c_0^2\delta_1(D_k+2D_g-2)),\, 1/a_1(c_0),\, m_1/(m_2\lambda_1),\, 2m_4/\lambda_1 m_2\big\},\\ \lambda < M^{1-q}\lambda_1 m_2/(2^{q+1}m_3). \end{gathered} \tag{5.7}$$

By the same argument as Case 1, we obtain that (5.5)-(5.6) still hold. So, for every λ satisfying (5.7), we can take $M > 0$ such that $\overline{u}_\lambda$ is a supersolution of (1.1).

On the other hand, let

$$\underline{u}_\lambda := ma_2(c_0)\psi(K^2(c_0\varphi_1)) \quad \text{in } \Omega,$$

where m is a positive constant to be determined.

Next, by choosing a suitable $m > 0$, we prove $\underline{u}_\lambda$ is a subsolution of (1.1). Indeed, by (5.2), we arrive at

$$\sup_{x\in\Omega} a(x)(\psi'(K^2(c_0\varphi_1(x))))^{q-1}K^q(c_0\varphi_1(x))k^{q-2}(c_0\varphi_1(x)) < \infty.$$

Hence, we can take sufficiently small $0 < m < \min\{M,\, 1/a_2(c_0)\}$ such that for each $q \in (0,2]$

$$\begin{aligned} &-\Delta\underline{u}_\lambda - \lambda a(x)|\nabla\underline{u}_\lambda|^q - \sigma(x)\\ &\leq a_2(c_0)k^2(c_0\varphi_1)\psi'(K^2(c_0\varphi_1))\Big(mc_0^2Q_1(c_0,\cdot)|\nabla\varphi_1|^2 + 2m\lambda_1\varphi_1 K(c_0\varphi_1)/k(c_0\varphi_1)\\ &\quad + |\lambda|(2mc_0)^q(a_2(c_0))^{q-1}a(x)(\psi'(K^2(c_0\varphi_1)))^{q-1}K^q(c_0\varphi_1)k^{q-2}(c_0\varphi_1)|\nabla\varphi_1|^q\Big)\\ &\leq b(x)\psi'(K^2(c_0\varphi_1))\Big(4mc_0^2(D_k+2D_g-2)\sup_{x\in\Omega}|\nabla\varphi_1(x)|^2\\ &\quad + \sup_{x\in\Omega} 2mc_0\lambda_1\big(K(c_0\varphi_1(x))/k(c_0\varphi_1(x))\big) + |\lambda|(2mc_0)^q(a_2(c_0))^{q-1} \end{aligned}$$

$$\times \sup_{x\in\Omega} a(x)(\psi'(K^2(c_0\varphi_1(x))))^{q-1}K^q(c_0\varphi_1(x))k^{q-2}(c_0\varphi_1(x))|\nabla\varphi_1(x)|^q\Big)$$
$$\le b(x)g(\psi(K^2(c_0\varphi_1)))$$
$$\le b(x)g(ma_2(c_0)\psi(K^2(c_0\varphi_1))).$$

This implies that $\underline{u}_\lambda$ is a subsolution of (1.1).

It follows by [15, Lemma 3] that (1.1) possesses a classical solution u_λ satisfying

$$\underline{u}_\lambda \le u_\lambda \le \overline{u}_\lambda \text{ in } \Omega;$$

i.e.,

$$u_\lambda(x) \approx \psi(K^2(c_0\varphi_1(x))),\ x \in \Omega.$$

Let $\tilde{k} \in C^1((0,\infty))$ is a positive extension of $k \in C^1((0,\delta_0])$ and $\tilde{K}(t) = \int_0^t \tilde{k}(s)ds,\ t > 0$. By Lemma 3.1 (ii), Lemma 3.4 (ii) and Proposition 2.6, we have

$$\psi \circ \tilde{K}^2 \in NRVZ_{2(1-D_g)/D_k}.$$

Since $\psi \circ \tilde{K}^2 \in C^1((0,\infty))$, we can take a positive constant $\delta < \min\{\delta_0, \mathrm{diam}(\Omega)\}$ and a function $y \in C((0,\delta])$ with $\lim_{t\to 0^+} y(t) = 0$ such that

$$\psi \circ \tilde{K}^2(t) = \psi \circ K^2(t) = \bar{c}\, t^{2(1-D_g)/D_k} \exp\Big(\int_t^\delta \frac{y(s)}{s}ds\Big), \quad t \in (0,\delta],\ \bar{c} > 0.$$

On the other hand, by (1.12), we obtain that there exists $c_1 > 1$ such that

$$d(x)/c_1 \le \varphi_1(x) \le c_1 d(x),\ x \in \Omega.$$

In fact, we can adjust $c_0 > 0$ such that $c_0 < \min\{\delta, 1/c_1\}$.

Let $\beta = \max_{t\in(0,\delta]} |y(t)|$. Then we deduce that

$$\Big|\exp\Big(\int_{d(x)}^{c_0\varphi_1(x)} \frac{y(s)}{s}ds\Big)\Big| \le (c_1/c_0)^\beta,\ x \in \Omega_\delta.$$

Hence

$$(c_0/c_1)^{(2(1-D_g)/D_k)+\beta} \le \psi\circ K^2(c_0\varphi_1(x))/\psi\circ\tilde{K}^2(d(x)) \le (c_1c_0)^{2(1-D_g)/D_k}(c_1/c_0)^\beta,$$

for $x \in \Omega_\delta$. Let

$$M_1 = \max\Big\{(c_1c_0)^{2(1-D_g)/D_k}(c_1/c_0)^\beta,\ \sup_{x\in\Omega\setminus\Omega_\delta} \psi\circ K^2(c_0\varphi_1(x))/\inf_{x\in\Omega\setminus\Omega_\delta} \psi\circ\tilde{K}^2(d(x))\Big\},$$

$$M_2 = \min\big\{(c_0/c_1)^{(2(1-D_g)/D_k)+\beta},\ \inf_{x\in\Omega\setminus\Omega_\delta} \psi\circ K^2(c_0\varphi_1(x))/\sup_{x\in\Omega\setminus\Omega_\delta} \psi\circ\tilde{K}^2(d(x))\big\}.$$

Then we obtain that for $x \in \Omega$,

$$M_2\psi\circ\tilde{K}^2(d(x)) \le \psi\circ K^2(c_0\varphi_1(x)) \le M_1\psi\circ\tilde{K}^2(d(x)),$$

i.e., (1.22) holds.

Step 2 (Uniqueness) Since uniqueness is an easy consequence of the relationship $v \le w$ whenever $v \le w$ on $\partial\Omega$, we prove only this relationship, where v and w are two solutions of (1.1) in Ω. Suppose $\min_{x\in\Omega}(w(x)-v(x)) < 0$, then there exists $x_0 \in \Omega$ such that $w(x_0) - v(x_0) = \min_{x\in\Omega}(w(x)-v(x))$. At the point, we have by the basic fact

$$\nabla(w-v) = 0 \text{ and } -\Delta(w-v) \le 0.$$

On the other hand, we see by (H1) and (G3) that

$$-\Delta(w-u) = b(x_0)(g(w(x_0)) - g(v(x_0)))) > 0,$$

which is a contradiction. Hence, $w \geq v$ in Ω. The proof is complete $\square$

Proof of Theorem 1.9. For $0 < c < \delta_0$, let

$$\theta(c,x) = \int_{c\varphi_1(x)}^{\delta_0} k^2(s)s^{-\gamma}ds$$

and define

$$\begin{aligned}
&Q_1(c,x)\\
&= -\Big(\frac{(c(1+\gamma))^2(c\varphi_1(x))^{1+\gamma}\theta(c,x)\psi''((c\varphi_1(x))^{1+\gamma}\theta(c,x))}{\psi'((c\varphi_1(x))^{1+\gamma}\theta(c,x))} + c^2(1+\gamma)\gamma\Big)\\
&\quad\times \frac{(c\varphi_1(x))^{\gamma-1}\theta(c,x)}{k^2(c\varphi_1(x))} - \frac{(c\varphi_1(x))^{1+\gamma}\theta(c,x)\psi''((c\varphi_1(x))^{1+\gamma}\theta(c,x))}{\psi'((c\varphi_1(x))^{1+\gamma}\theta(c,x))}\\
&\quad\times \frac{(c\varphi_1(x))^{1-\gamma}k^2(c\varphi_1(x))c^2}{\theta(c,x)}\\
&\quad+ \frac{2c^2(1+\gamma)(c\varphi_1(x))^{1+\gamma}\theta(c,x)\psi''((c\varphi_1(x))^{1+\gamma}\theta(c,x))}{\psi'((c\varphi_1(x))^{1+\gamma}\theta(c,x))}\\
&\quad+ (1+\gamma)c^2 + \frac{2k'(c\varphi_1(x))c\varphi_1(x)c^2}{k(c\varphi_1(x))} + c^2;
\end{aligned}$$

$$\begin{aligned}
Q_2(c,x) &= k^2(c\varphi_1(x))\psi'((c\varphi_1(x))^{1+\gamma}\theta(c,x))\\
&\quad\times \lambda_1\Big(\frac{(1+\gamma)(c\varphi_1(x))^{1+\gamma}\theta(c,x)}{k^2(c\varphi_1(x))} - (c\varphi_1(x))^2\Big);\\
Q_3(c,x) &= \Big|a_1(c)\psi'((c\varphi_1(x))^{1+\gamma}\theta(c,x))\Big(c(1+\gamma)(c\varphi_1(x))^{\gamma}\theta(c,x)\\
&\quad - c(c\varphi_1(x))k^2(c\varphi_1(x))\Big)\Big||\nabla\varphi_1(x)|.
\end{aligned}$$

By (H6), Lemmas 3.1 (iv) and 3.4 (iii)-(iv), we can take a sufficiently small $0 < c_0 < \delta_0$ such that for $x \in \Omega$,

$$((2-D_k\gamma-(1+\gamma)^2ED_k)c_0^2)/2D_k < Q_1(c_0,x) < ((2-D_k\gamma-(1+\gamma)^2ED_k)3c_0^2)/2D_k$$

and $Q_2(c_0,x)$, $Q_3(c_0,x) > 0$.

Let

$$\overline{u}_\lambda := Ma_1(c_0)\psi((c_0\varphi_1)^{1+\gamma}\theta(c_0,\cdot)) \quad \text{in } \Omega,$$

where M is a positive constant to be determined. As before, by choosing a suitable constant $M > 0$, we prove $\overline{u}_\lambda$ is a supersolution of (1.1).

By a straightforward calculation,

$$\begin{aligned}
-\Delta\overline{u}_\lambda &= Ma_1(c_0)k^2(c_0\varphi_1)\psi'((c_0\varphi_1)^{1+\gamma}\theta(c_0,\cdot))Q_1(c_0,\cdot)|\nabla\varphi_1|^2 + Ma_1(c_0)Q_2(c_0,\cdot)\\
&\geq MI + Ma_1(c_0)Q_2(c_0,\cdot),
\end{aligned}$$

where

$$I = (1/2D_k)(2-D_k\gamma-(1+\gamma)^2ED_k)c_0^2a_1(c_0)k^2(c_0\varphi_1)\psi'((c_0\varphi_1)^{1+\gamma}\theta(c_0,\cdot))|\nabla\varphi_1|^2.$$

By Hopf's maximum principle, there exist $\omega \Subset \Omega$ and a constant $\delta_1 > 0$ such that

$$|\nabla\varphi_1| \geq \delta_1, \text{ in } \Omega \setminus \omega.$$

Let

$$M > \max\big\{4D_k/((2-D_k\gamma-(1+\gamma)^2ED_k)\delta_1c_0^2),\, 1/a_1(c_0)\big\}.$$

Combining with (B3) and (G3), for $x \in \Omega \setminus \omega$, we obtain

$$MI(x)/2 \geq b(x)g(Ma_1(c_0)\psi((c_0\varphi_1(x))^{1+\gamma}\theta(c_0,x))). \tag{5.8}$$

On the other hand, by (H3), (1.12), Proposition 2.3 and Lemma 3.4 (vi) we see that

$$\begin{aligned}
&\lim_{d(x)\to 0}\Big[\Big(\frac{a(x)}{(d(x))^{-\rho_1}\hat{k}_1(d(x))}\Big)(d(x))^{-\rho_1}\hat{k}_1(d(x))\Big((k^2(c_0\varphi_1(x)))^{-1}(c_0\varphi_1(x))^{q\gamma}\\
&\times(\psi'((c_0\varphi_1(x))^{1+\gamma}\theta(c_0,x)))^{q-1}\theta^q(c_0,x)+(k^2(c_0\varphi_1(x)))^{q-1}(c_0\varphi_1(x))^q\\
&\times(\psi'((c_0\varphi_1(x))^{1+\gamma}\theta(c_0,x)))^{q-1}\Big)+\Big(\frac{\sigma(x)}{(d(x))^{-\rho_2}\hat{k}_2(d(x))}\Big)(d(x))^{-\rho_2}\hat{k}_2(d(x))\\
&\times\big(\psi'((c_0\varphi_1(x))^{1+\gamma}\theta(c_0,x))k^2(c_0\varphi_1(x))\big)^{-1}\Big]=0.
\end{aligned} \tag{5.9}$$

Hence, there exists $\omega' \Subset \Omega$ satisfying $\omega \Subset \omega'$ and $\operatorname{dist}(\omega', \partial\Omega) < \delta_0$ such that for $x \in \Omega \setminus \omega'$,

$$\begin{aligned}
MI(x)/2 &\geq k^2(c_0\varphi_1(x))\psi'((c_0\varphi_1(x))^{1+\gamma}\theta(c_0,x))\Big[\lambda M^q(a_1(c_0))^q|\nabla\varphi_1(x)|^q\\
&\quad\times\Big(\frac{a(x)}{(d(x))^{-\rho_1}\hat{k}_1(d(x))}\Big)(d(x))^{-\rho_1}\hat{k}_1(d(x))\Big((2c_0(1+\gamma))^q(k^2(c_0\varphi_1(x)))^{-1}\\
&\quad\times(c_0\varphi_1(x))^{q\gamma}(\psi'((c_0\varphi_1(x))^{1+\gamma}\theta(c_0,x)))^{q-1}\theta^q(c_0,x)\\
&\qquad+(2c_0)^q(k^2(c_0\varphi_1(x)))^{q-1}(c_0\varphi_1(x))^q\\
&\quad\times(\psi'((c_0\varphi_1(x))^{1+\gamma}\theta(c_0,x)))^{q-1}\Big)\\
&\quad+\Big(\frac{\sigma(x)}{(d(x))^{-\rho_2}\hat{k}_2(d(x))}\Big)(d(x))^{-\rho_2}\hat{k}_2(d(x))\big(k^2(c_0\varphi_1(x))\\
&\qquad\times\psi'((c_0\varphi_1(x))^{1+\gamma}\theta(c_0,x))\big)^{-1}\Big]\\
&\geq \lambda a(x)M^qQ_3^q(c_0,x)+\sigma(x).
\end{aligned} \tag{5.10}$$

This fact, combined with (5.8), shows that $\overline{u}_\lambda$ is a supersolution of Eq. (1.1) in $\Omega \setminus \omega'$.

Now, by taking a suitable constant $M > 0$, we prove $\overline{u}_\lambda$ is a supersolution of (1.1) in ω'.

As before, we define

$$m_1 := \sup_{x\in\omega'} b(x)g(\psi((c_0\varphi_1(x))^{1+\gamma}\theta(c_0,x))); \quad m_2 := \inf_{x\in\omega'}(a_1(c_0)/2)Q_2(c_0,x);$$
$$m_3 := \sup_{x\in\omega'} a(x)Q_3^q(c_0,x); \quad m_4 := \sup_{x\in\omega'}\sigma(x).$$

Let $M > \max\big\{m_1/m_2,\, 1/a_1(c_0)\big\}$. Using the monotonicity of g, we obtain that

$$\begin{aligned} M(a_1(c_0)/2)Q_2(c_0,x) &\geq Mm_2 \geq m_1 \\ &\geq b(x)g(Ma_1(c_0)\psi((c_0\varphi_1(x))^{1+\gamma}\theta(c_0,x))), \quad x \in \omega'. \end{aligned} \tag{5.11}$$

Here, we distinguish the following cases.

Case 1: $q \in [0,1)$**.** Put

$$M > \max\big\{\big(2m_3\max\{0,\lambda\}/m_2\big)^{1/(1-q)},\, 2m_4/m_2\big\}.$$

We obtain

$$M(a_1(c_0)/4)Q_2(c_0,x) \geq Mm_2/2 \geq \max\{0,\lambda\}M^q m_3 \geq \lambda M^q a(x)Q_3^q(c_0,x), \tag{5.12}$$

for $x \in \omega'$, and

$$M(a_1(c_0)/4)Q_2(c_0,x) \geq Mm_2/2 \geq m_4 \geq \sigma(x), \quad x \in \omega'. \tag{5.13}$$

Thus, $\overline{u}_\lambda$ is a supersolution of (1.1) in ω'. Finally, combining with (5.8), (5.10)-(5.13), we conclude by choosing

$$\begin{aligned} M > \max\big\{&4D_k/((2-D_k\gamma-(1+\gamma)^2ED_k)\delta_1c_0^2),\, 1/a_1(c_0),\, m_1/m_2, \\ &\big(2m_3\max\{0,\lambda\}/m_2\big)^{1/(1-q)},\, 2m_4/m_2\big\} \end{aligned}$$

that $\overline{u}_\lambda$ is a supersolution of (1.1).

Case 2: $q \in [1,2]$**.** Put

$$\begin{gathered} M > \max\big\{4D_k/((2-D_k\gamma-(1+\gamma)^2ED_k)\delta_1c_0^2),\, 1/a_1(c_0),\, m_1/m_2,\, 2m_4/m_2\big\}, \\ \lambda < (M^{1-q}m_2)/2m_3. \end{gathered} \tag{5.14}$$

It follows by a direct calculation that (5.12)-(5.13) still hold. So, for every λ satisfying (5.14), we can take $M > 0$ such that $\overline{u}_\lambda$ is a supersolution of (1.1).

On the other hand, let

$$\underline{u}_\lambda := ma_2(c_0)\psi((c_0\varphi_1)^{1+\gamma}\theta(c_0,\cdot)) \quad \text{in } \Omega,$$

where m is a positive constant to be determined.

Next, by choosing a suitable $m > 0$, we prove that $\underline{u}_\lambda$ is a subsolution of (1.1). By (5.9), we arrive at

$$\begin{aligned} &\sup_{x\in\Omega}\big[a(x)(\psi'((c_0\varphi_1(x))^{1+\gamma}\theta(c_0,x)))^{q-1}(k^2(c_0\varphi_1(x)))^{-1} \\ &\times\big|c_0(1+\gamma)(c_0\varphi_1(x))^\gamma\theta(c_0,x) - c_0(c_0\varphi_1(x))k^2(c_0\varphi_1(x))\big|^q\big] < \infty. \end{aligned}$$

Moreover, by Lemma 3.4 (vi), we obtain

$$\sup_{x\in\Omega}\big[(c_0\varphi_1(x))^\gamma(k^2(c_0\varphi_1(x)))^{-1}\theta(c_0,x)\big] < \infty.$$

Using a similar proof as for Theorem 1.8, we can take a small enough $0 < m < \min\{M,\, 1/a_2(c_0)\}$ such that for any $q \in (0,2]$

$$\begin{aligned} &-\Delta\underline{u}_\lambda(x) - \lambda a(x)|\nabla\underline{u}_\lambda(x)|^q - \sigma(x) \\ &\leq a_2(c_0)k^2(c_0\varphi_1)\psi'((c_0\varphi_1)^{1+\gamma}\theta(c_0,\cdot))\Big[m|\nabla\varphi_1|^2((2-D_k\gamma \\ &\quad -(1+\gamma)^2ED_k)3c_0^2)/2D_k + m\lambda_1\Big(\frac{(1+\gamma)(c_0\varphi_1(x))^{1+\gamma}\theta(c_0,x)}{k^2(c_0\varphi_1(x))} - (c_0\varphi_1(x))^2\Big) \\ &\quad + |\lambda|m^q(a_2(c_0))^{q-1}(\psi'((c_0\varphi_1(x))^{1+\gamma}\theta(c_0,x)))^{q-1}(k^2(c_0\varphi_1(x)))^{-1} \end{aligned}$$

$$\times \Big|c_0(1+\gamma)(c_0\varphi_1(x))^\gamma\theta(c_0,x) - c_0(c_0\varphi_1(x))k^2(c_0\varphi_1(x))\Big|^q\Big]$$
$$\le b(x)g(\psi(k^2(c_0\varphi_1(x))\theta(c_0,x)))$$
$$\le b(x)g(ma_2(c_0)\psi(k^2(c_0\varphi_1(x))\theta(c_0,x))).$$

Hence, by [15, Lemma 3], problem (1.1) has a classical solution u_λ satisfying

$$\underline{u}_\lambda \le u_\lambda \le \overline{u}_\lambda \quad \text{in } \Omega,$$

i.e.,

$$u_\lambda(x) \approx \psi((c_0\varphi_1)^{1+\gamma}\theta(c_0,x)),\ x\in\Omega.$$

Since the function $t\mapsto \psi(t^{1+\gamma}\int_t^{\delta_0} k^2(s)s^{-\gamma}ds)$ belongs to $NRVZ_1\cap C^1((0,\delta_0])$, we can take $\delta>0$ satisfying

$$c_0<\delta<\min\big\{\delta_0,\ \operatorname{diam}(\Omega)\big\}$$

and a function $y\in C((0,\delta])$ with $\lim_{t\to 0^+} y(t)=0$, such that

$$\psi\Big((c_0\varphi_1(x))^{1+\gamma}\int_{c_0\varphi_1(x)}^{\delta} k^2(s)s^{-\gamma}ds\Big) = \bar c\, c_0\varphi_1(x)\int_{c_0\varphi_1(x)}^{\delta}\frac{y(s)}{s}ds,\ x\in\Omega,\ \bar c>0.$$

Let $\widetilde{k}\in C^1((0,l))$ be a positive extension of $k\in C^1((0,\delta_0])$. As in the proof of Theorem 1.8, we can take $M_1>M_2>0$ such that

$$M_2\psi\Big((d(x))^{1+\gamma}\int_{d(x)}^{l}\tilde k(s)s^{-\gamma}ds\Big) \le \psi\Big((c_0\varphi_1(x))^{1+\gamma}\theta(c_0,x)\Big)$$
$$\le M_1\psi\Big((d(x))^{1+\gamma}\int_{d(x)}^{l}\tilde k(s)s^{-\gamma}ds\Big).$$

The proof is complete.

Proof of Theorem 1.10. We note that this proof is essentially the same as the proofs of Theorems 1.8 and 1.9, so we only provide an outline. For $0<c<\delta_0$, we define

$$Q_1(c,x) = -\frac{\psi''\big(\int_0^{c\varphi_1(x)}\frac{k(s)}{s}ds\big)k(c\varphi_1(x))}{\psi'\big(\int_0^{c\varphi_1(x)}\frac{k(s)}{s}ds\big)} - \frac{c\varphi_1 k'(c\varphi_1)}{k(c\varphi_1)} + 1;$$
$$Q_2(c,x) = \lambda_1\psi'\Big(\int_0^{c\varphi_1(x)}\frac{k(s)}{s}ds\Big)k(c\varphi_1(x));$$
$$Q_3(c,x) = \Big|a_1(c)\psi'\Big(\int_0^{c\varphi_1(x)}\frac{k(s)}{s}ds\Big)\frac{k(c\varphi_1(x))}{\varphi_1(x)}\Big||\nabla\varphi_1|.$$

As before, by Lemmas 3.1 (iv), 3.3 and 3.4 (iii), we can take a small enough $c_0>0$ such that

$$1/2 < Q_1(c_0,x) < 3/2.$$

Define

$$\overline{u}_\lambda(x) := Ma_1(c_0)\psi\Big(\int_0^{c_0\varphi_1(x)}\frac{k(s)}{s}ds\Big),\ x\in\Omega,$$

where M is a positive constant to be determined.

A straightforward calculation shows that

$$-\Delta\overline{u}_\lambda = Ma_1(c_0)\psi'\Big(\int_0^{c_0\varphi_1}\frac{k(s)}{s}ds\Big)(c_0\varphi_1)^{-2}k(c_0\varphi_1)c_0^2Q_1(c_0,\cdot)|\nabla\varphi_1|^2$$
$$+ Ma_1(c_0)Q_2(c_0,\cdot)$$

$$\geq MI + Ma_1(c_0)Q_2(c_0,\cdot),$$

where

$$I = (1/2)a_1(c_0)\psi'\Big(\int_0^{c_0\varphi_1} \frac{k(s)}{s}ds\Big)(c_0\varphi_1)^{-2}k(c_0\varphi_1)c_0^2.$$

By Hopf's maximum principle, there exists $\omega \Subset \Omega$ and a constant $\delta_1 > 0$ such that

$$|\nabla\varphi_1| \geq \delta_1, \text{ in } \Omega\setminus\omega.$$

Let

$$M > \big\{4/(c_0^2\delta_1),\, 1/a_1(c_0)\big\}.$$

Combining with (B4) and (G3), we have for any $x \in \Omega\setminus\omega$

$$\begin{aligned} MI(x)/2 &\geq a_1(c_0)(c_0\varphi_1)^{-2}k(c_0\varphi_1(x))g\Big(\psi\Big(\int_0^{c_0\varphi_1(x)} \frac{k(s)}{s}ds\Big)\Big) \\ &\geq b(x)g\Big(Ma_1(c_0)\psi\Big(\int_0^{c_0\varphi_1(x)} \frac{k(s)}{s}ds\Big)\Big). \end{aligned} \tag{5.15}$$

On the other hand, by (H4), (1.12), Proposition 2.3 and Lemma 3.4 (vii) we see that

$$\begin{aligned} &\lim_{d(x)\to 0}\Big[\Big(\frac{a(x)}{(d(x))^{-\rho_1}\hat{k}_1(d(x))}\Big)(d(x))^{-\rho_1}\hat{k}_1(d(x))(c_0\varphi_1(x))^{2-p}\Big(k(c_0\varphi_1(x)) \\ &\times \psi'\Big(\int_0^{c_0\varphi_1(x)} \frac{k(s)}{s}ds\Big)\Big)^{q-1} + \Big(\frac{\sigma(x)}{(d(x))^{-\rho_2}\hat{k}_2(d(x))}\Big)(d(x))^{-\rho_2}\hat{k}_2(d(x))(c_0\varphi_1(x))^2 \\ &\times \Big(k(c_0\varphi_1(x))\psi'\Big(\int_0^{c_0\varphi_1(x)} \frac{k(s)}{s}ds\Big)\Big)^{-1}\Big] = 0. \end{aligned}$$

By the same arguments as for Theorems 1.9 and 1.10, we know that there exists $\omega' \Subset \Omega$ satisfying $\omega \Subset \omega'$ and $\operatorname{dist}(\omega', \partial\Omega) < \delta_0$ such that for $x \in \Omega\setminus\omega'$

$$MI(x)/2 \geq \lambda a(x)M^q Q_3^q(c_0,x) + \sigma(x). \tag{5.16}$$

It follows by (5.15) and (5.16) that $\overline{u}_\lambda$ is a supersolution of (1.1) in $\Omega\setminus\omega'$.

Define

$$m_1 := \sup_{x\in\omega'} b(x)g\Big(\psi\Big(\int_0^{c_0\varphi_1(x)} \frac{k(s)}{s}ds\Big)\Big); \quad m_2 := \inf_{x\in\omega'}(a_1(c_0)/2)Q_2(c_0,x);$$
$$m_3 := \sup_{x\in\omega'} a(x)Q_3^q(c_0,x); \quad m_4 := \sup_{x\in\omega'}\sigma(x).$$

As in the proof of Theorem 1.10, when $q \in [0,1)$, we can take

$$M \geq \max\big\{4/(c_0^2\delta_1),\, 1/a_1(c_0),\, m_1/m_2,\, \big(2m_3\max\{0,\lambda\}/m_2\big)^{1/(1-q)},\, 2m_4/m_2\big\}$$

such that $\overline{u}_\lambda$ is a supersolution of (1.1).

When $q \in [1,2]$, let

$$M \geq \max\big\{4/(c_0^2\delta_1),\, 1/a_1(c_0),\, m_1/m_2,\, 2m_4/m_2\big\} \text{ and } \lambda < M^{1-q}m_2/2m_3.$$

A simple calculation shows that $\overline{u}_\lambda$ is a supersolution of (1.1).

On the other hand, by choosing a small $m > 0$, we show that

$$\underline{u}_\lambda := ma_2(c_0)\psi\Big(\int_0^{c_0\varphi_1(x)} \frac{k(s)}{s}ds\Big)$$

is a subsolution of (1.1) with $q \in (0,2]$.

Hence, by [15, Lemma 3], problem (1.1) possesses a classical solution u_λ satisfying

$$\underline{u}_\lambda \le u_\lambda \le \overline{u}_\lambda \quad \text{in } \Omega,$$

i.e.,

$$u_\lambda(x) \approx \psi\Big(\int_0^{c_0\varphi_1(x)} \frac{k(s)}{s} \Big), \quad x \in \Omega.$$

As in the proof of Theorem 1.8, we obtain that (1.24) holds. □

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