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EXISTENCE AND CONVERGENCE THEOREMS FOR EVOLUTIONARY HEMIVARIATIONAL INEQUALITIES OF SECOND ORDER

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Abstract. This article concerns with a class of evolutionary hemivariational inequalities in the framework of evolution triple. Based on the Rothe method, monotonicity-compactness technique and the properties of Clarke's generalized derivative and gradient, the existence and convergence theorems to these problems are established. The main idea in the proof is using the time difference to construct the approximate problems. The work generalizes the existence results on evolution inclusions and hemivariational inequalities of second order.

1. Introduction

Let V be a separable and reflexive Banach space and V^* be its dual space. Let H be a separable Hilbert space which is identified with its dual H^*. We assume that V, H and V^* form an evolution triple, i.e., $V \subset H \subset V^*$ with the embeddings being dense and continuous.

Setting $T > 0$ and $I = [0, T]$, the evolutionary hemivariational inequalities (EHI) considered in this article is stated as follows: find a pair (z, v) with $z(0) = z_0, v(0) = v_0$ such that for a.e. $t \in I$, $v(t) \in B(z'(t))$ and

$$\begin{aligned} &\big\langle v'(t) + A(t, z'(t)) + E(z(t)) - f(t), w\big\rangle \\ &+ \int_\Omega j^\circ(x, z(x,t), z'(x,t); w(x), w(x))dx \geq 0 \end{aligned} \tag{1.1}$$

holds for all $w \in V$. Here $A(t, \cdot) : V \mapsto V^*$ is a nonlinear and pseudomonotone operator, $B : H \to 2^H$ is the subdifferential of convex functional Ψ defined on H, $E(\cdot) : V \mapsto V^*$ is a linear and bounded operator, $\Omega \in \mathbb{R}^N$ and $j^\circ(x, z(x,t), z'(x,t); w(x), w(x))$ denotes the Clarke's generalized direction derivative of a locally Lipschitz function $j(x, \cdot, \cdot)$.

The notation of hemivariational inequality was introduced by Panagiotopoulos in the 1980s and 1990s as the variational formulations of the problems in Mechanics and Engineering Science (cf. [26, 27]). These variational forms involve nonconvex and nonsmooth energy functionals and express the principle of virtual work in their inequality forms. In last several decades, plenty of monographs are concerned with

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the study in this field, see, e.g., Naniewicz and Panagiotopoulos [25], Motreanu and Panagiotopoulos [22], Motreanu and Radulescu [23], Carl and Heikkila [4], Migorski, Ochal and Sofonea [21]. As to related papers, Motreanu and Radulescu [24], Radulescu and Repovs [32] have studied the existence results for elliptic inequality problems lack of convexity; Goeleven, Motreanu and Panagiotopoulos [10], Ciulcu, Motreanu and Radulescu [3] have dealt with existence and multiplicity of solutions to elliptic hemivariational inequalities. For recent studies on the first-order evolution inclusions and parabolic hemivariational inequalities, we refer readers to Papageorgiou et al [28], Migorski and Ochal [19], Liu [15, 16], Carl and Motreanu [6], Carl, Le and Motreanu [5], Peng and Liu [30, 31], Kalita [11, 12]. Besides, the evolution inclusions and hemivariational inequalities of second order have been considered by Papageorgiou and Yannakakis [29], Gasinski and Smolka[9], Migorski [17], Migorski and Ochal [18], Li and Liu [13] and so on.

The study of hemivariational problem (EHI) is connected with the nonlinear evolution inclusion

$$\begin{gathered}\big(B(z'(t))\big)' + A(t,z'(t)) + E(z(t)) + G(z(t),z'(t)) \ni f(t)\\ v_0 \in B(z'(0)), \quad z(0) = z_0,\end{gathered}\tag{1.2}$$

where $B = \partial\Psi$ and $G : H \times H \mapsto 2^H$ is a multivalued operator with nonempty, closed and convex values. Particularly, if $\Psi(u) = \frac{1}{2}\|u\|_H^2$, $B(u) = u$. As a result, (1.2) reduces to the following second order evolution inclusion:

$$\begin{gathered}z''(t) + A(t,z'(t)) + E(z(t)) + G(z(t),z'(t)) \ni f(t),\\ z'(0) = v_0, \quad z(0) = z_0.\end{gathered}\tag{1.3}$$

This kind of evolution inclusions and their applications to dynamic viscoelastic contact problems were studied in such references as [13, 17, 18, 29]. Usually, by introducing an integration operator, this inclusion was transformed into a first order integro-differential inclusion to which the existence theorem is based on the classical surjectivity result for parabolic inclusions; see, e.g., [28, Theorem 2.1]. Other methods such as upper and lower method (see, e.g., [5, 6]) and Rothe method (see [12]) are also used to parabolic hemivariational inequalities and inclusions. In this paper, Rothe method is adopted and the reason is two folds: one is this method is more effective to (1.2) than others because of the nonlinearity of B. Another reason lies that it is more constructive in numerical sense (cf. [11, 12]). The main idea in this paper is more or less similar to [12, 30, 31] due to the same structure of Rothe method. However, the difference and difficulties usually lie in the concrete estimates and limit procedure. As the same strategy in [13, 17, 18, 29], (1.2) will be studied by being transformed into the first order integral and differential inclusion.

The rest of this paper is organized as follows. Section 2 is concerned with some definitions and Lemmas. The hypotheses and the existence theorem to (1.2) (Theorem 3.2) are presented in Section 3; Section 4 and 5 are concerned with the estimates and convergence of Rothe sequences and the proof the main results. Finally, we solve the hemivariational inequality problem (EHI) in Section 6 by using the existence theorem in Section 3 and the properties of Clarke's generalized directional derivative and gradient.

2. Preliminaries

In this section, we introduce the necessary preliminary material on convex, nonsmooth analysis and monotone operators in Banach spaces. We refer the reader to monographs and textbooks, e.g., [5, 7, 8, 25] for the proofs of the results presented in this section.

Let $\Gamma_0(X)$ denote the set of proper, convex and lower semicontinuous functionals defined on a Banach space X. For $\phi \in \Gamma_0(X)$, the subdifferential $\partial\phi$ is the subset of X^*, defined by

$$\partial\phi(x) := \{x^* \in X^* : \langle x^*, v - x\rangle_X \le \phi(v) - \phi(x) \text{ for all } v \in X\}.$$

The convex conjugate of ϕ is defined as

$$\phi^*(v) := \sup_{u\in X}\{\langle v, u\rangle_X - \phi(u)\}, \quad \text{for all } v \in X^*.$$

It is well known that ϕ^* belongs to $\Gamma_0(X^*)$, and for each pair $(u, w) \in X \times X^*$, the following three conditions are equivalent to each other:

$$w \in \partial\phi(u); \quad \phi^*(w) = \langle w, u\rangle_X - \phi(u); \quad u \in \partial\phi^*(w). \tag{2.1}$$

Next, we recall some basic tools from nonsmooth analysis.

Definition 2.1. Let X be a Banach space and let $\varphi\colon X \to \mathbb{R}$ be a locally Lipschitz function. The generalized (Clarke) directional derivative of φ at $x \in X$ in the direction $v \in X$, denoted by $\varphi^0(x; v)$, is defined by

$$\varphi^0(x; v) := \limsup_{y\to x,\lambda\downarrow 0} \frac{\varphi(y + \lambda v) - \varphi(y)}{\lambda}$$

and the generalized gradient (subdifferential) of φ at x, denoted by $\partial\varphi(x)$, is a subset of a dual space X^* given by

$$\partial\varphi(x) := \{x^* \in X^* : \langle x^*, v\rangle_X \le \varphi^0(x; v) \text{ for all } v \in X\}.$$

It is well known, that for every $x \in X$, the set $\partial\varphi(x)$ is nonempty, convex and w^*-compact in X^*. If $\varphi\colon X \to \mathbb{R}$ is a convex and continuous function, then φ is locally Lipschitz and the generalized subdifferential of φ coincides with the subdifferential in the sense of convex analysis;

Definition 2.2. Let $A : X \mapsto X^*$ be a bounded operator. We say it is pseudomonotone if for any sequence $\{u_n\}_{n\ge 1}$ weakly convergent to u in X , from $\limsup_{n\to\infty}\langle A(u_n), u_n - u\rangle_X \le 0$, it follows that

$$\liminf_{n\to\infty}\langle A(u_n), u_n - v\rangle_X \ge \langle A(u), u - v\rangle_X, \quad \forall v \in X. \tag{2.2}$$

Recall that in the above definition the inequality (2.2) could be equivalently replaced by $\lim_{n\to\infty}\langle A(u_n), u_n-u\rangle_X = 0$ and $A(u_n) \to A(u)$ weakly in X^* as $n \to \infty$. Pseudomonotone operator, introduced by Brézis [1], is a significant generalization of monotone operator. It contains a variety of variational-type operators as special cases and play an important role in the studies of nonlinear problems (see, for example, [14, 5]). Moreover, this notation was generalized by Browder and Hess [2] to multi-valued case and it has been widely applied to dealing with hemivariational or variational-hemivariational inequality problems.

Definition 2.3. A multi-valued operator $G : X \to 2^{X^*}$ is said to be pseudomonotone if the following are satisfied:

(*i*) for each $u \in X$, $G(u) \subset X^*$ is nonempty, bounded, closed and convex;
(*ii*) the restriction of G to each finite-dimensional subspace F of X is weakly upper semicontinuous as an operator from F to X^* ;
(*iii*) if for any sequence $\{u_n\}_{n\geq 1}$ weakly convergent to u in X, from $\limsup_{n\to\infty} \langle u_n^*, u_n - u\rangle_X \leq 0$ with $u_n^* \in G(u_n)$, it follows: to each element $v \in X$, there exists $u^*(v) \in G(u)$ such that

$$\liminf_{n\to\infty} \langle u_n^*, u_n - v\rangle_X \geq \langle u^*(v), u - v\rangle_X.$$

Lemma 2.4 ([20, Proposition 2]). *Let X and Y be Banach spaces, and let $F : (0,T) \times X \to 2^Y$ be a multifunction such that*

(1) *the values of F are nonempty, closed and convex subsets of Y;*
(2) *for each $x \in X$, $F(\cdot, x)$ is measurable;*
(3) *for a.e. $t \in (0,T)$, $F(t,\cdot)$ is upper semicontinuous from X to w-Y .*

Let $x_n : (0,T) \to X$, $y_n : (0,T) \to Y, n \in N$, be measurable functions such that x_n converges almost everywhere on $(0,T)$ to a function $x : (0,T) \to X$ and y_n converges weakly in $L^1(0,T;Y)$ to $y : (0,T) \to Y$. If $y_n(t) \in F(t, x_n(t))$ for all $n \in N$ and almost all $t \in (0,T)$, then $y(t) \in F(t, x(t))$ for a.e. $t \in (0,T)$.

Following is an existence lemma to elliptic inclusions governed by the sum of a maximal monotone operator and a multi-valued pseudomonotone operator [25, Theorem 2.11]. This lemma shall be used to show the existence of solutions to the approximate problems in Section 4.

Lemma 2.5. *Let V be a reflexive Banach Space and $\tilde{T} : V \to 2^{V^*}$ a maximal monotone mapping with $u^0 \in D(\tilde{T})$. Let T be a bounded and pseudomonotone mapping from V to 2^{V^*}. Suppose that there exists a function $c : \mathcal{R}^+ \mapsto \mathcal{R}$ with $c(r) \to +\infty$ as $r \to +\infty$ such that for $(u^*, u) \in$Graph(T), $\langle u^*, u - u^0\rangle_V \geq c(\|u\|_V)\|u\|_V$. Then $R(\tilde{T} + T) = V^*$.*

3. Hypotheses and main theorems

Now we are in a position to present the hypotheses of the operators involving in our problems.

$A : [0,T] \times V$ to V^* is such that

(A1) for all $u \in V$, $t \to A(t,u)$ is measurable;
(A2) $A(t,\cdot) : V \to V^*$ is demicontinuous and pseudomonotone for a.e. $t \in I$ and there exists a constant $c_1 > 0$ such that

$$\|A(t,u)\|_{V^*} \leq c_1(1 + \|u\|_V), \quad \text{a.e. } t \in I;$$

(A3) there exists a constant $c_2 > 0$ such that

$$\langle A(t,u), u\rangle \geq c_2\|u\|_V^2 - 1, \quad \text{for all } u \in V \text{ and a.e. } t \in I.$$

$B = \partial\Psi$ where $\Psi \in \Gamma_0(H)$ and it is finite and continuous at 0. We assume that there exist $c_3, c_4 > 0$ such that

(B1) $\|\xi\|_H \leq c_3\,(1 + \|u\|_H)$ for all $u \in H$, $\xi \in \partial\Psi(u)$.
(B2) for all u_1, $u_2 \in H$ and $v_1 \in \partial\Psi(u_1)$, $v_2 \in \partial\Psi(u_2)$, one has

$$\langle v_1 - v_2, u_1 - u_2\rangle \geq c_4\|u_1 - u_2\|_H^2.$$

Without loss of generality, we assume $\Psi(0) = 0$ and thus $\Psi^*(v) \geq 0$ for all $v \in H$. Let i and i^* denote the injection from V to H and H to V^*, respectively. From the chain rule ([8, Proposition 5.7] it follows that $\partial(\Psi \circ i) = i^* \circ \partial\Psi \circ i$. So if we restrict the domain of B on V, then we have $B = \partial(\Psi \circ i)$. For simplicity, we sometimes omit notations i and i^*.

Let $E : V \to V^*$ be a linear, bounded, symmetric and monotone operator, namely, one has

- (E1) $E \in \mathcal{L}(V, V^*)$, $\|E(u)\|_{V^*} \leq c_5\|u\|_V$ for all $u \in V$ for some constant $c_5 > 0$.
- (E2) $\langle E(u), v\rangle = \langle E(v), u\rangle$, $\langle E(u), u\rangle \geq 0$ for all $u, v \in V$.

$G : H \times H \to 2^H$ is a multi-valued operator with nonempty, convex and closed values and satisfies

- (G1) $\|\eta\|_H \leq c_6(1 + \|u\|_H + \|v\|_H)$ for all $u, v \in H$, $\eta \in G(u, v)$ with $c_6 > 0$;
- (G2) the graph of G, i.e., $(u, v, G(u, v))$, is sequentially closed in $H \times H \times H_w$ (here by H_w we denote the Hilbert space H equipped with the weak topology).

Also we assume that

- (H0) $f \in L^2(I; V^*)$ and $v_0 \in B(u_0)$ for some $u_0 \in V$.
- (H1) the embedding operator i is compact and $c_6c_0^2 < c_2$ where c_0 is the embedding constant from V to H.

Definition 3.1. Given $f \in L^2(I; V^*), z_0 \in V, v_0 \in V^*$, a triple (z, v, g) of functions from I to $V \times V^* \times V^*$ is said to be a solution to (1.2) if:

- (a) $z \in H^1(I; V)$, $v \in L^\infty(I; H) \cap H^1(I; V^*)$ and $g \in L^2(I; H)$;
- (b) $v'(t) + A(t, z'(t)) + E(z(t)) + g(t) = f(t)$ in V^* a.e. $t \in I$;
- (c) $z(0) = z_0$, $v(0) = v_0$, $v(t) \in B(z'(t))$, $g(t) \in G(z(t), z'(t))$ a.e. $t \in I$.

Theorem 3.2. *Under the hypotheses* (A1)–(A3), (B1)–(B2), (E1)–(E2), (G1)–(G2), (H0)–(H1)*, the nonlinear evolution inclusion* (1.2) *admits at least one solution.*

To prove this theorem, we set $u(t) = z'(t), K(u)(t) = z_0 + \int_0^t u(s)ds$ and transform (1.2) into the first-order evolution inclusion

$$\begin{gathered} B'(u(t)) + A(t, u(t)) + E(K(u)(t)) + G(K(u)(t), u(t)) \ni f(t), \\ v_0 \in B(u(0)). \end{gathered} \tag{3.1}$$

As we can see, (3.1) is an integro-differential inclusion where the integration term involves in both E and G, which complicates the study.

Theorem 3.3. *Under assumptions* (A1)–(A3),(B1)–(B2),(E1)–(E2), (G1)–(G2), (H0)–(H1)*, there exist $u \in L^2(I; V)$, $v \in L^\infty(I; H) \cap H^1(I; V^*)$ and $g \in L^2(I; H)$ such that*

$$\begin{gathered} v'(t) + A(t, u(t)) + E(K(u)(t)) + g(t) = f(t) \quad \text{in } V^* \text{ a.e. } t \in I, \\ v(0) = v_0, \quad v(t) \in B(u(t)), \quad g(t) \in G(K(u)(t), u(t)) \quad \text{a.e. } t \in I. \end{gathered} \tag{3.2}$$

Note that $v(0) = v_0$ makes sense, because $v, v' \in L^2(I; V^*)$ implies $v \in C(I; V^*)$ by possibly modifying the values on a set of null measure.

4. Rothe method and a priori estimates

4.1. **Rothe method and approximate problems.** In the sequel, the Rothe method, also known as implicit time-discritization method, is applied for proving Theorem 3.3.

Let $m \in \mathbb{Z}^+$ and $(t_i)_{0\le i\le m}$ be a subdivision of I whose step is $\mu = T/m$. Setting $u_\mu^0 = u_0, v_\mu^0 = v_0$ and $g_\mu^0 = 0$; for $n = 0, 1, \dots, m-1$, we aim to find $(u_\mu^{n+1}, v_\mu^{n+1}, g_\mu^{n+1}) \in V \times V^* \times V^*$ such that

$$\begin{gathered} \frac{v_\mu^{n+1} - v_\mu^n}{\mu} + A_\mu^n(u_\mu^{n+1}) + E\big(\mu \sum_{j=0}^{n+1} u_\mu^j\big) + g_\mu^{n+1} = f_\mu^n \quad \text{in } V^*, \\ v_\mu^{n+1} \in B(u_\mu^{n+1}), \quad g_\mu^{n+1} \in G\Big(\mu \sum_{j=0}^{n+1} u_\mu^j, u_\mu^{n+1}\Big). \end{gathered} \tag{4.1}$$

where we set

$$t_\mu^n = n\mu; \quad A_\mu^n(u) = \frac{1}{\mu}\int_{t_\mu^n}^{t_\mu^{n+1}} A(t,u)dt; \quad f_\mu^n = \frac{1}{\mu}\int_{t_\mu^n}^{t_\mu^{n+1}} f(t)dt.$$

As is seen, the integration term $K(u)(t)$ in (3.1) is approximated by $\mu \sum_{j=0}^{n+1} u_\mu^j$ for $t \in (t_\mu^n, t_\mu^{n+1}]$. Before proceeding further, we present a proposition which is useful to limit procedure of evolutionary problems with pseudomonotone operator. We refer the reader to [31, Lemma 4.1] for its proof.

Proposition 4.1. *Suppose that* (A1)–(A3) *hold,* $u_n \to u$ *weakly in* $L^2(I;V)$ *as* $n \to \infty$ *and* $\limsup_{n\to\infty} \langle Au_n, u_n - u\rangle_{L^2(I;V)} \le 0$*. If we further have* $u_n \to u$ *in* $L^2(I;H)$*, then* $\lim_{n\to\infty}\langle Au_n, u_n\rangle = \langle Au, u\rangle$ *and* $Au_n \to Au$ *weakly in* $L^2(I;V^*)$ *as* $n \to \infty$*.*

Theorem 4.2. *Provided that* (A1)–(A3) *are satisfied, then operator* A_μ^n *is bounded and pseudomonotone from* V *to* V^* *for each* n*,* $n = 0, 1, \dots, m-1$*.*

This theorem can be proved by using Proposition 4.1, we also refer reader to [31] for its proof and omit it here.

Theorem 4.3. *Let us define* $F_\mu^n : V \to 2^{V^*}$ *by* $F_\mu^n(u) = A_\mu^n(u) + \mu E(u) + G\big(\mu \sum_{j=0}^n u_\mu^j + \mu u, u\big), \forall u \in V$*. Then* F_μ^n *is pseudomonotone and bounded.*

Proof. We shall verify the items of Definition 2.3 one by one. As (i) is obviously true, we focus on the second and third ones. To prove (ii), it suffice to show $(F_\mu^n)^{-1}(D) := \{u \in V : F_\mu^n(u) \cap D \ne \emptyset\}$ is closed in V for any weakly closed subset D of V^*. Indeed, letting $\{u_k\} \subset (F_\mu^n)^{-1}(D)$ and $u_k \to u$ in V as $k \to \infty$, there exist sequences $\eta_k \in V^*$ and $w_k \in G\big(\mu \sum_{j=0}^n u_\mu^j + \mu u_k, u_k\big)$ such that $\eta_k = A_\mu^n(u_k) + \mu E(u_k) + w_k$. From (A2) and (G1), it follows $A_\mu^n(u_k)$ and w_k are bounded in V^* and H, respectively, and thus we may assume $A_\mu^n(u_k) \to \xi$ weakly in V^*, $w_k \to w$ weakly in H as $k \to \infty$. By Theorem 4.2, one has $A_\mu^n(u) = \xi$. Meanwhile, $w \in G\big(\mu \sum_{j=0}^n u_\mu^j + \mu u, u\big)$ follows from (G2). From $E \in \mathcal{L}(V, V^*)$, $E(u_k) \to E(u)$. Therefore, $\xi + \mu E(u) + w \in F_\mu^n(u)$. Since $\eta_k \in D, \eta_k \to \xi + \mu E(u) + w$ weakly in V^* as $k \to \infty$ and D is a weakly closed set in V^*, we have $\xi + \mu E(u) + w \in D$. This, together with $\xi + \mu E(u) + w \in F_\mu^n(u)$, implies that $(F_\mu^n)^{-1}(D)$ is closed in V.

We proceed to check condition (iii). To this end, we assume that $u_k \to u$ weakly in V, $\eta_k \to \eta$ weakly in V^* as $k \to \infty$ such that $\eta_k = A_\mu^n(u_k) + \mu E(u_k) + w_k$ with $w_k \in G\big(\mu \sum_{j=0}^n u_\mu^j + \mu u_k, u_k\big)$ and $\limsup_{k\to\infty}\langle \eta_k, u_k - u\rangle_V \le 0$. Owing to $V \subset H$ compactly, one has $u_k \to u$ in H. Similar as before, we assume that $A_\mu^n(u_k) \to \xi$ weakly in V^*, $w_k \to w$ weakly in H as $k \to \infty$. By (G2), we have

$w \in G\big(\mu\sum_{j=0}^n u_\mu^j + \mu u, u\big)$. On the other hand, due to $\langle E(u_k - u), u_k - u\rangle \geq 0$ and $u_k \to u$ weakly in V as $k \to \infty$, one has

$$\liminf_{k\to\infty} \mu\langle E(u_k), u_k - u\rangle = \liminf_{k\to\infty} \big(\mu\langle E(u_k - u), u_k - u\rangle + \mu\langle E(u), u_k - u\rangle\big) \geq 0.$$

Moreover, taking $\langle w_k, u_k - u\rangle_V = \langle w_k, u_k - u\rangle_H \to 0$ as $k \to \infty$ into account, one has $\limsup_{k\to\infty}\langle A_\mu^n(u_k), u_k - u\rangle_V \leq 0$. Again by Theorem 4.2, one has $A_\mu^n(u) = \xi$ and $\lim_{k\to\infty}\langle A_\mu^n(u_k), u_k - u\rangle = 0$. Besides, from $E \in \mathcal{L}(V, V^*)$, $u_k \to u$ weakly in V and the fact that a bounded and linear operator is weakly continuous, we have $E(u_k) \to E(u)$ weakly in V^*. Consequently, $\eta = \xi + \mu E(u) + w \in F_\mu^n(u)$. Moreover, using the above results, it is easy to check that

$$\liminf_{k\to\infty}\langle \eta_k, u_k - v\rangle_V \geq \langle \eta, u - v\rangle_V, \quad \forall v \in V.$$

Therefore, $F_\mu^n : V \to 2^{V^*}$ is pseudomonotone. On the other hand, by virtue of (A2), (E1) and (G1), we can obtain that F_μ^n is bounded. The proof is complete. $\square$

Now we are in a position to show the existence of solutions to the discrete problem (4.1) for $n = 1, 2, \dots, m-1$. Firstly, (4.1) can be equivalently written in the following form: find u_μ^{n+1} such that

$$B(u_\mu^{n+1}) + \mu F_\mu^n(u_\mu^{n+1}) \ni \mu f_\mu^n + \mu^2\sum_{j=0}^n E(u_\mu^j) + v_\mu^n. \tag{4.2}$$

We have F_μ^n is a bounded and pseudomonotone mapping from V to 2^{V^*} by Theorem 4.3. So, μF_μ^n is also bounded and pseudomonotone for given $\mu > 0$. Since the subdifferential operator is maximal monotone, B is a maximal monotone operator from V to 2^{V^*}. Next, we check that F_μ^n, so as to μF_μ, is coercive for small and fixed μ. Let $u \in V$ and $u^* \in F_\mu^n(u)$. So, there exists $g \in G\big(\mu\sum_{j=0}^n u_\mu^j + \mu u, u\big)$ such that $u^* = A_\mu^n(u) + \mu E(u) + g$. By (G1), we have

$$\|g\|_H \leq c_6\big(1 + \mu\sum_{j=0}^n \|u_\mu^j\|_H + \mu\|u\|_H + \|u\|_H\big).$$

Furthermore, by using (A3) and (E1) we have

$$\begin{aligned} &\langle u^*, u\rangle_V \\ &= \langle A_\mu^n(u) + \mu E(u), u\rangle_V + \langle g, u\rangle_H \\ &\geq c_2\|u\|_V^2 - 1 - \mu c_5\|u\|_V^2 - c_6(1+\mu)\|u\|_H^2 - c_6\Big(\mu\sum_{j=0}^n \|u_\mu^j\|_H + 1\Big)\|u\|_H \\ &\geq (c_2 - \mu c_5 - c_0^2 c_6 - \mu c_0^2 c_6)\|u\|_V^2 - c_6 c_0\Big(\mu\sum_{j=0}^n \|u_\mu^j\|_H + 1\Big)\|u\|_V - 1 \end{aligned} \tag{4.3}$$

In view of $c_6 c_0^2 < c_2$, let us choose $\mu < \mu_0 = (c_5 + c_6 c_0^2)^{-1}(c_2 - c_6 c_0^2)$. By taking $u^0 = 0 \in D(B)$, $T = \mu F_\mu^n$ and $\tilde{T} = B$, it is easy to see the coercive condition in Lemma 2.5 is satisfied for $\mu < \mu_0$. So, by Lemma 2.5, we have the following conclusion:

Theorem 4.4. *Under the hypotheses of Theorem 3.3, there exists at least one solution to the discrete problem* (4.1) *for* $\mu < \mu_0$ *and* $n = 0, 1, \dots m-1$.

In what follows, we always assume $\mu < \mu_0$.

4.2. **A priori estimates.** Let the functions u_μ and v_μ be defined as follows:

$$u_\mu(t) = u_\mu^{n+1}, \quad t \in (t_\mu^n, t_\mu^{n+1}], \quad u_\mu(0) = u_0;$$
$$v_\mu(t) = v_\mu^{n+1}, \quad t \in (t_\mu^n, t_\mu^{n+1}], v_\mu(0) = v_0.$$

Let $\widehat{v}_\mu$ denote the linear time-interpolate function of v_μ, i.e.,

$$\widehat{v}_\mu(t) = v_\mu^n + \frac{t - n\mu}{\mu}(v_\mu^{n+1} - v_\mu^n) \quad \text{for } t \in (t_\mu^n, t_\mu^{n+1}]; \quad \widehat{v}_\mu(0) = v_0.$$

Clearly, the time derivative of $\widehat{v}_\mu$ on (t_μ^n, t_μ^{n+1}) is $\frac{1}{\mu}(v_\mu^{n+1} - v_\mu^n)$. It follows from Theorem 4.4 that

$$\frac{d}{dt}\widehat{v}_\mu(t) + A_\mu(t) + E_\mu(t) + g_\mu(t) = f_\mu(t) \text{ in } V^*, \quad \text{for a.e. } t \in I, \tag{4.4}$$

where $A_\mu, E_\mu(t)$, g_μ and $f_\mu : I \mapsto V^*$ take values $A_\mu^n(u_\mu^{n+1})$, $E\big(\mu \sum_{k=0}^{n+1} u_\mu^k\big)$, g_μ^{n+1} and f_μ^n, respectively for $t \in (t_\mu^n, t_\mu^{n+1})$. We remark that f_μ is a possible approximate choice that converges to f in $L^2(I;V^*)$ and satisfies $\|f_\mu\|_{L^2(I;V^*)} \leq \|f\|_{L^2(I;V^*)}$. Multiplying the equation in (4.1) by u_μ^{n+1} , we have

$$\frac{1}{\mu}\langle v_\mu^{n+1} - v_\mu^n, u_\mu^{n+1}\rangle + \big\langle A_\mu^n(u_\mu^{n+1}) + E\big(\mu \sum_{k=0}^{n+1} u_\mu^k\big) + g_\mu^{n+1}, u_\mu^{n+1}\big\rangle = \langle f_\mu^n, u_\mu^{n+1}\rangle.$$

Noting that $(\Psi \circ i)^* \in \Gamma_0(V^*)$ and $u_\mu^{n+1} \in \partial(\Psi \circ i)^*(v_\mu^{n+1})$, we have

$$(\Psi \circ i)^*(v_\mu^{n+1}) - (\Psi \circ i)^*(v_\mu^n) \leq \langle v_\mu^{n+1} - v_\mu^n, u_\mu^{n+1}\rangle.$$

Applying this inequality to the above equation and summing it from $n = 0$ to $j, 0 < j \leq m-1$, we deduce that

$$\begin{aligned} &(\Psi \circ i)^*(v_\mu^{j+1}) + \mu \sum_{n=0}^{j} \big\langle A_\mu^n(u_\mu^{n+1}) + E\big(\mu \sum_{k=0}^{n+1} u_\mu^k\big) + g_\mu^{n+1}, u_\mu^{n+1}\big\rangle \\ &\leq \mu \sum_{n=0}^{j} \langle f_\mu^n, u_\mu^{n+1}\rangle + (\Psi \circ i)^*(v_0). \end{aligned} \tag{4.5}$$

By (A2), (E1), (G1) and $\|f_\mu\|_{L^2(I;V^*)} \leq \|f\|_{L^2(I;V^*)}$, we deduce that

$$\begin{aligned} &(\Psi \circ i)^*(v_\mu^{j+1}) + \mu(c_2 - c_0^2 c_6 - \mu c_5 - \mu c_0^2 c_6) \sum_{n=0}^{j} \|u_\mu^{n+1}\|_V^2 \\ &\leq \mu \sum_{n=0}^{j} \|u_\mu^{n+1}\|_V \big(\mu c_5 \sum_{k=0}^{n} \|u_\mu^k\|_V + \mu c_0 c_6 \sum_{k=0}^{n} \|u_\mu^k\|_H + c_0 c_6\big) \\ &\quad + \|f\| \Big(\mu \sum_{n=0}^{j} \|u_\mu^{n+1}\|_V^2\Big)^{1/2} + C \\ &\leq \mu \sum_{n=0}^{j} \|u_\mu^{n+1}\|_V (c_5 + c_0^2 c_6)((n+1)\mu)^{1/2} \big(\mu \sum_{k=0}^{n} \|u_\mu^k\|_V^2\big)^{1/2} \\ &\quad + c_0 c_6 \mu \sum_{n=0}^{j} \|u_\mu^{n+1}\|_V + \|f\| \Big(\mu \sum_{n=0}^{j} \|u_\mu^{n+1}\|_V^2\Big)^{1/2} + C \end{aligned}$$

$$
\begin{aligned}
&\leq (c_5+c_0^2c_6)\sqrt{T}\Big(\mu\sum_{n=0}^{j}\|u_\mu^{n+1}\|_V^2\Big)^{1/2}\Big(\mu\sum_{n=0}^{j}\mu\sum_{k=0}^{n}\|u_\mu^k\|_V^2\Big)^{1/2}\\
&\quad+\big(c_0c_6\sqrt{T}+\|f\|\big)\Big(\mu\sum_{n=0}^{j}\|u_\mu^{n+1}\|_V^2\Big)^{1/2}+C,
\end{aligned}
$$

where the Hölder inequality and similar estimates to (4.3) are used. Here and in what follows, C denotes a positive constant that only depends on $c_0, c_1, \dots, c_7$, $\|f\|_{L^2(I;V^*)}$, $(\Psi\circ i)^*(v_0)$, and may change from line to line. Using the Young inequality, we have

$$
(\Psi\circ i)^*(v_\mu^{j+1})+\mu\sum_{n=0}^{j}\|u_\mu^{n+1}\|_V^2\leq C\Big(1+\mu\sum_{n=0}^{j}\mu\sum_{k=0}^{n}\|u_\mu^k\|_V^2\Big). \tag{4.6}
$$

Recall that $(\Psi\circ i)^*(v)\geq 0, \forall v\in V^*$. By applying the discrete Gronwall lemma, we deduce that

$$
\|u_\mu\|_{L^2(I;V)}^2=\mu\sum_{n=0}^{m-1}\|u_\mu^{n+1}\|_V^2\leq C. \tag{4.7}
$$

Taking into account (A2), we can deduce that

$$
\|Au_\mu\|_{L^2(I;V^*)}^2=\sum_{n=0}^{m-1}\int_{t_\mu^n}^{t_\mu^{n+1}}\|A(t,u_\mu^{n+1})\|_{V^*}^2dt\leq C. \tag{4.8}
$$

$$
\|A_\mu\|_{L^2(I;V^*)}^2=\mu\sum_{n=0}^{m-1}\|A_\mu^n(u_\mu^{n+1})\|_{V^*}^2\leq C. \tag{4.9}
$$

Moreover, from (E1) and (G1), we compute

$$
\begin{aligned}
\|E_\mu\|_{L^2(I;V^*)}^2&=\int_0^T\|E_\mu(t)\|_{V^*}^2dt=\mu\sum_{n=0}^{m-1}\|E(\mu\sum_{k=0}^{n+1}u_\mu^k)\|_{V^*}^2\\
&\leq \mu\sum_{n=0}^{m-1}(n+1)\mu c_5^2\mu\sum_{k=0}^{n+1}\|u_\mu^k\|_V^2\\
&\leq Tc_5^2\mu\sum_{n=0}^{m-1}\mu\sum_{k=0}^{n+1}\|u_\mu^k\|_V^2\leq C,
\end{aligned} \tag{4.10}
$$

and

$$
\begin{aligned}
\|g_\mu\|_{L^2(I;H)}^2&=\int_0^T\|g_\mu(t)\|_H^2dt=\mu\sum_{n=0}^{m-1}\|G(\mu\sum_{k=0}^{n+1}u_\mu^k,u_\mu^{n+1})\|_H^2\\
&\leq 2\mu c_6^2\sum_{n=0}^{m-1}\big(1+T\mu\sum_{k=0}^{n+1}\|u_\mu^k\|_V^2+\|u_\mu^{n+1}\|_V^2\big)\leq C.
\end{aligned} \tag{4.11}
$$

Thus, from the above four inequalities and (4.4), it follows that

$$
\|\widehat{v}_\mu'\|_{L^2(I;V^*)}\leq C. \tag{4.12}
$$

It follows from (4.7), (4.8) and $v_\mu^0=v_0$ that fo rall $\mu>0$,

$$
(\Psi\circ i)^*(v_\mu^j)\leq C,\ j\in\{0,1,\dots,m-1\}.
$$

Furthermore, for any $w \in B(u) = i^* \circ \partial\Psi \circ i(u)$ with $u \in V$, we have

$$(\Psi \circ i)^*(w) = \langle w, u\rangle_V - (\Psi \circ i)(u) = \langle w, u\rangle_H - \Psi(u) = \Psi^*(w).$$

Hence, we have $\Psi^*(v_\mu(t)) = (\Psi \circ i)^*(v_\mu(t)) \le C$, $t \in I$. By the definition of the conjugate functional, we have

$$\Psi^*(v_\mu(t)) \ge \langle v_\mu(t), u\rangle - \Psi(u) \quad \forall u \in H,\ t \in I.$$

Choosing $u = \|v_\mu(t)\|_H^{-1} v_\mu(t)$, we have $\|u\|_H = 1$ and

$$\Psi^*(v_\mu(t)) \ge \|v_\mu(t)\|_H - \Psi(u).$$

Consequently,

$$\begin{aligned}
\|v_\mu(t)\|_H &\le \Psi^*(v_\mu(t)) + \Psi(u)\\
&\le \Psi^*(v_\mu(t)) + \langle \xi, u\rangle \quad (\xi \in \partial\Psi(u), \Psi(0) = 0)\\
&\le \Psi^*(v_\mu(t)) + c_3\|u\|_H(1 + \|u\|_H),
\end{aligned}$$

which implies

$$\|v_\mu\|_{L^\infty(I;H)} \le C, \quad \|\widehat{v}_\mu\|_{C(0,T;H)} \le C. \tag{4.13}$$

5. Convergence and limit procedure

By the a priori estimates(4.7)-(4.12), (4.13), there exist $u \in L^2(I;V)$, $v \in L^\infty(I;H) \cap H^1(I;V^*)$ and $g \in L^2(I;H)$ such that

$$u_\mu \to u \quad \text{weakly in } L^2(I;V) \tag{5.1}$$
$$A_\mu, Au_\mu \to \xi \quad \text{weakly in } L^2(I;V^*) \tag{5.2}$$
$$E_\mu \to \eta \quad \text{weakly in } L^2(I;V^*) \tag{5.3}$$
$$g_\mu \to g \quad \text{weakly in } L^2(I;H) \tag{5.4}$$
$$\widehat{v}'_\mu \to v' \quad \text{weakly in } L^2(I;V^*) \tag{5.5}$$
$$v_\mu, \widehat{v}_\mu \to v \quad \text{weakly star in } L^\infty(I;H), \tag{5.6}$$

by possibly taking subsequences. We remark that (5.2) follows from (4.8), (4.9) and the fact that $\lim_{\mu\to 0}\langle A_\mu - Au_\mu, v\rangle = 0$ for all $v \in L^2(I;V)$; (5.6) is followed by (4.13), the density embedding of V into H and the fact that

$$\|\widehat{v}_\mu - v_\mu\|^2_{L^2(I;V^*)} \le \frac{1}{3}\mu^2\big\|\widehat{v}'_\mu\big\|^2_{L^2(I;V^*)} \longrightarrow 0, \quad \text{as } \mu \to 0. \tag{5.7}$$

In view of (5.1)-(5.5), passing to the limit in (4.4) we have

$$v'(t) + \xi(t) + \eta(t) + g(t) = f(t) \quad \text{in } V^* \text{ a.e. } t \in I. \tag{5.8}$$

Note that $\widehat{v}_\mu \to v$ in $L^2(I;V^*)$ due to (5.6), (5.6) and the compact embedding of $H \subset V^*$. On the other hand, we shall show that u_μ converges to u in $L^2(I;H)$ in the subsequent Theorem 5.1.

It follows from (5.7) that $v_\mu \to v$ in $L^2(I;V^*)$ as $\mu \to 0$. Observing that B is a maximal monotone operator from $L^2(I;V)$ to $L^2(I;V^*)$, from (5.1) and $v_\mu \in B(u_\mu)$, we conclude that $v \in B(u)$, i.e., $v(t) \in B(u(t))$ a.e. $t \in I$.

By its construction, $\widehat{v}_\mu \in C(0,T;V^*)$ for any $\mu > 0$. Owing to $v \in W^{1,q}(0,T;V^*)$, we may assume $v \in C(0,T;V^*)$ by modifying the values on a set of null measure. Since $\widehat{v}_\mu \to v$ in $L^2(I;V^*)$, we obtain that passing to a subsequence, $\widehat{v}_\mu(t) \to v(t)$ in V^* for all $t \in I$. Therefore, $\widehat{v}_\mu(0) = v_0$ leads to $v(0) = v_0$.

Next, we aim to show that $\eta(t) = E(K(u)(t))$ a.e. $t \in I$. For convenience, we define $\tilde{E}_\mu, \tilde{E}_0 \in L^2(I; V^*)$ as $\tilde{E}_\mu(t) = E(K(u_\mu)(t))$ and $\tilde{E}_0(t) = E(K(u)(t))$ respectively, a.e. $t \in I$. We check that

$$
\begin{aligned}
&\|E_\mu - \tilde{E}_\mu\|^2_{L^2(I;V^*)} \\
&= \sum_{n=0}^{m-1} \int_{t_\mu^n}^{t_\mu^{n+1}} \big\|E(\mu \sum_{k=0}^{n+1} u_\mu^k) - E(\mu \sum_{k=0}^{n} u_\mu^k + (t - n\mu) u_\mu^{n+1})\big\|^2_{V^*} dt \\
&\le \sum_{n=0}^{m-1} \int_{t_\mu^n}^{t_\mu^{n+1}} \mu^2 \|E(u_\mu^{n+1})\|^2_{V^*} dt \\
&\le \mu^2 c_5^2 \mu \sum_{n=0}^{m-1} \|u_\mu^{n+1})\|^2_V \le \mu^2 C.
\end{aligned} \tag{5.9}
$$

Since $u_\mu \to u$ weakly in $L^2(I; V)$, we have $\tilde{E}_\mu(t) = K(u_\mu)(t) \to \tilde{E}_0(t) = K(u)(t)$ weakly in V^* for all $t \in I$. On the other hand,

$$
\int_0^T \|\tilde{E}_\mu(t)\|^2_{V^*} dt = \int_0^T \|E(K(u_\mu)(t))\|^2_{V^*} dt \le 2c_5^2 T\big(\|u_0\|^2_V + \|u_\mu\|^2_{L^2(I;V)}\big).
$$

Thus, we can apply the dominated convergence result to get $\tilde{E}_\mu \to \tilde{E}_0$ weakly in $L^2(I; V^*)$, which, together with (5.3), (5.9), implies $\eta = \tilde{E}_0$, i.e., $\eta(t) = E(K(u)(t))$ a.e. $t \in I$.

In the sequel, a compact argument with respect to u_μ in $L^2(I; H)$ is developed with the aid of compactness conclusion in Simon [33].

Theorem 5.1. *Suppose that $\{u_\mu\}_{\mu>0}$ is generated by (4.1) under the assumptions of Theorem 3.3. Then it is relatively compact in $L^2(I; H)$.*

Proof. First of all, $\{u_\mu\}_{\mu>0}$ is bounded in $L^2(I; V)$ from (4.7). Since $V \subset H$ compactly, according to [33, Theorem 3], to prove this theorem it suffices to show that

$$
\int_0^{T-h} \|u_\mu(t+h) - u_\mu(t)\|^2_H dt \to 0, \quad \text{as } h \to 0, \text{ uniformly for } \mu > 0. \tag{5.10}
$$

We first show

$$
\int_0^{T-h} \langle v_\mu(t+h) - v_\mu(t), u_\mu(t+h) - u_\mu(t)\rangle dt \le C h^{1/2}. \tag{5.11}
$$

Actually, for all $h > 0$, we can assume that $h = k\mu + \tau$, $k \in \{0, 1, \dots, m-1\}$, $0 \le \tau < \mu$. We compute

$$
\begin{aligned}
&\int_0^{T-h} \langle v_\mu(t+h) - v_\mu(t), u_\mu(t+h) - u_\mu(t)\rangle \, dt \\
&= \sum_{n=0}^{m-k-1} \int_{n\mu}^{(n+1)\mu-\tau} \langle v_\mu(t+h) - v_\mu(t), u_\mu(t+h) - u_\mu(t)\rangle \, dt \\
&\quad + \sum_{n=1}^{m-k-1} \int_{n\mu-\tau}^{n\mu} \langle v_\mu(t+h) - v_\mu(t), u_\mu(t+h) - u_\mu(t)\rangle \, dt \\
&= (\mu - \tau) \sum_{n=0}^{m-k-1} \big\langle v_\mu^{n+k+1} - v_\mu^{n+1}, u_\mu^{n+k+1} - u_\mu^{n+1} \big\rangle
\end{aligned}
$$

$$+\tau\sum_{n=1}^{m-k-1}\left\langle v_\mu^{n+k+1}-v_\mu^n, u_\mu^{n+k+1}-u_\mu^n\right\rangle.$$

On the other hand, for $k\in\{0,1,\dots,m-1\}$ and $n\in\{0,1,\dots,m-k-1\}$,

$$\begin{aligned}
\|v_\mu^{n+k+1}-v_\mu^{n+1}\|_{V^*} &\le \mu\sum_{l=1}^{k}\|\frac{v_\mu^{l+n+1}-v_\mu^{l+n}}{\mu}\|_{V^*}\\
&= \mu\sum_{l=1}^{k}\|f_\mu^{l+n}-A_\mu^{l+n}(u_\mu^{l+n+1})-E\big(\mu\sum_{j=0}^{l+n+1}u_\mu^j\big)-g_\mu^{l+n+1}\|_{V^*}\\
&\le \Big(\mu\sum_{l=1}^{k}1^2\Big)^{1/2}\big(\|f_\mu-A_\mu-E_\mu-g_\mu\|_{L^2(I;V^*)}\big)\\
&\le C(k\mu)^{1/2}.
\end{aligned}$$

Similar computation gives

$$\|v_\mu^{n+k+1}-v_\mu^n\|_{V^*}\le C\big(k\mu\big)^{1/2}\le Ch^{1/2}.$$

Consequently, we deduce that

$$\begin{aligned}
&\int_0^{T-h}\langle v_\mu(t+h)-v_\mu(t), u_\mu(t+h)-u_\mu(t)\rangle\,dt\\
&\le Ch^{1/2}\big((\mu-\tau)\sum_{n=0}^{m-k-1}\|u_\mu^{n+k+1}-u_\mu^{n+1}\|_V+\tau\sum_{n=1}^{m-k-1}\|u_\mu^{n+k+1}-u_\mu^n\|_V\big)\\
&\le Ch^{1/2}\big(2\mu\sum_{n=0}^{m-1}\|u_\mu^{n+1}\|_V\big).
\end{aligned}$$

Thus we obtain (5.11) from (4.7) and Hölder inequality. Recalling that $v_\mu(s)\in B(u_\mu(s))$, $s\in I$, we conclude from (5.11) and (B2) that

$$\int_0^{T-h}\|u_\mu(t+h)-u_\mu(t)\|_H^2dt\le Ch^{\frac12},\quad \forall\mu>0,\ \forall h\in I,$$

which implies (5.10). This completes the proof. $\square$

We proceed to prove Theorem 3.3. Taking $j=m-1$ in (4.5), and noticing

$$\begin{aligned}
\langle A_\mu,u_\mu\rangle &= \sum_{n=0}^{m-1}\int_{t_\mu^n}^{t_\mu^{n+1}}\langle A_\mu^n(u_\mu^{n+1}),u_\mu^{n+1}\rangle_V\,dt\\
&= \mu\sum_{n=0}^{m-1}\langle A_\mu^n(u_\mu^{n+1}),u_\mu^{n+1}\rangle_V\\
&= \sum_{n=0}^{m-1}\int_{t_\mu^n}^{t_\mu^{n+1}}\langle A(t,u_\mu^{n+1}),u_\mu^{n+1}\rangle_V\,dt\\
&= \langle Au_\mu,u_\mu\rangle,
\end{aligned}$$

we have

$$(\Psi\circ i)^*(v_\mu(T))+\langle Au_\mu+E_\mu+g_\mu,u_\mu\rangle\le\langle f_\mu,u_\mu\rangle+(\Psi\circ i)^*(v_0).$$

In view of (5.1)-(5.5) and $f_\mu \to f$ in $L^q(0,T;V^*)$, we further deduce that

$$\begin{aligned}
&\limsup_{\mu\to 0}\langle Au_\mu + E_\mu + g_\mu, u_\mu - u\rangle \\
&= \limsup_{\mu\to 0}\big(\langle f_\mu, u_\mu\rangle - (\Psi\circ i)^*(v_\mu(T)) - \langle Au_\mu + E_\mu + g_\mu, u\rangle\big) + (\Psi\circ i)^*(v_0) \\
&= \langle f-\xi-\eta-g, u\rangle + (\Psi\circ i)^*(v_0) - \liminf_{\mu\to 0}(\Psi\circ i)^*(v_\mu(T)).
\end{aligned} \tag{5.12}$$

On the other hand, multiplying by $u(t)$ in (5.8) and integrating over I, we have

$$\int_0^T \langle \frac{dv(t)}{dt}, u(t)\rangle dt = \langle f-\xi-\eta-g, u\rangle. \tag{5.13}$$

Since $u(t) \in \partial(\Psi\circ i)^*(v(t))$, a.e. $t\in I$, we conclude from the chain rule (see e.g., [30, Lemma 1]) that

$$\frac{d}{dt}(\Psi\circ i)^*(v(t)) = \langle \frac{dv(t)}{dt}, u(t)\rangle, \quad \text{a.e. } t\in I.$$

Integrating over I and using (5.13), we have

$$(\Psi\circ i)^*(v(T)) - (\Psi\circ i)^*(v_0) = \langle f-\xi-g, u\rangle. \tag{5.14}$$

Observing that $v_\mu(T) \to v(T)$ weakly in V^*, we have

$$\liminf_{\mu\to 0}(\Psi\circ i)^*(v_\mu(T)) \ge (\Psi\circ i)^*(v(T)). \tag{5.15}$$

Finally, from (5.12), (5.14) and (5.15), we deduce that

$$\limsup_{\mu\to 0}\langle Au_\mu + E_\mu + g_\mu, u_\mu - u\rangle \le 0. \tag{5.16}$$

On the other hand, we have

$$\langle E_\mu, u_\mu - u\rangle = \langle E_\mu - \tilde{E}_\mu, u_\mu - u\rangle + \langle \tilde{E}_\mu - \eta, u_\mu - u\rangle + \langle \eta, u_\mu - u\rangle.$$

Note that

$$\begin{aligned}
&\langle \tilde{E}_\mu - \eta, u_\mu - u\rangle \\
&= \int_0^T \langle E(K(u_\mu)(t)) - E(K(u)(t)), u_\mu(t) - u(t)\rangle dt \\
&\ge \int_0^T \langle E(K(u_\mu)(t) - K(u)(t)), \frac{d}{dt}(K(u_\mu)(t) - K(u)(t))\rangle dt \\
&= \frac{1}{2}\int_0^T \frac{d}{dt}\langle E(K(u_\mu)(t)) - E(K(u)(t)), K(u_\mu)(t) - K(u)(t)\rangle dt \\
&= \langle E(K(u_\mu)(T)) - E(K(u)(T)), K(u_\mu)(T) - K(u)(T)\rangle \ge 0,
\end{aligned}$$

So, this inequality, together with (5.9) and $u_\mu \to u$ weakly in $L^2(I;V)$ implies

$$\limsup_{\mu\to 0}\langle E_\mu, u_\mu - u\rangle \ge 0. \tag{5.17}$$

It follows from (5.4) and $u_\mu \to u$ in $L^2(I;H)$ that

$$\limsup_{\mu\to 0}\langle g_\mu, u_\mu - u\rangle = \lim_{\mu\to 0}\langle g_\mu, u_\mu - u\rangle_{L^2(I;H)} = 0. \tag{5.18}$$

Hence, we conclude from (5.16), (5.17) and (5.18) that

$$\limsup_{\mu\to 0}\langle Au_\mu, u_\mu - u\rangle \le 0. \tag{5.19}$$

Recalling that (5.2) and $u_\mu \to u$ in $L^2(I;H)$, we have $Au = \xi$ and $\lim_{\mu\to 0}\langle Au_\mu, u_\mu - u\rangle = 0$ by applying Proposition 4.1.

To finish the proof, we still need to show that $g(t) \in G(K(u)(t), u(t))$ a.e. $t \in I$. Define $\tilde{K}_\mu \in L^2(I;V)$ as $\tilde{K}_\mu(t) = \mu\sum_{k=0}^{n+1} u_\mu^k$, $t \in (t_\mu^n, t_\mu^{n+1}]$; $\tilde{K}_\mu(0) = z_0$. Then, we have

$$
\begin{aligned}
\|\tilde{K}_\mu - K(u)\|_{L^2(I;H)}^2 &\le 2\big(\|\tilde{K}_\mu - K(u_\mu)\|_{L^2(I;H)}^2 + \|K(u_\mu) - K(u)\|_{L^2(I;H)}^2\big)\\
&\le 2\sum_{n=0}^{m-1}\int_{t_\mu^n}^{t_\mu^{n+1}} \big\|\mu\sum_{k=0}^{n+1} u_\mu^k - \big(\mu\sum_{k=0}^{n} u_\mu^k + (t-n\mu)u_\mu^{n+1}\big)\big\|_H^2 dt\\
&\quad + 2\int_0^T \big\|\int_0^t (u_\mu(s) - u(s))ds\big\|_H^2 dt\\
&\le 2\mu\sum_{n=0}^{m-1}\mu^2\|u_\mu^{n+1}\|_H^2 + 2T\big(\int_0^T \|u_\mu(s) - u(s)\|_H ds\big)^2\\
&\le 2\mu^2 c_0^2\mu\sum_{n=0}^{m-1}\|u_\mu^{n+1}\|_V^2 + 2T^3\int_0^T \|u_\mu(s) - u(s)\|_H^2 ds\\
&= 2\mu^2 c_0^2\|u_\mu\|_{L^2(I;V)}^2 + 2T^3\|u_\mu - u\|_{L^2(I;H)}^2.
\end{aligned}
$$

It implies $\tilde{K}_\mu \to K(u)$ in $L^2(I;H)$. Thus we have $\tilde{K}_\mu(t) \to K(u)(t)$ in H a.e. $t \in I$ by possibly taking a subsequence. Recall that $g_\mu \to g$ weakly in $L^2(I;H)$, $g_\mu(t) \in G(\tilde{K}_\mu(t), u_\mu(t))$, and $u_\mu(t) \to u(t)$ in H a.e. $t \in I$ by possibly taking a subsequence. Besides, by (G1)–(G2), it is easy to show that G is upper semicontinuous from $H\times H$ to H_w. Consequently, by Lemma 2.4, we have $g(t) \in G(K(u)(t), u(t))$ a.e. $t \in I$. So, we finally get

$$
\begin{gathered}
\frac{d}{dt}v(t) + A(t,u(t)) + E(u(t)) + g(t) = f(t) \quad \text{in } V^* \text{ a.e. } t \in I,\\
v(0) = v_0, \quad v(t) \in B(u(t)), \quad g(t) \in G(K(u)(t), u(t)) \text{ a.e. } t \in I.
\end{gathered} \tag{5.20}
$$

This completes the proof of Theorem 3.3.

Remark 5.2. Suppose that u, v, g are any weak accumulation points of u_μ, v_μ, g_μ in $L^2(I;V)$, $L^\infty(0,T;H)$ and $L^2(I;H)$, respectively, then, the triple (u,v,g) is a solution to (3.1).

6. Hemivariational inequality (EHI)

We turn our attention to the second order nonlinear evolution inclusion (1.2) and the evolutionary hemivariational inequality problem (EHI).

In view of Theorem 3.3, the nonlinear evolution inclusion (3.1) admits at least one solution (u,v,g), where $u \in L^2(I;V), v \in L^\infty(0,T;H) \cap H^1(I;V^*)$, $g \in L^2(I;H)$ and (3.2) holds. Since $u(t) = z'(t), K(u)(t) = z_0 + \int_0^t u(s)ds$, it is easy to deduce that the triple (z,v,g) satisfies the conditions in definition 3.1 by taking $z(t) = K(u)(t)$ for every $t \in I$. Therefore, (z,v,g) is a solution to (1.2) which completes the proof of Theorem 3.2.

Assume that Ω is an open and bounded subset of $\mathbb{R}^N$ and $V = H_0^1(\Omega), H = L^2(\Omega), V^* = H^{-1}(\Omega)$. In the following, the hypotheses on j are given to investigate the weak solution of the evolutionary hemivariational inequality problems (EHI).

Let $j : \Omega\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ be a function such that

(J1) $j(\cdot,\xi,\eta)$ is measurable for all $\xi,\eta\in\mathbb{R}$, and $j(\cdot,0,0)\in L^1(\Omega)$;
(J2) $j(x,\cdot,\cdot)$ is locally Lipschitz continuous on $\mathbb{R}\times\mathbb{R}$ for all $x\in\Omega$;
(J3) there exists a constant c_7 such that $|\partial j(x,\xi_1,\xi_2)|_{\mathbb{R}\times\mathbb{R}}\le c_7(1+|\xi_1|+|\xi_2|)$ for all $x\in\Omega$.

Note that the function j, in particular, can take the form of $j(x,\xi,\eta)=j_1(x,\xi)+j_2(x,\eta)$, where both $j_1,j_2:\Omega\times\mathbb{R}\to\mathbb{R}$ are locally Lipschitz continuous functions. Let us introduce integral function $J:H\times H\mapsto\mathbb{R}$, defined by

$$J(u,v)=\int_\Omega j(x,u(x),v(x))dx,\quad \text{for } u,v\in H.$$

Observing that J is well defined and finite at $(0,0)$ by $(j1)$ and $(j2)$. Thus, by conditions (J1)-(J3) and [7, Theorem 2.7.5], one has J is uniformly Lipschitz continuous on each bounded subset of $H\times H$. The Clarke's generalized gradient $\partial J:H\times H\to 2^{H\times H}$ is well defined and satisfies

$$\partial J(u,v)\subset\int_\Omega \partial j(x,u(x),v(x))dx,\quad \text{for } u,v\in H.$$

Moreover, the Clarke's generalized direction derivative of J satisfies

$$J^\circ(u,v;z,w)\le\int_\Omega j^\circ(x,u(x),v(x);z(x),w(x))dx,\quad \forall z,w\in H. \tag{6.1}$$

We define the multivalued operator $G:H\times H\to 2^H$ as

$$G(u,v):=\big\{\eta_1+\eta_2:(\eta_1,\eta_2)\in\partial J(u,v)\big\},\quad \forall u,v\in H. \tag{6.2}$$

Thus, by (J3), (G1) is satisfied with c_6 depending on $|\Omega|$ and c_7.

Theorem 6.1. *Under hypotheses* (A1)–(A3), (B1)–(B2), (E1)–(E2), (J1)–(J3) *and* (H0)–(H1)*, there exist $z\in H^1(I;V), v\in L^\infty(I;H)\cap H^1(I;V^*)$ with $v(0)=v_0$, $z(0)=z_0$ such that the inequality* (1.1) *holds and $v(t)\in B(z'(t))$ for a.e. $t\in I$.*

Note that since B is a nonlinear and multi-valued operator, this theorem generalizes existence results in [18].

Proof. Let the multivalued function G be defined as (6.2). We check the assumption (G1)–(G2). First of all, G takes nonempty, convex and closed values since the Clarke's generalized gradient is nonempty convex and closed. Since (G1) has been proved by (J3), it remains to show (G2). It suffices to show that $\eta\in G(u,v)$ whenever the sequences $u_n\to u$, $v_n\to v$ in H and $\eta_n\to\eta$ weakly in H as $n\to\infty$ with $\eta_n\in G(u_n,v_n)$. In fact, by $\eta_n\in G(u_n,v_n)$, there exist sequences η_{1n} and η_{2n} with $\eta_n=\eta_{1n}+\eta_{2n}$ and $(\eta_{1n},\eta_{2n})\in\partial J(u_n,v_n)$ for each n. Since (η_{1n},η_{2n}) is bounded, up to a subsequence, we may assume $\eta_{1n}\to\eta_1$, $\eta_{2n}\to\eta_2$ weakly in H. Consequently, $\eta_{1n}+\eta_{2n}\to\eta_1+\eta_2$ weakly in H. On the other hand, it follows from the weak closeness properties of the Clarke's generalized gradient [7, Proposition 2.1.5 (b)], we have $(\eta_1,\eta_2)\in\partial J(u,v)$. Thus, we have $\eta=\eta_1+\eta_2\in G(u,v)$, i.e., the graph of G is sequentially closed in $H\times H\times H_w$.

Therefore, by Theorem 3.2, there exists a triple (z,v,g) with $z\in H^1(I;V)$, $v\in L^\infty(I;H)\cap H^1(I;V^*)$ and $g\in L^2(I;H)$ such that

$$v'(t)+A(t,z'(t))+E(z(t))+g(t)=f(t)\quad \text{in } V^* \text{ a.e. } t\in I, \tag{6.3}$$

where $g(t)\in G(z(t),z'(t))$, $v(t)\in B(z'(t))$, a.e. $t\in I$ and $z(0)=z_0$, $v(0)=v_0$. It follows from the definition of $G(\cdot,\cdot)$, there exist $g_1,g_2\in L^2(I;H)$ with $g(t)=$

$g_1(t) + g_2(t)$ and $(g_1(t), g_2(t)) \in \partial J(z(t), z'(t))$ for a.e. $t \in I$. Consequently, for all $w \in V$ and a.e. $t \in I$, one has

$$\begin{aligned} \big\langle (g_1(t), g_2(t)), (w, w)\big\rangle_{H\times H} &\leq J^0(z(t), z'(t); w, w) \\ &\leq \int_\Omega j^\circ(x, z(x,t), z'(x,t); w(x), w(x))dx, \end{aligned} \tag{6.4}$$

by the definition of Clarke generalized gradient and (6.1). Furthermore, since

$$\big\langle g(t), w\big\rangle_V = \big\langle g_1(t) + g_2(t), w\big\rangle_H = \big\langle (g_1(t), g_2(t)), (w, w)\big\rangle_{H\times H}$$

we have

$$\big\langle g(t), w\big\rangle_V \leq \int_\Omega j^\circ(x, z(x,t), z'(x,t); w(x), w(x))dx. \tag{6.5}$$

Finally, multiplying by $w \in V$ on both sides of equation in (6.3), then hemivariational inequality (1.1) follows from (6.5) immediately. We complete the proof of Theorem 6.1. $\square$

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