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NONHOMOGENEOUS ELLIPTIC PROBLEMS OF KIRCHHOFF TYPE INVOLVING CRITICAL SOBOLEV EXPONENTS

SAFIA BENMANSOUR, MOHAMMED BOUCHEKIF

Abstract. This article concerns the existence and the multiplicity of solutions for nonhomogeneous elliptic Kirchhoff problems involving the critical Sobolev exponent, defined on a regular bounded domain of $\mathbb{R}^3$. Our approach is essentially based on Ekeland's Variational Principle and the Mountain Pass Lemma.

1. Introduction

In this work we study the existence and the multiplicity of solutions for the problem

$$\begin{gathered} -(a\int_\Omega |\nabla u|^2dx+b)\Delta u=|u|^4u+f \quad \text{in } \Omega,\\ u=0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

where Ω is a smooth bounded domain of $\mathbb{R}^3$, a, b are positive constants and f belongs to H^{-1} (the topological dual of $H_0^1(\Omega)$) satisfying suitable conditions.

The original one-dimensional Kirchhoff equation was introduced by Kirchhoff [8] in 1883. His model takes into account the changes in length of the strings produced by transverse vibrations.

Problem (1.1) is called nonlocal because of the presence of the integral over the entire domain Ω, which implies that the equation in (1.1) is no longer a pointwise identity.

Problem (1.1) is related to the stationary analog of the Kirchhoff equation

$$\begin{gathered} u_{tt}-(a\int_\Omega |\nabla u|^2dx+b)\triangle u=h(x,u) \quad \text{in } \Omega\times(0,\ T),\\ u=0 \quad \text{in } \partial\Omega\times(0,\ T),\\ u(x,0)=u_0(x), \quad u_t(x,0)=u_1(x), \end{gathered}$$

where T is a positive constant, u_0 and u_1 are given functions. It can be seen as a generalization of the classical D'Alembert wave equation for free vibrations of elastic strings. For such problems, u denotes the displacement, $h(x, u)$ the external force, b is the initial tension and a is related to the intrinsic properties of the strings

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(such as Young's modulus). For more details, we refer the readers to the work of D'Ancona and Shibata [6] and the references therein.

Nonlocal problems arise not only from mathematical and physical fields but also from several other branches. When they appear in biological systems, u describes a process depending on the average of itself, as population density. Their theoretical study has attracted a lot of interests from mathematicians for a long time and many works have been done. We quote in particular the famous article of Lions [10]. However in most of papers, the used approach relies on topological methods.

In the last two decades, many authors have considered the stationary elliptic problem

$$\begin{gathered} -\Big(a\int_\Omega |\nabla u|^2dx + b\Big)\Delta u = h(x,u) \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.2}$$

where $\Omega \subset \mathbb{R}^N$ and $h(x,u)$ is a continuous function, see for example [1]. Alves and colleagues were the first to obtain existence results via variational methods. After this breakthrough, many works have been done in this direction. One can quote [2] for the case where $h(x,u)$ is asymptotically linear at infinity.

Problem (1.2) has also been extensively studied in the whole space when the potential function has a subcritical or critical growth, for more details see [9].

In the case of a bounded domain of $\mathbb{R}^N$ with $N \geq 3$, Tarantello [11] proved, under a suitable condition on f, the existence of at least two solutions to (1.2) for $a = 0$, $b = 1$ and $h(x,u) = |u|^{4/(N-2)}u + f$.

A natural and interesting question is whether results in [11] remain valid for $a > 0$. Our answer is affirmative and given for $N = 3$. To our best knowledge, this kind of problems has not been considered before.

We will use the following notation: S is the best Sobolev constant for the embedding from $H_0^1(\Omega)$ to $L^6(\Omega)$; $\|\cdot\|$ is the norm of $H_0^1(\Omega)$ induced by the product $(u,v) = \int_\Omega \nabla u \nabla v dx$; $\|\cdot\|_-$ and $|\cdot|_p := (\int_\Omega |\cdot|^p dx)^{1/p}$ are the norms in $H^{-1}$and $L^p(\Omega)$ for $1 \leq p < \infty$ respectively; we denote the space $H_0^1(\Omega)$ by H and the integrable $\int_\Omega udx$ by $\int u$; B_c^r is the ball of center c and radius r; $o_n(1)$ denotes any quantity which tends to zero as n tends to infinity, $O(\varepsilon^\alpha)$ means that $|O(\varepsilon^\alpha)\varepsilon^{-\alpha}| \leq K$ for some constant $K > 0$ and $o(\varepsilon^\alpha)$ means $|o(\varepsilon^\alpha)\varepsilon^{-\alpha}| \to 0$ as $\epsilon \to 0$.

In what follows, we fix $b > 0$ and consider a as a positive parameter. To state our main results, we need the following hypothesis

(H1) $|\int fv| < K_a(v)$, for all $v \in H$ such that $|v|_6 = 1$, where

$$K_a(v) := 10^{-5/2}[12a^2\|v\|^8 + 80b\|v\|^2 + 4a\|v\|^4A_a(v)][3a\|v\|^4 + A_a(v)]^{1/2}$$

with $A_a(v) := \|v\|(9a^2\|v\|^6 + 20b)^{1/2}$.

We shall prove the following results.

Theorem 1.1. *Assume that $f \neq 0$ satisfies* (H1). *Then problem* (1.1) *admits at least one weak solution in H. It is nonnegative if f is also nonnegative.*

Theorem 1.2. *Under hypothesis of Theorem 1.1 and for a a small positive number, problem* (1.1) *admits at least two weak solutions in H. They are nonnegative if f is also nonnegative.*

Remark 1.3. In dimension 1 and 2, our problem becomes subcritical and standard compactness argument applies to get the existence of solutions. This also happens for $f \equiv 0$. For dimensions higher than three, the problem under consideration turns out to be "supercritical" thus no existence result is suspected directly via variational methods.

Theorem 1.1 remains valid when f satisfies

$$\Big|\int fv\Big| \le K_a(v), \text{ for all } v \in H \quad \text{such that } |v|_6 = 1.$$

These remarks clarify the purpose of restricting this study to dimension three in this paper. This work is organized as follows: in Section 2 we give the definition of Palais-Smale condition and some preliminary results which we will use later. Section 3 is devoted to the proofs of Theorems 1.1 and 1.2.

2. Preliminary results

We define the energy functional corresponding to problem (1.1) by

$$I_a(u) = \frac{1}{2}\widehat{M}(\|u\|^2) - \frac{1}{6}|u|_6^6 - \int fu, \quad \text{for all } u \in H$$

where $\widehat{M}(t)$ is the primitive of $M(t) = at + b$ with $\widehat{M}(0) = 0$. It is clear that I_a is well defined and of C^1 on H and its critical points are weak solutions of problem (1.1) i.e. they satisfy:

$$(a\|u\|^2 + b)\int \nabla u \nabla v - \int |u|^4 uv - \int fv = 0, \text{ for all } v \in H.$$

The functional I_a is not bounded from below on H but it is on a subset of H. A good candidate for an appropriate subset of H is the so called Nehari manifold defined by

$$\mathcal{N} = \{u \in H\backslash\{0\} : \langle I_a'(u), u\rangle = 0\}.$$

Let $h_u(t) = I_a(tu)$ for $t \in \mathbb{R}^*$ and $u \in H\backslash\{0\}$. These maps are known as fibering maps and were first introduced by Drábek and Pohozaev [7]. The set $\mathcal{N}$ is closely linked to the behavior of $h_u(t)$, for more details see for example [5].

It is natural to split $\mathcal{N}$ into three subsets:

$$\mathcal{N}^+ := \{u \in \mathcal{N} : h_u''(1) > 0\}, \quad \mathcal{N}^0 := \{u \in \mathcal{N} : h_u''(1) = 0\},$$
$$\mathcal{N}^- := \{u \in \mathcal{N} : h_u''(1) < 0\},$$

where $h_u''(t) = -5|u|_6^6 t^4 + 3a\|u\|^4 t^2 + b\|u\|^2$. These subsets correspond to local minima, points of inflexion and local maxima of I_a respectively.

Definition 2.1. A sequence (u_n) is said to be a Palais-Smale sequence at level c ($(\text{P-S})_c$ in short) for I in H if

$$I(u_n) = c + o_n(1) \text{ and } I'(u_n) = o_n(1) \text{ in } H^{-1}.$$

We say that I satisfies the Palais-Smale condition at level c if any $(\text{P-S})_c$ sequence for I has a convergent subsequence in H.

Put

$$H_u(t) = h_u'(t) + \int fu = -|u|_6^6 t^5 + a\|u\|^4 t^3 + b\|u\|^2 t.$$

The function $H_u(t)$ attains its maximum $\widetilde{K}_a(u)$ at the point $t_{a,\max}^u$ where

$$\widetilde{K}_a(u) := 10^{-5/2}|u|_6^{-9}[12a^2\|u\|^8 + 80b|u|_6^6\|u\|^2 + 4a\|u\|^4\widetilde{A}_a(u)][3a\|u\|^4 + \widetilde{A}_a(u)]^{1/2}$$

and

$$t^u_{a,\max} = 10^{-1/2}|u|_6^{-3}(3a\|u\|^4 + \widetilde{A}_a(u))^{1/4}$$

with $\widetilde{A}_a(u) := \|u\|(9a^2\|u\|^6 + 20b|u|_6^6)^{1/2}$.

For $a \geq 0$, let

$$\widetilde{\mu}_{a,f} := \inf_{v\in H\setminus\{0\}}\{\widetilde{K_a}(v) - |\int fv|\}, \quad \mu_{a,f} := \inf_{|v|_6=1}\{K_a(v) - \int fv\}.$$

Remark 2.2. (i) If $\widetilde{\mu}_{a,f} > 0$ then $\mu_{a,f} > 0$.

(ii) We have, for $a > 0$, $\widetilde{\mu}_{a,f} \geq \widetilde{\mu}_{0,f}$. Under the hypothese (H1) with $a = 0$, Tarantello has proved that $\mu_{0,f} > 0$. Thus we deduce that $\widetilde{\mu}_{a,f} > 0$.

The following lemmas play crucial roles in the sequel.

Lemma 2.3. *Suppose* (H1) *holds. Then, for any $u \in H\setminus\{0\}$, there exist three unique values $t_1^+ = t_1^+(u)$, $t^- = t^-(u) \neq 0$ and $t_2^+ = t_2^+(u)$ such that:*

(i) $t_1^+ < -t^u_{a,\max}$, $t_1^+u \in \mathcal{N}^-$, *and* $I_a(t_1^+u) = \max_{t\leq -t^u_{a,\max}} I_a(tu)$,

(ii) $-t^u_{a,\max} < t^- < t^u_{a,\max}$, $t^-u \in \mathcal{N}^+$ *and* $I_a(t^-u) = \min_{|t|\leq t^u_{a,\max}} I_a(tu)$

(iii) $t_2^+ > t^u_{a,\max}, t_2^+u \in \mathcal{N}^-$ *and* $I_a(t_2^+u) = \max_{t\geq t^u_{a,\max}} I_a(tu)$.

Proof. An easy computation shows that $H_u(t)$ is concave for $t > 0$ and attains its maximum $\widetilde{K}_a(u)$ at $t^u_{a,\max}$. As $H_u(t)$ is odd and under the hypothesis (H1) we obtain the desired results. □

For $t > 0$, we have

$$\Psi(tu) = t\Psi(u), \quad \text{where } \Psi(u) = \widetilde{K_a}(u) - |\int fu|,$$

and for a given $\gamma > 0$, we derive that

$$\inf_{|u|_6\geq\gamma} \Psi(u) \geq \gamma\widetilde{\mu}_{a,f}. \tag{2.1}$$

In particular if f satisfies (H1) this infimum is bounded away from zero.

Lemma 2.4. *If f satisfies* (H1)*, then $\mathcal{N}^0 = \emptyset$.*

Proof. Arguing by contradiction we assume that there exists $u \in \mathcal{N}^0$, i.e.,

$$3a\|u\|^4 + b\|u\|^2 = 5|u|_6^6; \tag{2.2}$$

thus, we obtain:

$$\widetilde{A}_a(u) = 3a\|u\|^4 + 2b\|u\|^2, \text{ and } (t^u_{a,\max})^2 = 1\,.$$

Consequently,

$$\Psi(u) = \widetilde{K_a}(u) - |\int fu| \leq \widetilde{K_a}(u) - \int fu = H_u(1) - \int fu = h'_u(1) = 0\,. \tag{2.3}$$

Condition (2.2) implies that

$$|u|_6 \geq (\frac{b}{5}S)^{1/4} := \gamma.$$

From (2.1) and (2.3) we obtain

$$0 < \gamma\widetilde{\mu}_{a,f} \leq \Psi(u) = 0,$$

which yields a contradiction. □

Lemma 2.5. *Suppose that $f \neq 0$ satisfies* (H1), *then for each $u \in \mathcal{N}$, there exist $\varepsilon > 0$ and a differentiable function $t : B(0,\varepsilon) \subset H \to \mathbb{R}^+$ such that $t(0) = 1$, $t(v)(u-v) \in \mathcal{N}$ for $\|v\| < \epsilon$ and*

$$\langle t'(0), v\rangle = \frac{2(2a\|u\|^2 + b)\int \nabla u \nabla v - 6b\int |u|^4 uv - \int fv}{3a\|u\|^4 + b\|u\|^2 - 5|u|_6^6}. \tag{2.4}$$

Proof. Define the map $F : \mathbb{R} \times H \to \mathbb{R}$, by

$$F(s,w) = as^3\|u-w\|^4 + bs\|u-w\|^2 - s^5|u-w|_6^6 - \int f(u-w).$$

Since $F(1,0) = 0$, $\frac{\partial F}{\partial s}(1,0) = 3a\|u\|^4 + b\|u\|^2 - 5|u|_6^6 \neq 0$ and applying the implicit function theorem at the point $(1,0)$, we get the desired result. □

Define

$$c_0 = \inf_{v\in\mathcal{N}^+} I_a(v), \quad c_1 = \inf_{v\in\mathcal{N}^-} I_a(v). \tag{2.5}$$

Moreover if u_0 is a local minimum for I_a then we have $3a\|u_0\|^4 + b\|u_0\|^2 - 5|u_0|_6^6 \geq 0$ and since $\mathcal{N}^0 = \emptyset$, we obtain $u_0 \in \mathcal{N}^+$. Consequently $c_0 = \inf_{u\in\mathcal{N}} I_a(u)$.

Lemma 2.6. *The functional I_a is coercive and bounded from below on $\mathcal{N}$.*

Proof. For $u \in \mathcal{N}$, we have $a\|u\|^4 + b\|u\|^2 = |u|_6^6 + \int fu$. Therefore, we get

$$\begin{aligned} I_a(u) &= \frac{a}{12}\|u\|^4 + \frac{b}{3}\|u\|^2 - \frac{5}{6}\int fu \\ &\geq \frac{b}{3}\|u\|^2 - \frac{5}{6}\|f\|_-\|u\|, \\ &\geq \frac{-25}{48b}\|f\|_-^2, \end{aligned}$$

Thus I_a is coercive and bounded from below on $\mathcal{N}$. □

In particular, we have $c_0 \geq \frac{-25}{48b}\|f\|_-^2$. To prove that $c_0 < 0$, we need an upper bound for c_0. For this, consider $v \in H$ the unique solution of the equation $-\Delta u = f$. Then for $f \not\equiv 0$ we have $\int fv = \|v\|^2 = \|f\|_-^2$.

Let $t_0 = t^-(v)$, $v \in H\backslash\{0\}$ defined as in Lemma 2.3. So $t_0 v \in \mathcal{N}^+$ and consequently we have

$$\begin{aligned} I_a(t_0 v) &= -\frac{3a}{4}t_0^4\|v\|^4 - \frac{b}{2}t_0^2\|v\|^2 + \frac{5}{6}t_0^6|v|_6^6 \\ &\leq -\frac{a}{4}t_0^4\|v\|^4 - \frac{b}{3}t_0^2\|v\|^2 < 0, \end{aligned}$$

thus $c_0 < 0$.

Lemma 2.7. *Let f verifying* (H1), *then there exist minimizing sequences $(u_n) \subset \mathcal{N}^+$ and $(v_n) \subset \mathcal{N}^-$ such that*

(i) *$I_a(u_n) < c_0 + \frac{1}{n}$ and $I_a(w) \geq I_a(u_n) - \frac{1}{n}\|w - u_n\|$ for all $w \in \mathcal{N}^+$.*
(ii) *$I_a(v_n) < c_1 + \frac{1}{n}$ and $I_a(w) \geq I_a(v_n) - \frac{1}{n}\|w - v_n\|$ for all $w \in \mathcal{N}^-$.*

Proof. It is easy to prove that I_a is bounded in $\mathcal{N}$, then by using the Ekeland Variational Principle to minimization problems (2.5), we get minimizing sequences $(u_n) \subset \mathcal{N}^+$ and $(v_n) \subset \mathcal{N}^-$ satisfying (i) and (ii) respectively. □

Let $(u_n) \subset \mathcal{N}^+$ be the minimizing sequence obtained in the above lemma. For n large enough, we have

$$I_a(u_n) = \frac{a}{12}\|u_n\|^4 + \frac{b}{3}\|u_n\|^2 - \frac{5}{6}\int f u_n < c_0 + \frac{1}{n} \le -\frac{b}{3}t_0^2\|f\|_-^2,$$

this implies

$$\int f u_n \ge \frac{2}{5} b t_0^2 \|f\|_-^2 > 0, \tag{2.6}$$

and consequently we have

$$\frac{2}{5} b t_0^2 \|f\|_- \le \|u_n\| \le \frac{5}{2b}\|f\|_-. \tag{2.7}$$

So, we deduce that (u_n) is bounded in H.

Lemma 2.8. *Let f verifying* (H1)*, then $\|I_a'(u_n)\|$ tends to 0 as n tends to $+\infty$.*

Proof. Assume that $\|I_a'(u_n)\| > 0$ for n large, by applying Lemma 2.5 with $u = u_n$ and $w = \delta \frac{I_a'(u_n)}{\|I_a'(u_n)\|}$, $\delta > 0$ small, we find $t_n(\delta) := t[\delta \frac{I_a'(u_n)}{\|I_a'(u_n)\|}]$, such that

$$w_\delta = t_n(\delta)\big[u_n - \delta \frac{I_a'(u_n)}{\|I_a'(u_n)\|}\big] \in \mathcal{N}.$$

From the Ekeland Variational Principle, we have

$$\begin{aligned} \frac{1}{n}\|w_\delta - u_n\| &\ge I_a(u_n) - I_a(w_\delta) \\ &= (1 - t_n(\delta))\langle I_a(w_\delta), u_n\rangle + \delta t_n(\delta)\langle I_a'(w_\delta), \frac{I_a'(u_n)}{\|I_a'(u_n)\|}\rangle + o_n(\delta). \end{aligned}$$

Dividing by δ and passing to the limit as δ goes to zero, we get

$$\frac{1}{n}(1 + |t_n'(0)|\|u_n\|) \ge -t_n'(0)\langle I_a'(u_n),\ u_n\rangle + \|I_a'(u_n)\| = \|I_a'(u_n)\|,$$

where $t_n'(0) = \langle t'(0), \frac{I_a'(u_n)}{\|I_a'(u_n)\|}\ \rangle$. Thus from (2.7), we conclude that

$$\|I_a'(u_n)\| \le \frac{C}{n}(1 + |t_n'(0)|).$$

We claim that $|t_n'(0)|$ is bounded uniformly on n; indeed, since (u_n) is a bounded sequence, from (2.4) and the estimate (2.7), we have

$$|t_n'(0)| \le \frac{C}{|3a\|u_n\|^4 + b\|u_n\|^2 - 5|u_n|_6^6|}.$$

Hence we must prove that $|3a\|u_n\|^4 + b\|u_n\|^2 - 5|u_n|_6^6|$ is bounded away from zero. Arguing by contradiction, assume that for a subsequence still called (u_n), we have

$$3a\|u_n\|^4 + b\|u_n\|^2 - 5|u_n|_6^6 = o_n(1). \tag{2.8}$$

From (2.7) and (2.8) we derive that

$$|u_n|_6 \ge \gamma, \text{ for a suitable constant } \gamma$$

In addition (2.8) and the fact that $u_n \in \mathcal{N}$ also give

$$\int f u_n = -2a\|u_n\|^4 + 4|u_n|_6^6 + o_n(1),$$

which together with the definition of $\widetilde{\mu}_{a,f}$ imply that

$$\begin{aligned}0 < \gamma\widetilde{\mu}_{a,f} &\le \gamma(\widetilde{K_a}(u_n) - \int fu_n) + o_n(1)\\ &= \gamma h'_{u_n}(1) + o_n(1) = o_n(1).\end{aligned}$$

which is absurd. Thus $\|I'_a(u_n)\|$ tends to 0 as n tends to ∞. □

3. Proofs of Theorems 1.1 and 1.2

3.1. **Existence of a local minimizer on $\mathcal{N}^+$.** In this subsection, we prove that I_a achieves a local minimum in $\mathcal{N}^+$ by the Ekeland Variational Principle.

Proof of Theorem 1.1. Since (u_n) is bounded in H, passing to a subsequence if necessary, we have $u_n \rightharpoonup u_0$ weakly in H, then we get $\langle I'_a(u_0), w\rangle = 0$, for all $w \in H$. So u_0 is a weak solution for (1.1).

From (2.6), we deduce that $\int fu_0 > 0$, then $u_0 \in H\backslash\{0\}$ and in particular $u_0 \in \mathcal{N}$. Thus

$$c_0 \le I_a(u_0) = \frac{a}{12}\|u_0\|^4 + \frac{b}{3}\|u_0\|^2 - \frac{5}{6}\int fu_0 \le \liminf_{n\to\infty} I_a(u_n) = c_0,$$

then $c_0 = I_a(u_0)$. It follows that (u_n) converges strongly to u_0 in H and necessarily $u_0 \in \mathcal{N}^+$. To conclude that u_0 is a local minimum of I_a, let us recall that for every $u \in H$, we have

$$I_a(su) \ge I_a(t^- u) \quad \text{for every } 0 < s < t^u_{a,\max},$$

in particular for $u = u_0 \in \mathcal{N}^+$, we have $t^- = 1 < t^{u_0}_{a,\max}$. Choose $\varepsilon > 0$ sufficiently small to have $1 < t^{u_0-w}_{a,\max}$ and $t(w)$ satisfying $t(w)(u_0 - w) \in \mathcal{N}$ for every $\|w\| < \varepsilon$. Since $t(w) \to 1$ as $\|w\| \to 0$, we can always assume that

$$t(w) < t^{u_0-w}_{a,\max} \quad \text{for every } w \text{ such that } \|w\| < \varepsilon,$$

so $t(w)(u_0 - w) \in \mathcal{N}^+$ and for $0 < s < t^{u_0-w}_{a,\max}$, we have

$$I_a(s(u_0 - w)) \ge I_a(t(w)(u_0 - w)) \ge I_a(u_0),$$

taking $s = 1$, we conclude that $I_a(u_0 - w) \ge I_a(u_0)$, for all $w \in H$ such that $\|w\| < \varepsilon$. □

To see that $u_0 \ge 0$ when $f \ge 0$, it suffices to take $t_0 = t^-(|u_0|)$ such that $t_0|u_0| \in \mathcal{N}^+$. This implies that necessarily

$$I_a(t_0|u_0|) \le I_a(|u_0|) \le I_a(u_0).$$

Consequently, we can always take $u_0 \ge 0$.

3.2. **Existence of a local minimizer on $\mathcal{N}^-$.** This subsection is devoted to the existence of a second solution u_1 in $\mathcal{N}^-$ via Mountain Pass Lemma such that $c_1 = I_a(u_1)$. First we determine the good level for covering the Palais-Smale condition.

The best Sobolev constant S is attained in $\mathbb{R}^3$ by

$$U_{\varepsilon,x_0}(x) = \varepsilon^{1/2}(\varepsilon^2 + |x - x_0|^2)^{-1/2},$$

where $x_0 \in \Omega$ and $\varepsilon > 0$. We have the following important result.

Lemma 3.1. *Let f satisfying* (H1), *then I_a satisfies the (P-S)$_c$ condition for*

$$c < c^* = \frac{ab}{4}S^3 + \frac{a^3}{24}S^6 + \frac{b}{6}SE_1 + \frac{a^2}{24}S^4E_1 + c_0,$$

where $E_1 = (a^2S^4 + 4bS)^{1/2}$.

Proof. Let (u_n) be a (P-S)$_c$ sequence with $c < c^*$, then (u_n) is a bounded sequence in H. Thus it has a subsequence still denoted (u_n) such that $u_n \rightharpoonup u$ in H, $u_n \to u$ strongly in $L^s(\Omega)$ for all $1 \le s < 6$ and $u_n \to u$ a.e. in Ω.

Let $w_n = u_n - u$. From the Brezis-Lieb Lemma [4], one has:

$$\|u_n\|^2 = \|w_n\|^2 + \|u\|^2 + o_n(1), \quad \|u_n\|^4 = \|w_n\|^4 + 2\|w_n\|^2\|u\|^2 + \|u\|^4 + o_n(1),$$
$$|u_n|_6^6 = |w_n|_6^6 + |u|_6^6 + o_n(1).$$

Since $I_a(u_n) = c + o_n(1)$, we get

$$\frac{a}{4}\|w_n\|^4 + \frac{b}{2}\|w_n\|^2 + \frac{a}{2}\|w_n\|^2\|u\|^2 - \frac{1}{6}|w_n|_6^6 = I_a(u_n) - I_a(u) = c - I_a(u) + o_n(1)\,.$$

By the fact that $I_a'(u_n) = o_n(1)$ and $\langle I_a'(u), u\rangle = 0$, we obtain

$$a\|w_n\|^4 + b\|w_n\|^2 + 2a\|w_n\|^2\|u\|^2 - |w_n|_6^6 = o_n(1)\,.$$

Assume that $\|w_n\| \to l$ with $l > 0$, it follows that

$$|w_n|_6^6 = al^4 + bl^2 + 2al^2\|u\|^2 + o_n(1)\,.$$

From the definition of S, we have

$$\|w_n\|^2 \ge S|w_n|_6^2, \quad \text{for all } n\,..$$

As $n \to +\infty$, we deduce that

$$l^2 \ge \frac{a}{2}S^3 + \frac{1}{2}S\big(a^2S^4 + 4S(b + 2a\|u\|^2)\big)^{1/2}.$$

Consequently we obtain

$$\begin{aligned} c &= \frac{a}{12}l^4 + \frac{b}{3}l^2 + \frac{a}{6}l^2\|u\|^2 + I_a(u) \\ &\ge \frac{a}{12}l^4 + \frac{b}{3}l^2 + c_0 \\ &\ge \frac{ab}{4}S^3 + \frac{a^3}{24}S^6 + \frac{b}{6}SE_1 + \frac{a^2}{24}S^4E_1 + c_0 = c^* \end{aligned}$$

which is a contradiction. Therefore $l = 0$, then $u_n \to u$ strongly in H. □

Now, we shall give some useful estimates of the extremal functions. Let $\phi \in C_0^\infty(\Omega)$ such that $\phi(x) = 1$ for $x \in B_{x_0}^r$, $\phi(x) = 0$ for $x \in \mathbb{R}^3 \backslash B_{x_0}^{2r}$, $0 \le \phi \le 1$ and $|\nabla\phi| \le C$.

Set $u_{\varepsilon,x_0}(x) = \phi(x)U_{\varepsilon,x_0}(x)$. The following estimates are obtained in [3], as ε tends to 0:

$$|u_{\varepsilon,x_0}|_6^6 = A + O(\varepsilon^3) \quad \text{and} \quad \|u_{\varepsilon,x_0}\|^2 = B + O(\varepsilon),$$

where

$$A = \int_{\mathbb{R}^3} (1 + |x - x_0|^2)^{-3}, \quad B = \int_{\mathbb{R}^3} |\nabla U_{1,x_0}(x)|^2,$$

and from [11], we have $\int u_{\varepsilon,x_0}^5 u_0 = O(\varepsilon^{1/2}) + o(\varepsilon^{1/2})$.

In the search of our second solution, it is natural to show that $c_1 < c^*$. For this let $\Omega' \subset \Omega$ a be set of positive measure such that $u_0 > 0$ on Ω' (if not replace u_0 and f by $-u_0$ and $-f$ respectively), where u_0 is given in Theorem 1.1.

Lemma 3.2. *Assume that the hypothesis* (H1) *is satisfied, then there exist a_0 and ε_0 small enough such that for every $0 < \varepsilon < \varepsilon_0$ and $0 < a < a_0$ we have $I_a(u_0 + tu_{\varepsilon,x_0}) < c^*$ for all $t > 0$.*

Proof. From the above estimates and the Holder Inequality, we obtain

$$\begin{aligned}
&I_a(u_0 + tu_{\varepsilon,x_0})\\
&= I_a(u_0) + \frac{a}{4}t^4\|u_{\varepsilon,x_0}\|^4 + \frac{b}{2}t^2\|u_{\varepsilon,x_0}\|^2 - \frac{1}{6}t^6|u_{\varepsilon,x_0}|_6^6 - \frac{t^5}{6}\int u_\varepsilon^5 u_0\\
&\quad + at^2\Big[\Big(\int \nabla u_0 \nabla u_\varepsilon\Big)^2 + \|u_\varepsilon\|^2\Big(\frac{1}{2}\|u_0\|^2 + t\int \nabla u_0 \nabla u_\varepsilon\Big)\Big] + o(\varepsilon^{1/2})\\
&\le I_a(u_0) + \frac{a}{4}t^4B^2 + \frac{b}{2}t^2B - \frac{1}{6}t^6A - \frac{t^5}{6}O(\varepsilon^{1/2}) +\\
&\quad + at^2\big[\frac{3}{2}\|u_0\|^2B + tB^{3/2}\|u_0\|\big] + o(\varepsilon^{1/2})\\
&= c_0 + Q_\varepsilon(t) + R(t),
\end{aligned}$$

where

$$Q_\varepsilon(t) = -\frac{1}{6}At^6 + \frac{a}{4}B^2t^4 + \frac{b}{2}Bt^2 - \frac{t^5}{6}O(\varepsilon^{1/2}) + o(\varepsilon^{1/2}),$$

and

$$R(t) = a\big[\frac{3}{2}t^2\|u_0\|^2B + t^3B^{3/2}\|u_0\|\big].$$

We know that $\lim_{t\to+\infty} Q_\varepsilon(t) = -\infty$, and $Q_\varepsilon(t) > 0$ for t near 0, so $\sup_{t\ge 0} Q_\varepsilon(t)$ is achieved for $t = T_\varepsilon > 0$ and T_ε satisfies:

$$-AT_\varepsilon^5 + aB^2T_\varepsilon^3 + bBT_\varepsilon = O(\varepsilon^{1/2}).$$

Also $Q_0(t)$ attains its maximum at T_0 given by

$$T_0^2 = \frac{aB^2 + (a^2B^4 + 4bAB)^{1/2}}{2A}.$$

It is clear that T_ε tends to T_0 as ε goes to 0. Write $T_\varepsilon = T_0(1 \pm \delta_\varepsilon)$, hence δ_ε tends to 0 as ε goes to 0.

Moreover, since $I_a(u_0 + tu_\varepsilon) \to -\infty$ as t approaches ∞, there exists $T_\varepsilon < T_1$ such that

$$I_a(u_0 + tu_{\varepsilon,x_0}) \le c^* + Q_\varepsilon(T_\varepsilon) + \sup_{t<T_1} R(t).$$

On the other hand, we have

$$\begin{aligned}
Q_\varepsilon(T_\varepsilon) &= -\frac{1}{6}AT_\varepsilon^6 + \frac{a}{4}B^2T_\varepsilon^4 + \frac{b}{2}BT_\varepsilon^2 - O(\varepsilon^{1/2}) + o(\varepsilon^{1/2})\\
&= -\frac{1}{6}AT_0^6 + \frac{a}{4}B^2T_0^4 + \frac{b}{2}BT_0^2 \pm aT_0^4B^2\delta_\varepsilon \pm bT_0^2B\delta_\varepsilon \mp T_0^6A\delta_\varepsilon\\
&\quad - O(\varepsilon^{1/2}) + o(\varepsilon^{1/2})\\
&= -\frac{1}{6}AT_0^6 + \frac{a}{4}B^2T_0^4 + \frac{b}{2}BT_0^2 - O(\varepsilon^{1/2}) + o(\varepsilon^{1/2}).
\end{aligned} \tag{3.1}$$

Now substituting the expression of T_0 in (3.1), we obtain

$$\begin{aligned} Q_\varepsilon(T_\varepsilon) &= \frac{abB^3}{4A} + \frac{b(a^2B^6+4bB^3A)^{1/2}}{6A} + \frac{a^3B^6}{24A^2} + \frac{a^2(a^2B^{12}+4bB^9A)^{1/2}}{24A^2} \\ &\quad - O(\varepsilon^{1/2}) + o(\varepsilon^{1/2}) \\ &= \frac{ab}{4}S^3 + \frac{a^3}{24}S^6 + (\frac{b}{6}S + \frac{a^2}{24}S^4)(a^2S^4+4bS)^{1/2} - O(\varepsilon^{1/2}) + o(\varepsilon^{1/2}) \\ &\le c^* - c_0 - O(\varepsilon^{1/2}) + o(\varepsilon^{1/2}). \end{aligned}$$

Thus we have

$$\begin{aligned} I_a(u_0 + tu_{\varepsilon,x_0}) &\le c^* - O(\varepsilon^{1/2}) + o(\varepsilon^{1/2}) + \sup_{t<T_1} R(t) \\ &\le c^* - O(\varepsilon^{1/2}) + o(\varepsilon^{1/2}) + aK, \end{aligned}$$

where $K := \frac{3}{2}T_1^2\|u_0\|^2B + T_1^3B^{3/2}\|u_0\|$.

Consequently, there exist a_0 and ε_0 small enough such that $I_a(u_0+tu_{\varepsilon,x_0}) < c^*$ for every $0<\varepsilon<\varepsilon_0$ and $0<a<a_0$. □

Proof of Theorem 1.2. By Lemma 2.3, there exists an unique $t^+(u)>0$ such that $t^+(u)u \in \mathcal{N}^-$ and $I_a(t^+u) \ge I_a(tu)$, for all $|t| \ge t^u_{a,\max}$ and every $u \in H$ such that $\|u\|=1$.

The extremal property of $t^+(u)$ and its uniqueness give that it is a continuous function of u. Set

$$V_1 = \{0\} \cup \{u : \|u\| < t^+(\frac{u}{\|u\|})\}, \quad V_2 = \{u : \|u\| > t^+(\frac{u}{\|u\|})\}.$$

As in [11], we remark that under the condition (H1), we have $H\backslash\mathcal{N}^- = V_1 \cup V_2$ and $\mathcal{N}^+ \subset V_1$, $u_0 \in V_1$ and $u_0 + t_0u_\varepsilon \in V_2$ for a $t_0>0$, carefully chosen.

Let $\Gamma = \{h : [0,1] \to H \text{ continuous}, h(0)=u_0, h(1)=u_0+t_0u_\varepsilon\}$. It is obvious that $h:[0,1]\to H$ given by $h(t)=u_0+tt_0u_\varepsilon$ belongs to Γ. We conclude that

$$c = \inf_{h\in\Gamma}\max_{t\in[0,1]} I_a(h(t)) < c^*.$$

As the range of any $h\in\Gamma$ intersects $\mathcal{N}^-$, one has $c \ge c_1$.

Applying again the Ekeland Variational Principle, we obtain a minimizing sequence $(u_n) \subset \mathcal{N}^-$ such that

$$I_a(u_n) \to c_1 \quad \text{and} \quad \|I_a'(u_n)\| \to 0.$$

We also deduce that $c_1 < c^*$. Consequently, we get a subsequence (u_{n_k}) of (u_n) and $u_1 \in H$ such that

$$u_{n_k} \to u_1 \quad \text{strongly in } H.$$

This implies that u_1 is a critical point for I_a, $u_1 \in \mathcal{N}^-$ and $I_a(u_1)=c_1$. □

Finally for $f\ge 0$, let $t^+ = t^+(|u_1|) > 0$ satisfying $t^+|u_1| \in \mathcal{N}^-$. From Lemma 2.3 we have $I_a(u_1) = \max_{t\ge t_{a,\max}} I_a(tu_1) \ge I_a(t^+u_1) \ge I_a(t^+|u_1|)$. So we conclude that $u_1 \ge 0$.

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Safia Benmansour
Laboratoire des Systèmes Dynamiques et Applications. Faculté des Sciences, Université de Tlemcen. BP 119, 13000 Tlemcen, Algérie
E-mail address: safiabenmansour@hotmail.fr

Mohammed Bouchekif
Laboratoire des Systèmes Dynamiques et Applications. Faculté des Sciences, Université de Tlemcen. BP 119, 13000 Tlemcen, Algérie
E-mail address: m_bouchekif@yahoo.fr