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OSCILLATION OF SOLUTIONS TO FOURTH-ORDER TRINOMIAL DELAY DIFFERENTIAL EQUATIONS

JOZEF DŽURINA, BLANKA BACULÍKOVÁ, IRENA JADLOVSKÁ

Abstract. The objective of this article is to study the oscillation properties of the solutions to the fourth-order linear trinomial delay differential equation

$$y^{(4)}(t) + p(t)y'(t) + q(t)y(\tau(t)) = 0.$$

Applying suitable comparison principles, we present new criteria for oscillation. In contrast with the existing results, we establish oscillation of all solutions, and essentially simplify the examination process for oscillation. An example is included to illustrate the importance of results obtained.

1. Introduction

We consider the trinomial fourth-order differential equation with delay argument

$$y^{(4)}(t) + p(t)y'(t) + q(t)y(\tau(t)) = 0, \tag{1.1}$$

under the assumptions

(H1) $p(t), q(t) \in C([t_0,\infty))$, $p(t), q(t)$ are positive, $\tau(t) \in C^1([t_0,\infty))$, $\tau'(t) > 0$, $\tau(t) \le t$, $\lim_{t\to\infty} \tau(t) = \infty$.

By a solution we mean a function $y(t) \in C^4([T_y,\infty))$, $T_y \ge t_0$, which satisfies (1.1) on $[T_y,\infty)$. We consider only those solutions $y(t)$ of (1.1) which satisfy $\sup\{|y(t)| : t \ge T\} > 0$ for all $T \ge T_y$. We assume that (1.1) possesses such a solution. A solution of (1.1) is called oscillatory if it has arbitrarily large zeros on $[T_y,\infty)$ and otherwise it is called to be nonoscillatory. Equation (1.1) is said to be oscillatory if all its solutions are oscillatory.

The investigation of linear fourth-order differential equations, firstly originated with the vibrating rod problem of mathematical physics [5], is generally of great practical importance. Such equations form part of an immense collection of higher-order differential equations and are encountered in various fields of science and engineering as the more basic mathematical models. For instance, it is well known that the problem of beam deflection in linear theory of elasticity is described by classical linear fourth order equation

$$y^{(4)} + q(t)y = 0,$$

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where $y(t)$ approximates the shape of a beam, deflected from the equilibrium due to some external forces. Another particularly interesting model of physiological systems represented by fourth order differential equations with time delay concerns the oscillatory movements of muscles which can arise due to the interaction of a muscle with its inertial load [20].

In view of the above, the study on oscillations of the fourth-order differential equations has received considerable portion of attention and some profound results have been obtained. By establishing comparison theorems of Sturm's type, Leighton and Nehari [18], Howard [10] studied extensively the nature and behavior of solutions of a self-adjoin linear fourth-order differential equations of the form

$$(r(t)y'')'' \pm q(t)y = 0$$

and their results obtained were fundamental in the next research.

Thereafter, the problem of obtaining sufficient conditions for oscillatory and non-oscillatory properties of different classes of two-term fourth-order differential equations, including, for instance, delay and neutral delay dynamic equations on time scales, partial differential equations and difference equations has been and still is receiving intensive attention. We refer the reader to [1] - [22] and the references cited therein.

The problem of the oscillation of trinomial differential equations has been widely studied by many authors who have provided various techniques especially for lower order. The motivation for this work was twofold: a continuation of the pioneering work of Hou and Chengmin [11] and on the other hand, the thought of a missing analogy with investigation of third order trinomial differential equations taking advantage of existing results, which will be briefly stated.

A systematic study of asymptotic behavior of solutions of the third-order differential equation with damping of the form

$$y'''(t) + p(t)y'(t) + q(t)y(t) = 0,$$

has been made by [9], followed by [6, 7, 16, 12], to mention just a few.

The articles [2, 3, 21] deal with the third order delay nonlinear differential equation of the form

$$(r_2(t)(r_1(t)y'(t))')' + p(t)y'(t) + q(t)f(y(\tau(t))) = 0 \tag{1.2}$$

Using a generalized Riccati transformation and integral averaging technique, authors establish some sufficient conditions which ensure that any solution of oscillates or converges to zero. The another oscillation criteria have been obtained by establishing a useful comparison principle with either first or second order delay differential inequality, given in [1]. The key assumption in the above papers is the existence of a positive solution of the auxiliary second-order linear differential equation

$$(r_2(t)v'(t))' + \frac{p(t)}{r_1(t)}v(t) = 0. \tag{1.3}$$

Recently, the present authors have established in [4] new comparison theorems that reduce examination of the properties of the partial case of equation (1.2); that is,

$$y'''(t) + p(t)y'(t) + q(t)y(\tau(t)) = 0 \tag{1.4}$$

to the study of the properties of associated first order delay differential equations. This is possible by rewriting equation into the binomial form

$$\Big(v^2(t)\big(\frac{1}{v(t)}y'(t)\big)'\Big)' + v(t)q(t)y(\tau(t)) = 0,$$

making use of the solution (1.3). It is worth to note that another approaches exist, for example, in [8] it is generalized Lazer's result [16] and established new criteria depending on the sign of particular functional without requirement of an additional information of related second-order equation. Contrary to most known results, we stress that the technique used in the paper [4] has established oscillation of *all* solutions. The equation (1.4) can be viewed either as the lowest possible prototype of a higher-order trinomial differential equation

$$y^{(n)}(t) + p(t)y^{(n-2)}(t) + q(t)y(\tau(t)) = 0, \tag{1.5}$$

or

$$y^{(n)}(t) + p(t)y'(t) + q(t)y(\tau(t)) = 0 \tag{1.6}$$

While for equations of the first type the same approach holds, there is a few literature concerning the asymptotic and oscillatory properties of equations of the second type.

The authors in [11] studied equation (1.1) by means of Riccati transformation and presented conditions under which every nonoscillatory solution tends to zero as $t \to \infty$. They also indicated an interesting application in column-beam theory, where the middle term is incorporated to control the slope of a beam. Their crucial theorem ensures a constant sign first-derivative $y'(t)$ when an auxiliary third-order differential equation

$$z'''(t) + p(t)z(t) = 0 \tag{1.7}$$

has increasing solution.

In this article, we are dealing with the oscillation and asymptotic behavior of the solutions of the fourth-order delay trinomial differential equation (1.1). Establishing oscillatory criteria for fourth-order trinomial differential equations is far from easy, because the presence of the middle term $p(t)y'(t)$ causes the structure of possible nonoscillatory solutions to be unclear. Our technique permits us to rewrite trinomial equations as binomial differential equations with quasiderivative.

We offer a new approach, which uses a decreasing solution of an auxiliary differential equation (1.7) (which always exists) and a positive solution of related second-order differential equation (we provide condition under which it exists) in order to obtain its associated binomial form. Furthermore, by comparison with a couple of first-order delay differential equations, we establish oscillation of all solutions of (1.1).

2. Preliminary results

Before giving the main results, we will state Lemmas, which permit us to rewrite the studied trinomial equation (1.1) into the binomial equation.

Consider the operator

$$L_y = y^{(4)}(t) + p(t)y'(t).$$

Lemma 2.1. *Let $z(t)$ be a positive solution of*

$$z'''(t) + p(t)z(t) = 0. \tag{2.1}$$

Then

$$L_y = \Big[\frac{1}{z(t)}\Big(z^2(t)\big(\frac{y'(t)}{z(t)}\big)'\Big)'\Big]' + z''(t)\big(\frac{y'(t)}{z(t)}\big)'. \tag{2.2}$$

Proof. Simple computations show that the right hand side of (2.2) equals

$$\begin{aligned}
&\big[\frac{1}{z(t)}\big(y''(t)z(t) - y'(t)z'(t)\big)'\big]' + z''(t)\big(\frac{y'(t)}{z(t)}\big)' \\
&= \big[y'''(t) - \frac{y'(t)z''(t)}{z(t)}\big]' + z''(t)\big(\frac{y'(t)}{z(t)}\big)' \\
&= y^{(4)}(t) - y'(t)\frac{z'''(t)}{z(t)},
\end{aligned} \tag{2.3}$$

which in view of (2.1) leads to

$$L_y = y^{(4)}(t) + p(t)y'(t) = \Big[\frac{1}{z(t)}\Big(z^2(t)\big(\frac{y'(t)}{z(t)}\big)'\Big)'\Big]' + z''(t)\big(\frac{y'(t)}{z(t)}\big)'.$$

The proof is complete. □

Remark 2.2. It follows from Chanturia and Kiguradze's result [13], that (2.1) always possesses a positive decreasing solution, so-called Kneser solution.

We recall the another result of Chanturia and Kiguradze [13], that will be useful in the sequel.

Lemma 2.3. *Assume that*

$$\limsup_{t\to\infty} t^3 p(t) < \frac{2}{3\sqrt{3}}, \tag{2.4}$$

then all solutions of (2.1) *are non-oscillatory.*

It is important to note that the above transformation does not reduce (1.1) into the binomial form, as desired. However, it permits us to decrement a difference in the derivative order between the first and the second term of (1.1). In other words, (1.1), which is exactly a case of (1.6), turns out to be a more general version of the second higher-order mentioned prototype (1.5).

Now, consider an another operator

$$M_y = \frac{1}{v(t)}\Big[\frac{v^2(t)}{z(t)}\Big(\frac{z^2(t)}{v(t)}\big(\frac{1}{z(t)}y'(t)\big)'\Big)'\Big]'.$$

Lemma 2.4. *Let $z(t)$ be a positive decreasing solution of* (2.1) *and let the equation*

$$\big(\frac{1}{z(t)}v'(t)\big)' + \frac{z''(t)}{z^2(t)}v(t) = 0 \tag{2.5}$$

possess a positive solution. Then

$$L_y = \frac{1}{v(t)}\Big[\frac{v^2(t)}{z(t)}\Big(\frac{z^2(t)}{v(t)}\big(\frac{1}{z(t)}y'(t)\big)'\Big)'\Big]'. \tag{2.6}$$

Proof. We shall show, that operators M_y and L_y are equivalent. It is easy to see that:

$$
\begin{aligned}
M_y &= \frac{1}{v(t)}\Big[-\frac{v'(t)}{z(t)}z^2(t)\big(\frac{1}{z(t)}y'(t)\big)' + v(t)\frac{1}{z(t)}\Big(z^2(t)\big(\frac{1}{z(t)}y'(t)\big)'\Big)'\Big]' \\
&= \frac{1}{v(t)}\Big[-(\frac{v'(t)}{z(t)})'z^2(t)\big(\frac{1}{z(t)}y'(t)\big)' - \frac{v'(t)}{z(t)}\Big(z^2(t)\big(\frac{1}{z(t)}y'(t)\big)'\Big)' \\
&\quad + v'(t)\frac{1}{z(t)}(z^2(t)\big(\frac{1}{z(t)}y'(t)\big)')' + v(t)\Big(\frac{1}{z(t)}\Big(z^2(t)\big(\frac{1}{z(t)}y'(t)\big)'\Big)'\Big)'\Big] \\
&= \Big(\frac{1}{z(t)}\Big(z^2(t)\big(\frac{1}{z(t)}y'(t)\big)'\Big)'\Big)' - \frac{z^2(t)}{v(t)}(\frac{v'(t)}{z(t)})'\big(\frac{1}{z(t)}y'(t)\big)'.
\end{aligned}
\tag{2.7}
$$

Applying (2.2) from Lemma 2.1, we obtain

$$
\begin{aligned}
M_y &= y^{(4)}(t) + p(t)y'(t) - z''(t)\big(\frac{1}{z(t)}y'(t)\big)' - \frac{z^2(t)}{v(t)}(\frac{v'(t)}{z(t)})'\big(\frac{1}{z(t)}y'(t)\big) \\
&= y^{(4)}(t) + p(t)y'(t) - \big(\frac{1}{z(t)}y'(t)\big)'\frac{z^2(t)}{v(t)}\big[(\frac{1}{z(t)}v'(t))' + \frac{z''(t)}{z^2(t)}v(t)\big].
\end{aligned}
$$

Since $v(t)$ is a solution of (2.5), the previous equality yields

$$M_y = y^{(4)}(t) + p(t)y'(t) = L_y.$$

The proof is complete. □

Lemmas 2.1 and 3.1 permit us to rewrite studied trinomial equation (1.1) in the binomial equation

$$\Big[\frac{v^2(t)}{z(t)}\Big(\frac{z^2(t)}{v(t)}\big(\frac{1}{z(t)}y'(t)\big)'\Big)'\Big]' + v(t)q(t)y(\tau(t)) = 0. \tag{2.8}$$

As already stated, principal theorems in this paper will relate properties of solutions of the fourth order delay differential equation (1.1) to those of solutions of a couple of auxiliary linear ordinary differential equations of the third and the second order. We need to explore conditions that guarantee existence of positive solutions to the auxiliary equation (2.5).

Moreover, for our next purposes, it is desirable to have (2.8) in a canonical form

$$\int_{t_0}^{\infty} \frac{z(t)}{v^2(t)}\,\mathrm{d}t = \infty, \tag{2.9}$$

$$\int_{t_0}^{\infty} \frac{v(t)}{z^2(t)}\,\mathrm{d}t = \infty, \tag{2.10}$$

$$\int_{t_0}^{\infty} z(t)\,\mathrm{d}t = \infty, \tag{2.11}$$

since properties of canonical equations are generally nicely explored.

We will assume throughout the remainder of the paper that (2.11) holds. In the next result we crack the problem of the existence of positive solution for (2.5).

Lemma 2.5. *Assume that all solutions of* (2.1) *are non-oscillatory, then* (2.5) *possesses a positive solution.*

Proof. It is clear that all solutions of (2.5) are either oscillatory or non-oscillatory. We admit that (2.5) has an oscillatory solution $v(t)$. Then $v(t)$ also satisfies

$$-z'(t)v'(t) + z(t)v''(t) + z''(t)v(t) = 0.$$

Differentiating the last equality, one can see that

$$z(t)v'''(t) + z'''(t)v(t) = 0.$$

But (2.1) implies that $z'''(t)/z(t) = -p(t)$. Therefore, $v(t)$ is the oscillatory solution of the differential equation

$$v'''(t) - p(t)v(t) = 0. \tag{2.12}$$

On the other hand, Chanturia and Kiguradze [13] have shown that all solutions of (2.1) are nonoscillatory if and only if all solutions of (2.12) so does. This contradicts to oscillation of $v(t)$ and we conclude that all solutions of (2.5) are non-oscillatory. $\square$

Combining Lemma 2.3 and 2.5, we get easily verifiable criterion for (2.5) to be non-oscillatory.

Corollary 2.6. *Assume that* (2.4) *hold. Then* (2.5) *possesses a positive solution.*

Now, we show that under assumption (2.4), the conditions (2.9) and (2.10) are always satisfied.

Lemma 2.7. *Let* (2.4) *hold. Then* (2.5) *always has a solution* $v(t)$ *such that* (2.9) *and* (2.10) *are satisfied.*

Proof. The existence of a positive solution $v(t)$ of (2.5) follows from Corollary 2.6. Moreover, the monotonicity properties of $v(t)$ and $z(t)$ implies that $0 < z(t) < c_1$ and $v(t) > c_2 > 0$. Thus (2.10) is satisfied. On the other hand, If $v(t)$ does not satisfy (2.9); i.e.,

$$\int^{\infty} \frac{z(s)}{v^2(s)}\,\mathrm{d}s < \infty,$$

then it is easy to see that $v_*(t)$ given by

$$v_*(t) = v(t)\int_t^{\infty} \frac{z(s)}{v^2(s)}\,\mathrm{d}s \tag{2.13}$$

satisfies

$$\begin{aligned}\Big(\frac{1}{z(t)}v_*'(t)\Big)' &= \Big(\frac{1}{z(t)}v'(t)\Big)'\int_t^{\infty} \frac{z(s)}{v^2(s)}\,\mathrm{d}s\\ &= -\frac{z''(t)}{z^2(t)}v(t)\int_t^{\infty} \frac{z(s)}{v^2(s)}\,\mathrm{d}s = -\frac{z''(t)}{z^2(t)}v_*(t).\end{aligned}$$

Thus $v_*(t)$ is another positive solution of (2.5). Moreover, $v_*(t)$ meets (2.9) by now. To see this, let us denote

$$\mathcal{V}(t) = \int_t^{\infty} \frac{z(s)}{v^2(s)}\,\mathrm{d}s,$$

then $\lim_{t\to\infty} \mathcal{V}(t) = 0$ and

$$\int_{t_0}^{\infty} \frac{z(t)}{v_*^2(t)}\,\mathrm{d}t = -\int_{t_0}^{\infty} \frac{\mathcal{V}'(t)}{\mathcal{V}^2(t)}\,\mathrm{d}t = \lim_{t\to\infty}\Big(\frac{1}{\mathcal{V}(t)} - \frac{1}{\mathcal{V}(t_0)}\Big) = \infty.$$

$\square$

An immediate consequence of the above reasoning is the following result.

Corollary 2.8. *Let* (2.11) *hold. Assume that* (2.4) *is fulfilled, then the trinomial equation* (1.1) *can always be rewritten in its binomial form* (2.8) *and what is more,* (2.8) *is in the canonical form.*

3. Oscillation of (1.1)

Now, we are ready to study the properties of (1.1) with the help of (2.8). Without loss of generality, we can consider only with the positive solutions of (2.8). The following result is a modification of Kiguradze's lemma [13].

Lemma 3.1. *Let* (H2) *hold. Assume that* $y(t)$ *is an eventually positive solution of* (2.8), *then* $\Big[\frac{v^2(t)}{z(t)}\Big(\frac{z^2(t)}{v(t)}\big(\frac{1}{z(t)}y'(t)\big)'\Big)'\Big]' < 0$ *and, moreover, either*

$$y(t) \in \mathscr{N}_1 \iff y'(t) > 0,\ \big(\frac{1}{z(t)}y'(t)\big)' < 0,\ \Big(\frac{z^2(t)}{v(t)}\big(\frac{1}{z(t)}y'(t)\big)'\Big)' > 0$$

or

$$y(t) \in \mathscr{N}_3 \iff y'(t) > 0,\ \big(\frac{1}{z(t)}y'(t)\big)' > 0,\ \Big(\frac{z^2(t)}{v(t)}\big(\frac{1}{z(t)}y'(t)\big)'\Big)' > 0.$$

Consequently, assuming (H2), the set $\mathscr{N}$ of all positive solutions of (1.1) has the decomposition

$$\mathscr{N} = \mathscr{N}_1 \cup \mathscr{N}_3.$$

To obtain oscillation of studied equation (1.1), we need to eliminate booth cases of possible non-oscillatory solutions.

Let us denote

$$Q_1(t) = \Big(\frac{v(t)}{z^2(t)}\int_{t_1}^{\tau(t)} z(s)\,\mathrm{d}s\Big)\Big(\int_t^\infty \frac{z(u)}{v^2(u)}\int_u^\infty v(s)q(s)\,\mathrm{d}s\mathrm{d}u\Big)$$

and

$$Q_2(t) = v(t)q(t)\int_{t_1}^{\tau(t)} z(s_1)\int_{t_1}^{s_1}\frac{v(u)}{z^2(u)}\int_{t_1}^{u}\frac{z(s)}{v^2(s)}\,\mathrm{d}s\mathrm{d}u\mathrm{d}s_1.$$

Theorem 3.2. *Let (H2) hold. Assume that both first-order delay differential equations*

$$x'(t) + Q_1(t)x(\tau(t)) = 0 \tag{3.1}$$

and

$$x'(t) + Q_2(t)x(\tau(t)) = 0. \tag{3.2}$$

are oscillatory. Then (1.1) *is oscillatory.*

Proof. Assume that $y(t)$ is an eventually positive solution of (1.1). Then $y(t)$ obeys also (2.8). It follows from Lemma 3.1 that either $y(t) \in \mathscr{N}_1$ or $y(t) \in \mathscr{N}_3$. At first, we admit that $y(t) \in \mathscr{N}_1$. Noting that $\frac{1}{z(t)}y'(t)$ is decreasing, we see that

$$y(t) \ge \int_{t_1}^t z(u)\,\frac{y'(u)}{z(u)}\,du \ge \frac{y'(t)}{z(t)}\int_{t_1}^t z(u)\,du. \tag{3.3}$$

Integrating (2.8) from t to ∞, we have

$$\frac{v^2(t)}{z(t)}\Big(\frac{z^2(t)}{v(t)}\big(\frac{1}{z(t)}y'(t)\big)'\Big)' \ge \int_t^\infty v(s)q(s)y(\tau(s))\,\mathrm{d}s. \tag{3.4}$$

Taking into account that $y(\tau(t))$ is increasing, the last inequality yields

$$\Big(\frac{z^2(t)}{v(t)}\big(\frac{1}{z(t)}y'(t)\big)'\Big)' \geq y(\tau(t))\,\frac{z(t)}{v^2(t)}\,\int_t^\infty v(s)q(s)\,\mathrm{d}s. \tag{3.5}$$

Integrating once more, we are led to

$$-\big(\frac{1}{z(t)}y'(t)\big)' \geq y(\tau(t))\,\frac{v(t)}{z^2(t)}\int_t^\infty \frac{z(u)}{v^2(u)}\int_u^\infty v(s)q(s)\,\mathrm{d}s\mathrm{d}u. \tag{3.6}$$

Combining the last inequality with (2.3), one gets

$$-\big(\frac{1}{z(t)}y'(t)\big)' \geq \frac{1}{z(\tau(t))}y'(\tau(t))Q_1(t). \tag{3.7}$$

Thus, the function $x(t) = \frac{y'(t)}{z(t)}$ is positive a solution of the differential inequality

$$x'(t) + Q_1(t)x(\tau(t)) \leq 0.$$

Hence, by Philos theorem [19], we conclude that the corresponding differential equation (3.1) also has a positive solution, which contradicts to assumptions of the theorem.

Now, we shall assume that $y(t) \in \mathscr{N}_3$. Since $\frac{v^2(t)}{z(t)}\big(\frac{z^2(t)}{v(t)}\big(\frac{1}{z(t)}y'(t)\big)'\big)'$ is decreasing, we are led to

$$\begin{aligned}
\frac{z^2(t)}{v(t)}\big(\frac{1}{z(t)}y'(t)\big)' &\geq \int_{t_1}^t \frac{z(s)}{v^2(s)}\frac{v^2(s)}{z(s)}\Big(\frac{z^2(s)}{v(s)}\big(\frac{1}{z(s)}y'(s)\big)'\Big)'\,du \\
&\geq \frac{v^2(t)}{z(t)}\Big(\frac{z^2(t)}{v(t)}\big(\frac{1}{z(t)}y'(t)\big)'\Big)'\int_{t_1}^t \frac{z(s)}{v^2(s)}\,\mathrm{d}s.
\end{aligned}$$

Integrating the above inequality, one can verify that

$$y'(t) \geq z(t)\frac{v^2(t)}{z(t)}\Big(\frac{z^2(t)}{v(t)}\big(\frac{1}{z(t)}y'(t)\big)'\Big)'\int_{t_1}^t \frac{v(u)}{z^2(u)}\int_{t_1}^t \frac{z(s)}{v^2(s)}\,\mathrm{d}s\mathrm{d}u.$$

Integrating once more, we see that $x(t) = \frac{v^2(t)}{z(t)}\Big(\frac{z^2(t)}{v(t)}\big(\frac{1}{z(t)}y'(t)\big)'\Big)'$ satisfies

$$y(t) \geq x(t)\int_{t_1}^t z(s_1)\int_{t_1}^{s_1}\frac{v(u)}{z^2(u)}\int_{t_1}^t \frac{z(s)}{v^2(s)}\,\mathrm{d}s\mathrm{d}u\mathrm{d}s_1.$$

Setting the last estimate into (2.8), we see that $x(t)$ is a positive solution of the differential inequality

$$x'(t) + Q_2(t)x(\tau(t)) \leq 0,$$

which in view of Philos theorem in [19] guarantees that the corresponding differential equation (3.2) has also a positive solution. This is a contradiction and the proof is complete now. $\square$

Applying suitable criteria for oscillation of (3.1), (3.2), we obtain immediately criteria for oscillation of (E). The first one is due to Ladde et al. [15], while the second one pertains to Kusano and Kitamura [14].

Corollary 3.3. *Let (H2) hold. Assume that for $i = 1, 2$*

$$\liminf_{t\to\infty}\int_{\tau(t)}^t Q_i(s)\mathrm{d}s > \frac{1}{\mathrm{e}} \tag{3.8}$$

hold. Then (1.1) *is oscillatory.*

Corollary 3.4. *Let* (H2) *hold. Assume that for* $i = 1, 2$

$$\liminf_{t\to\infty} \int_t^\infty Q_i(s)\mathrm{d}s = \infty \tag{3.9}$$

hold. Then (1.1) *is oscillatory.*

Example 3.5. Let us consider the fourth order delay differential equation

$$y^{(4)}(t) + \frac{0.231}{t^3}\, y'(t) + \frac{a}{t^4}\, y(\lambda t) = 0, \quad a > 0, \quad \lambda \in (0,1), \quad t \ge 1. \tag{3.10}$$

For considered equation (3.10), the auxiliary equation (2.1) takes the form

$$z'''(t) + \frac{0.231}{t^3}\, z(t) = 0$$

with positive solution $z(t) = t^{-0.1}$. On the other hand, (2.5) reduces to

$$(t^{0.1}\, v'(t))' + \frac{0.11}{t^{1.9}}\, v(t) = 0$$

which possesses the positive solution $v(t) = t^\alpha$, where $\alpha = \frac{0.9-\sqrt{0.37}}{2}$. It is easy to verify that (H2) holds. Moreover, simple computation shows that

$$Q_1(t) = \frac{a\lambda^{0.9}}{0.9(3-\alpha)(2.1+\alpha)}\,\frac{1}{t} - \frac{K}{t^{1.9}}, \quad K \in \mathbb{R}.$$

and

$$Q_2(t) = \frac{a\lambda^{3-\lambda}}{(3-\alpha)(2.1-\alpha)(0.9-2\alpha)}\,\frac{1}{t} - \frac{K_1}{t^{1.9-2\alpha}} - \frac{K_2}{t^{3.1-\alpha}} - \frac{K_2}{t^{4-\alpha}}, \quad K_i \in \mathbb{R}$$

Criteria (3.8) and (3.9) from Corollary 3.4 yield

$$a\lambda^{0.9} \ln\frac{1}{\lambda} > \frac{0.9(3-\alpha)(2.1+\alpha)}{\mathrm{e}} \tag{3.11}$$

and

$$a\lambda^{3-\alpha} \ln\frac{1}{\lambda} > \frac{(3-\alpha)(2.1-\alpha)(0.9-2\alpha)}{\mathrm{e}} \tag{3.12}$$

respectively. By Corollary 3.4, we conclude that (3.10) is oscillatory if (3.11) and (3.12) hold simultaneously. For e.g. $\lambda = 0.6$ it happens provided that $a > 10.4991$.

We note that criteria from [11], [17] does not provide any information about oscillation of(1.1).

In this article we have established a new approach for studying the oscillation of the fourth order trinomial delay differential equation. One of the key elements in the method is to study the associated binomial representation (2.8) of equation (1.1). Our technique is based on the existence of positive solutions of a couple of auxiliary differential equations of the second and third order. Furthermore, we establish some basic properties of related linear operator to ensure the canonical form of (2.8).

As consequence, employing some comparison principles, one can easily deduce oscillation of all solutions of studied equation. The presented technique is new and essentially simplifies investigation of oscillation of fourth-order trinomial differential equations.

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References

[1] R. Agarwal, M. Aktas, A. Tiryaki, Aydın; *On oscillation criteria for third order nonlinear delay differential equations*, Arch. Math. 45 (2009), No. 4 1–18.

[2] M. F. Aktaş, D. Çakmak, A. Tiryaki; *On the qualitative behaviors of solutions of third order nonlinear differential equations*, Computers & Mathematics with Applications, 62 (2011), 2029–2036.

[3] M. F. Aktaş, A. Tiryaki, A. Zafer; *Oscillation criteria for third-order nonlinear functional differential equations*, Applied Mathematics Letters 7 (2010), 756–762.

[4] B. Baculíková, J. Džurina, *Comparison theorems for the third-order delay trinomial differential equations*, Advances in Difference Equations, 2010 (2010), 1–9.

[5] Courant, R., Hilbert, D.; *Methods of mathematical physics*, Marcel Dekker, New York, 1996.

[6] L. Erbe; *Existence of oscillatory solutions and asymptotic behavior for a class of third order linear differential equations*, Pacific J. Math. 64 (1976), 369–385.

[7] M. Cecchi, Z. Došlá, M. Marini; *On third order differential equations with property A and B*, Journal of Mathematical Analysis and Applications 231 (1999), 509–525.

[8] J. Graef, S. H. Saker; *Oscillation theory of third-order nonlinear functional differential equations*, Hiroshima Mathematical Journal, 43 (2013), 49–72.

[9] M. Hanan; *Oscillation criteria for third-order linear differential equations*, Pacific Journal of Mathematics, 11 (1961), 919–944.

[10] H. Howard; *Oscillation criteria for fourth-order linear differential equations*, Transactions of the American Mathematical Society 96(1960), 296–311.

[11] Ch. Hou, S.S. Chengmin; *Asymptotic dichotomy in a class of fourth-order nonlinear delay differential equations with damping*, Abstract and Applied Analysis, 2009(2009),1–9.

[12] J. Jones; *An asymptotic property of solutions of* $y''' + py' + qy = 0$, Pacific Journal of Mathematics, 47(1973),135–138.

[13] I. T. Kiguradze, T. A. Chaturia; *Asymptotic Properties of Solutions of Nonatunomous Ordinary Differential Equations*, Kluwer Acad. Publ., Dordrecht 1993.

[14] Y. Kitamura, T. Kusano; *Oscillations of first-order nonlinear differential equations with deviating arguments,* Proc. Amer. Math. Soc., 78 (1980), pp. 64–68.

[15] G. S. Ladde, V. Lakshmikantham, B. G. Zhang; *Oscillation Theory of Differential Equations with Deviating Arguments*, Marcel Dekker, New York, 1987. Zbl 0832.34071

[16] A. C. Lazer; *The behavior of solutions of the differential equation,* Pacific J. Math, 17(1966), pp. 430–435.

[17] H. Liang; *Asymptotic behavior of solutions to higher order nonlinear delay differential equations,* Electron. J. Diff. Equ., 2014 (2014), No. 186, pp. 1–12.

[18] W. Leighton; *Quadratic functionals of second order,* Transactions of the American Mathematical Society, 151 (1970), pp. 309–322.

[19] Ch. G. Philos; *Ot the existence of nonoscillatory solutions tending to zero at* ∞ *for differential equations with positive delay*, Arch. Math. 36 (1981), 168–178. Zbl 0463.34050

[20] R. B. Stein; *An analysis of oscillations in neuro-muscular systems,* Journal of Mathematical Biology, 2 (1975), pp. 87–105.

[21] A. Tiryaki, M.F.Aktas; *Oscillation criteria of a certain class ot third order nonlinear delay differential equations with damping*, J. Math. Anal. Appl. 325(2007), 54–68. Zbl 1110.34048

[22] E. Thandapani, S. Pandian, R. Dhanasekaran, J. Graef; *Asymptotic results for a class of fourth order quasilinear difference equations*, Journal of Difference Equations and Applications 13(2007), 1085–1103.

Department of Mathematics, Faculty of Electrical Engineering and Informatics, Technical University of Košice, Letná 9, 042 00 Košice, Slovakia

E-mail address: `jozef.dzurina@tuke.sk`

E-mail address: `blanka.baculikova@tuke.sk`

E-mail address: `irena.jadlovska@student.tuke.sk`