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THEOREMS OF KIGURADZE-TYPE AND BELOHOREC-TYPE REVISITED ON TIME SCALES

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Abstract. This article concerns the oscillation of second-order nonlinear dynamic equations. By using generalized Riccati transformations, Kiguradze-type and Belohorec-type oscillation theorems are obtained on an arbitrary time scale. Our results cover those for differential equations and difference equations, and provide new oscillation criteria for irregular time scales. Some examples are given to illustrate our results.

1. Introduction

Consider the second-order nonlinear dynamic equation

$$\big(r(t)x^{\Delta}(t)\big)^{\Delta} + p(t)f(x^{\sigma}(t)) = 0, \quad t \geq t_0, \tag{1.1}$$

where the independent variable is in a time scale $\mathbb{T}$.

Since we are interested in the oscillatory behavior of solutions near infinity, we assume that $\sup \mathbb{T} = \infty$. Recall that a solution to (1.1) is a nontrivial real function $x(t)$ such that $x(t) \in C^1_{rd}[t_0,\infty)$, and $r(t)x^{\Delta}(t) \in C^1_{rd}[t_0,\infty)$ and satisfying (1.1) on $[t_0,\infty)$, where C_{rd} is the space of real-valued right-dense continuous functions (see [3, p. 7]). Throughout this paper, we shall restrict attention to those solutions of (1.1) which exist on some half line $[t_0,\infty)$ and satisfy $\sup\{|x(t)| : t > t_0\} > 0$. For simplicity of notation in the lemmas, theorems, and examples that follow, we use $[t_0,\infty) := [t_0,\infty)_{\mathbb{R}} \cap \mathbb{T}$, $x^{\alpha}(\sigma(t)) = \big(x^{\sigma}\big)^{\alpha} = x^{\alpha^{\sigma}}$ and $(x^{\alpha}(t))^{\Delta} = (x^{\alpha})^{\Delta} = x^{\alpha^{\Delta}}$, and so on.

We recall that a solution of (1.1) is called oscillatory if it has arbitrarily large zeros, otherwise, it is called nonoscillatory. Equation (1.1) is said to be oscillatory if every function that satisfies the equation eventually, that is, on the interval $[t_0, \infty)$, has arbitrarily large zeros. It is not sufficient only to know that some solutions have this behavior. The oscillation theory of dynamic equations has been developed extensively during the past several years. We refer the reader to the monographs [1, 3] and the references cited therein. Recently, there has been an increasing interest in studying the oscillation of nonlinear differential equations on time scales [2,4-16,19-22]. The oscillation problem for (1.1) and its various particular cases has been considered by many authors. For completeness, we review some earlier results:

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In 1955, Atkinson [2] considered the second-order nonlinear differential equation

$$x''(t) + p(t)|x(t)|^{\alpha-1}x(t) = 0, \quad t \geq t_0, \tag{1.2}$$

and established the following necessary and sufficient condition for the oscillation of (1.2).

Theorem 1.1. *Let $\alpha > 1$ and $p(t) \geq 0$. Then (1.2) is oscillatory if and only if*

$$\int_{t_0}^{\infty} sp(s)ds = \infty. \tag{1.3}$$

If $p(t)$ is allowed to take on negative values, Kiguradze [12] proved that condition (1.3) is still sufficient for all solutions of (1.2) to be oscillatory for the same case considered by Atkinson.

In 1961, Belohorec [5] considered the sublinear case of (1.2), and obtained the following result.

Theorem 1.2. *Let $0 < \alpha < 1$ and $p(t) \geq 0$. Then (1.2) is oscillatory if and only if*

$$\int_{t_0}^{\infty} s^{\alpha}p(s)ds = \infty. \tag{1.4}$$

If $p(t)$ is allowed to take on negative values, Belohorec [6] proved that (1.4) is sufficient for (1.2) to be oscillatory. These results have been further extended by Kwong and Wong [13].

In 1983, Hooker and Patula [8] considered the following second-order nonlinear difference equation

$$\Delta^2 x(n) + p(n)|x(n+1)|^{\alpha-1}x(n+1) = 0, \quad t \in \mathbb{N}. \tag{1.5}$$

and established the following necessary and sufficient condition for the oscillation of (1.5).

Theorem 1.3. *Let $\alpha > 1$ and $p(t) \geq 0$. Then (1.5) is oscillatory if and only if*

$$\sum_{n=1}^{\infty} np(n) = \infty. \tag{1.6}$$

For the sublinear analog, Mingarelli [15] proved the following result.

Theorem 1.4. *Let $0 < \alpha < 1$ be a quotient of odd positive integers, and $p(n) \geq 0$. Then (1.5) is oscillatory if and only if*

$$\sum_{n=1}^{\infty} n^{\alpha}p(n) = \infty. \tag{1.7}$$

In 2011, Jia et al [9] considered the following second-order superlinear dynamic equation

$$x^{\Delta\Delta}(t) + p(t)f(x^{\sigma}(t)) = 0, \quad t \in \mathbb{T}, \tag{1.8}$$

and obtained Kiguradze-type oscillation theorems for the dynamic equation (1.8). Jia et al [10] considered the following second-order sublinear dynamic equation

$$x^{\Delta\Delta}(t) + p(t)x^{\alpha}(\sigma(t)) = 0, \quad t \in \mathbb{T}. \tag{1.9}$$

where $0 < \alpha < 1$ is a quotient of odd positive integers and $p(t)$ is allowed to take on negative values. It was noted, more generally in [10] that (1.9) is oscillatory if there exists a real number β satisfying $0 < \beta \leq 1$, such that

$$\int_{t_0}^{\infty} \sigma^{\alpha\beta}(s)p(s)\Delta t = \infty.$$

However, these results of Jia et al [9, 10] were obtained for the case that $\mathbb{T}$ is a *regular* time scale. Let us recall that $\mathbb{T}$ is said to be a regular time scale provided it is a time scale with $\inf \mathbb{T} = t_0$ and $\sup \mathbb{T} = \infty$, and $\mathbb{T}$ is either an isolated time scale (all points in $\mathbb{T}$ are isolated) or $\mathbb{T}$ is the real interval $[t_0, \infty)$. An important tool in the study of the oscillatory behavior of solutions is the Riccati transformation

$$\omega(t) = r(t)\frac{x'(t)}{x(t)}, \quad t \in \mathbb{R}, \tag{1.10}$$

which goes back as far as the classical results of Kamenev [14] and Philos [16]. It is also a very useful tool for the half-linear or nonlinear differential equations when the coefficient function is nonnegative [19, 20].

In this article, we consider more general dynamic equations on arbitrary time scales when $p(t)$ is allowed to take on negative values. Our purpose here, is to establish oscillation criteria for all time scales. That is, we will not assume that $\mathbb{T}$ is a regular time scale. We will use the Riccati transformation (1.10) which, as noted above, is still applicable, even when the coefficient function is allowed to take on negative values. In the next section, we shall give some lemmas, which will be used to prove our main theorems. In Section 3, we shall give the Kiguradze-type oscillation theorems for (1.1) which can be applied to any time scale. We also obtain some new oscillation criteria for all bounded solutions of (1.1). In Section 4, we shall establish the Belohorec-type oscillation theorems which are also valid for all time scales. Finally, in Section 5, by means of several examples, we illustrate our results.

2. Some lemmas

On an arbitrary time scale $\mathbb{T}$, the usual chain rule from calculus is no longer valid. One form of the extended chain rule, due to Keller [11] and generalized to measure chains by Pötzsche [17], is as follows.

Lemma 2.1. *Assume $G : \mathbb{T} \to \mathbb{R}$ is delta differentiable on $\mathbb{T}$. Assume further that $F : \mathbb{R} \to \mathbb{R}$ is continuously differentiable. Then $F \circ G : \mathbb{T} \to \mathbb{R}$ is delta differentiable and satisfies*

$$[F \circ G]^{\Delta}(t) = \Big\{ \int_0^1 F'(G(t) + h\mu(t)G^{\Delta}(t))dh \Big\} G^{\Delta}(t).$$

We shall need the following second mean value theorem (see [9, 10]) and the differential inequality, which will be useful tools in the following proofs and which we formulate as follows.

Lemma 2.2. *Let h be a bounded function that is integrable on $[a, b]$. Let m_H and M_H be the infimum and supremum, respectively, of the function $H(t) := \int_a^t h(s)\Delta s$ on $[a, b]$. Suppose that g is nonincreasing with $g(t) \geq 0$ on $[a, b]$. Then there is* *some number Λ with $m_H \le \Lambda \le M_H$ such that*

$$\int_a^b h(t)g(t)\Delta t = g(a)\Lambda.$$

Lemma 2.3. *Assume $G : \mathbb{T} \to \mathbb{R}$ is delta differentiable on $\mathbb{T}$. Assume further that $F : \mathbb{R} \to \mathbb{R}$ and F'' is continuous. If the function F satisfies $F'(u) \ge 0$ and $F''(u) \le 0$. Then*

$$F'(G^\sigma(t))G^\Delta(t) \le [F \circ G]^\Delta(t) \le F'(G(t))G^\Delta(t).$$

Proof. Using Lemma 2.1, we obtain

$$[F \circ G]^\Delta(t) = \Big\{ \int_0^1 F'(G(t) + h\mu(t)G^\Delta(t))dh \Big\} G^\Delta(t). \tag{2.1}$$

If $G^\Delta(t) \ge 0$, it is easy to see that

$$G^\sigma(t) \ge G(t) + h\mu(t)G^\Delta(t) \ge G(t) \quad \text{for all } 0 \le h \le 1. \tag{2.2}$$

From (2.2) and $F''(u) \le 0$, we have

$$F'(G^\sigma(t)) \le F'\big(G(t) + h\mu(t)G^\Delta(t)\big) \le F'(G(t)) \quad \text{for all } 0 \le h \le 1. \tag{2.3}$$

From (2.3), $F'(u) \ge 0$ and $G^\Delta(t) \ge 0$, we obtain

$$F'(G^\sigma(t))G^\Delta(t) \le \Big\{ \int_0^1 F'(G(t) + h\mu(t)G^\Delta(t))dh \Big\} G^\Delta(t) \le F'(G(t))G^\Delta(t).$$

Therefore, from (2.1), we obtain

$$F'(G^\sigma(t))G^\Delta(t) \le [F \circ G]^\Delta(t) \le F'(G(t))G^\Delta(t).$$

If $G^\Delta(t) \le 0$, by a similar analysis, we obtain the same conclusion. This completes the proof. □

3. Kiguradze-type oscillation theorems

In this section, with respect to (1.1) we shall assume the following conditions hold:

- (H1) $p \in C_{rd}(\mathbb{T}, \mathbb{R})$ and $1/r \in C_{rd}(\mathbb{T}, \mathbb{R}^+)$ (see e.g. [18]);
- (H2) $f \in C(\mathbb{R}, \mathbb{R})$ satisfies $f'(x) \ge 0$, and $xf(x) > 0$ for $x \ne 0$;
- (H3) The superlinearity conditions:

$$0 < \int_\varepsilon^\infty \frac{1}{f(u)} du, quad \int_{-\varepsilon}^{-\infty} \frac{1}{f(u)} du < +\infty$$

  for all $\varepsilon > 0$.

Theorem 3.1. *Assume that conditions* (H1)–(H3) *hold. If there exists a function $\phi > 0$ satisfying $\phi^\Delta \ge 0$ and $[\phi^\Delta r]^\Delta \le 0$ such that*

$$\int_{t_0}^\infty \frac{1}{\phi(s)r(s)} \Delta s = \infty \quad \text{and} \quad \int_{t_0}^\infty \phi^\sigma(s)p(s)\Delta s = \infty. \tag{3.1}$$

Then (1.1) *is oscillatory.*

Proof. Suppose that $x(t)$ is a nonoscillatory solution of (1.1) on $[t_0,\infty)$. Without loss of generality, we may assume that $x(t)>0$ for $t\geq t_1\geq t_0$. By condition (H2), we shall only consider this case, since the substitution $z(t)=-x(t)$ transforms (1.1) into an equation of the same form. Define the following generalized Riccati transformation:

$$\omega(t):=\phi(t)\frac{r(t)x^\Delta(t)}{f(x(t))}\quad\text{for } t\geq t_1. \tag{3.2}$$

By the product rule and the quotient rule, we obtain

$$\begin{aligned}
\omega^\Delta &= [rx^\Delta]^\Delta[\frac{\phi}{f\circ x}]^\sigma + rx^\Delta[\frac{\phi}{f\circ x}]^\Delta\\
&= \phi^\sigma\frac{[rx^\Delta]^\Delta}{f\circ x^\sigma} + rx^\Delta[\frac{\phi^\Delta}{f\circ x^\sigma} - \frac{\phi[f\circ x]^\Delta}{[f\circ x][f\circ x^\sigma]}]\\
&= -\phi^\sigma p + \phi^\Delta r\frac{x^\Delta}{f\circ x^\sigma} - \frac{\phi r x^\Delta[f\circ x]^\Delta}{[f\circ x][f\circ x^\sigma]}.
\end{aligned}$$

Integrating the above equation from t_1 to t, we obtain that

$$\omega(t)-\omega(t_1) = -\int_{t_1}^t \phi^\sigma p\Delta s + \int_{t_1}^t \phi^\Delta r\frac{x^\Delta}{f\circ x^\sigma}\Delta s - \int_{t_1}^t \frac{\phi r x^\Delta[f\circ x]^\Delta}{[f\circ x][f\circ x^\sigma]}\Delta s. \tag{3.3}$$

From Lemma 2.1, we obtain

$$[f(x(t))]^\Delta = \Big\{\int_0^1 f'(x(t)+h\mu(t)x^\Delta(t))dh\Big\}x^\Delta(t).$$

So, from condition (H2), we have

$$x^\Delta(t)[f(x(t))]^\Delta\geq 0. \tag{3.4}$$

From condition (H1) and Lemma 2.2, we obtain

$$\int_{t_1}^t \phi^\Delta r\frac{x^\Delta}{f\circ x^\sigma}\Delta s = \phi^\Delta(s)r(s)|_{s=t_1}\Lambda(t)\quad\text{for } t\in[t_1,\infty), \tag{3.5}$$

where $m_x\leq\Lambda(t)\leq M_x$, and m_x and M_x denote the infimum and supremum, respectively, of the function $\int_{t_1}^s \frac{x^\Delta(\xi)}{f(x^\sigma(\xi))}\Delta\xi$ for $s\in[t_1,t]$. Letting

$$F(u)=\int_\varepsilon^u\frac{1}{f(v)}dv$$

and $G(\xi)=x(\xi)$, by Lemma 2.3, for all $\varepsilon>0$, we have

$$[F(G(\xi))]^\Delta = \Big[\int_\varepsilon^{x(\xi)}\frac{1}{f(v)}dv\Big]^\Delta \geq \frac{x^\Delta(\xi)}{f(x^\sigma(\xi))}. \tag{3.6}$$

Integrating the above equation from t_1 to s, we obtain

$$\begin{aligned}
\int_{t_1}^s\frac{x^\Delta(\xi)}{f(x^\sigma(\xi))}\Delta\xi &\leq \int_\varepsilon^{x(s)}\frac{1}{f(v)}dv - \int_\varepsilon^{x(t_1)}\frac{1}{f(v)}dv\\
&= \int_{x(t_1)}^{x(s)}\frac{1}{f(v)}dv < \int_{x(t_1)}^\infty\frac{1}{f(v)}dv.
\end{aligned} \tag{3.7}$$

So, from condition (H3) and (3.7), we have

$$\Lambda(t)\leq M_x\leq M(t_1):=\int_{x(t_1)}^\infty\frac{1}{f(v)}dv,\quad\text{for } t\in[t_1,\infty). \tag{3.8}$$

From (3.3), (3.5), and (3.8), we have that

$$\omega(t)-\omega(t_1)\le -\int_{t_1}^{t}\phi^\sigma p\Delta s-\int_{t_1}^{t}\frac{\phi r x^\Delta[f\circ x]^\Delta}{[f\circ x][f\circ x^\sigma]}\Delta s+\phi^\Delta(t_1)r(t_1)M(t_1). \tag{3.9}$$

Since $\int_{t_0}^{\infty}\phi^\sigma(s)p(s)\Delta s=\infty$, from (3.4) and (3.9), it is easy to see that there exists a sufficiently large $t_2\ge t_1$, such that for $t\ge t_2$

$$\begin{aligned}\omega(t)+\int_{t_2}^{t}\frac{\phi r x^\Delta[f\circ x]^\Delta}{[f\circ x][f\circ x^\sigma]}\Delta s&\le \omega(t)+\int_{t_1}^{t}\frac{\phi r x^\Delta[f\circ x]^\Delta}{[f\circ x][f\circ x^\sigma]}\Delta s\\&\le \omega(t_1)-\int_{t_1}^{t}\phi^\sigma p\Delta s+\phi^\Delta(t_1)r(t_1)M(t_1)\\&< -1.\end{aligned}\tag{3.10}$$

In particular, from (3.4) and (3.10), we have

$$x^\Delta(t)<0,\quad \text{for } t\ge t_2. \tag{3.11}$$

Define

$$y(t):=1+\int_{t_2}^{t}\frac{\phi r x^\Delta[f\circ x]^\Delta}{[f\circ x][f\circ x^\sigma]}\Delta s. \tag{3.12}$$

Obviously, $y(t_2)=1$. From (3.10), we obtain

$$-\omega(t)\ge y(t). \tag{3.13}$$

From (3.11), (3.12) and (3.13) we have

$$y^\Delta(t)=\frac{\phi(t)r(t)x^\Delta(t)[f(x(t))]^\Delta}{f(x(t))f(x(\sigma(t)))}>-y(t)\frac{[f(x(t))]^\Delta}{f(x(\sigma(t)))}. \tag{3.14}$$

It follows that $y^\Delta(t)f(x^\sigma(t))+y(t)[f(x(t))]^\Delta>0$. That is $[y(t)f(x(t))]^\Delta>0$. From (3.13), we obtain

$$-\frac{\phi(t)r(t)x^\Delta(t)}{f(x(t))}\ge y(t)>\frac{y(t_2)f(x(t_2))}{f(x(t))},\quad \text{for } t\ge t_2.$$

Therefore, we obtain

$$x^\Delta(t)<-\frac{y(t_2)f(x(t_2))}{\phi(t)r(t)}.$$

Integrating the above equation from t_2 to t, we obtain

$$x(t)-x(t_2)<-\int_{t_2}^{t}\frac{y(t_2)f(x(t_2))}{\phi(s)r(s)}\Delta s\to-\infty,\quad \text{as } t\to\infty,$$

which contradicts $x(t)>0$. This completes the proof of Theorem 3.1. $\square$

Let $\phi(s)=s^\beta$. As a consequence of Theorem 3.1, we can get the following theorem which is the main result of Jia et al [9] on a regular time scale when $r(t)=1$.

Theorem 3.2. *Assume that conditions* (H1)–(H3) *hold. Suppose there exists a real number* β *satisfying* $0<\beta\le 1$, *such that* $\int_{t_0}^{\infty}\sigma^\beta(s)p(s)\Delta t=\infty$. *Then* (1.1) *with* $r(t)=1$ *is oscillatory.*

If we do not assume the superlinearity condition (H3) in the previous theorems, then from (3.7), we can conclude that all bounded solutions are oscillatory which also includes the sublinear case. That is, we can state the following theorem.

Theorem 3.3. *Assume that conditions* (H1)–(H2) *hold. If there exists a function* $\phi > 0$ *satisfying* $\phi^\Delta \geq 0$ *and* $[\phi^\Delta r]^\Delta \leq 0$ *such that*

$$\int_{t_0}^{\infty} \frac{1}{\phi(s)r(s)}\Delta s = \infty \quad \text{and} \quad \int_{t_0}^{\infty} \phi^\sigma(s)p(s)\Delta s = \infty. \tag{3.15}$$

Then all bounded solutions of (1.1) *are oscillatory.*

4. Belohorec-type oscillation theorems

In this section, we consider the second-order sublinear dynamic equation

$$\big(r(t)x^\Delta(t)\big)^\Delta + p(t)x^\alpha(\sigma(t)) = 0, \quad t \geq t_0, \tag{4.1}$$

where the independent variable is in a time scale $\mathbb{T}$, $0 < \alpha < 1$ is a quotient of odd positive integers, $p \in C_{rd}(\mathbb{T}, \mathbb{R})$ and $1/r \in C_{rd}(\mathbb{T}, \mathbb{R}^+)$.

Theorem 4.1. *If there exists a function* $\phi > 0$ *satisfying* $\phi^\Delta \geq 0$ *and* $[r\phi^\Delta]^\Delta \leq 0$ *such that*

$$\int_{t_0}^{\infty} \frac{1}{r(s)\phi(s)}\Delta s = \infty \quad \text{and} \quad \int_{t_0}^{\infty} \phi^\alpha(\sigma(s))p(s)\Delta s = \infty. \tag{4.2}$$

Then (4.1) *is oscillatory.*

Proof. Suppose that $x(t)$ is a nonoscillatory solution of (4.1) on $[t_0, \infty)$. Without loss of generality, we may assume that $x(t) > 0$ for $t \geq t_1 \geq t_0$. By condition (H2), we shall only consider this case, since the substitution $z(t) = -x(t)$ transforms (4.1) into an equation of the same form. Let us make the substitution $x(t) = \phi(t)u(t)$ and define

$$\omega(t) := \frac{r(t)x^\Delta(t)}{x^\alpha(t)} \quad \text{for } t \geq t_1. \tag{4.3}$$

It follows that

$$\omega(t) = \frac{r(\phi u)^\Delta}{(\phi u)^\alpha} = \frac{r\phi u^\Delta}{\phi^\alpha u^\alpha} + \frac{r\phi^\Delta u^\sigma}{\phi^\alpha u^\alpha} \geq \frac{r\phi u^\Delta}{\phi^\alpha u^\alpha}. \tag{4.4}$$

By the product rule and the quotient rule, from (4.4) and (4.1), we obtain

$$\begin{aligned}
\omega^\Delta &= [rx^\Delta]^\Delta [\frac{1}{x^\alpha}]^\sigma + rx^\Delta [\frac{1}{x^\alpha}]^\Delta \\
&= [rx^\Delta]^\Delta [\frac{1}{x^\alpha}]^\sigma - rx^\Delta \frac{[x^\alpha]^\Delta}{x^\alpha x^{\alpha^\sigma}} \\
&= -p - rx^\Delta \frac{[x^\alpha]^\Delta}{x^\alpha x^{\alpha^\sigma}}.
\end{aligned} \tag{4.5}$$

By the substitution $x(t) = \phi(t)u(t)$ and (4.5), we have

$$\begin{aligned}
\omega^\Delta &= -p - r\frac{(\phi u)^\Delta [(\phi u)^\alpha]^\Delta}{(\phi u)^\alpha (\phi u)^{\alpha^\sigma}} \\
&= -p - r\frac{[\phi u^\Delta + \phi^\Delta u^\sigma][\phi^\alpha u^{\alpha^\Delta} + \phi^{\alpha^\Delta} u^{\alpha^\sigma}]}{\phi^\alpha u^\alpha \phi^{\alpha^\sigma} u^{\alpha^\sigma}} \\
&= -p - r\frac{\phi\phi^\alpha u^\Delta u^{\alpha^\Delta} + \phi^\alpha \phi^\Delta u^\sigma u^{\alpha^\Delta} + \phi\phi^{\alpha^\Delta} u^{\alpha^\sigma} u^\Delta + \phi^\Delta \phi^{\alpha^\Delta} u^\sigma u^{\alpha^\sigma}}{\phi^\alpha \phi^{\alpha^\sigma} u^\alpha u^{\alpha^\sigma}}.
\end{aligned} \tag{4.6}$$

Multiplying both sides of (4.6) by ϕ^{α^σ}, we obtain

$$\phi^{\alpha^\sigma}\omega^\Delta = -\phi^{\alpha^\sigma}p - r\frac{\phi\phi^\alpha u^\Delta u^{\alpha^\Delta} + \phi^\alpha\phi^\Delta u^\sigma u^{\alpha^\Delta} + \phi\phi^{\alpha^\Delta}u^{\alpha^\sigma}u^\Delta + \phi^\Delta\phi^{\alpha^\Delta}u^\sigma u^{\alpha^\sigma}}{\phi^\alpha u^\alpha u^{\alpha^\sigma}}. \tag{4.7}$$

Integrating (4.7) from t_1 to t, and using an integration by parts formula we obtain

$$\begin{aligned}\phi^\alpha\omega &= \phi^\alpha(t_1)\omega(t_1) + \int_{t_1}^t \phi^{\alpha^\Delta}\omega\Delta s - \int_{t_1}^t \phi^{\alpha^\sigma}p\Delta s\\ &\quad - \int_{t_1}^t r\frac{\phi\phi^\alpha u^\Delta u^{\alpha^\Delta} + \phi^\alpha\phi^\Delta u^\sigma u^{\alpha^\Delta} + \phi\phi^{\alpha^\Delta}u^{\alpha^\sigma}u^\Delta + \phi^\Delta\phi^{\alpha^\Delta}u^\sigma u^{\alpha^\sigma}}{\phi^\alpha u^\alpha u^{\alpha^\sigma}}\Delta s.\end{aligned} \tag{4.8}$$

From Lemma 2.3, it is easy to see that

$$\alpha u^{(\alpha-1)^\sigma}u^\Delta \le u^{\alpha^\Delta} \le \alpha u^{\alpha-1}u^\Delta. \tag{4.9}$$

Then from (4.9), we have

$$u^\Delta u^{\alpha^\Delta} \ge 0. \tag{4.10}$$

From (4.4), (4.8) and (4.10), we obtain

$$\begin{aligned}\frac{r\phi u^\Delta}{u^\alpha} &\le \phi^\alpha(t_1)\omega(t_1) - \int_{t_1}^t \phi^{\alpha^\sigma}p\Delta s + \int_{t_1}^t \phi^{\alpha^\Delta}[\frac{r\phi u^\Delta}{\phi^\alpha u^\alpha} + \frac{r\phi^\Delta u^\sigma}{\phi^\alpha u^\alpha}]\Delta s\\ &\quad - \int_{t_1}^t r\phi^\Delta\frac{u^\sigma u^{\alpha^\Delta}}{u^\alpha u^{\alpha^\sigma}}\Delta s - \int_{t_1}^t \frac{r\phi\phi^{\alpha^\Delta}}{\phi^\alpha}\frac{u^\Delta}{u^\alpha}\Delta s - \int_{t_1}^t \frac{r\phi^\Delta\phi^{\alpha^\Delta}}{\phi^\alpha}\frac{u^\sigma}{u^\alpha}\Delta s\\ &= \phi^\alpha(t_1)\omega(t_1) - \int_{t_1}^t \phi^{\alpha^\sigma}p\Delta s - \int_{t_1}^t r\phi^\Delta\frac{u^\sigma u^{\alpha^\Delta}}{u^\alpha u^{\alpha^\sigma}}\Delta s.\end{aligned} \tag{4.11}$$

From (4.9), (4.11) and Lemma 2.2, for $t \in [t_1,\infty)$, we obtain

$$\begin{aligned}\frac{r\phi u^\Delta}{u^\alpha} &\le \phi^\alpha(t_1)\omega(t_1) - \int_{t_1}^t \phi^{\alpha^\sigma}p\Delta s - \int_{t_1}^t \alpha r\phi^\Delta\frac{u^\Delta}{u^\alpha}\Delta s\\ &= \phi^\alpha(t_1)\omega(t_1) - \int_{t_1}^t \phi^{\alpha^\sigma}p\Delta s - r(t_1)\phi^\Delta(t_1)\Lambda(t),\end{aligned} \tag{4.12}$$

where $m_x \le \Lambda(t) \le M_x$, and m_x and M_x denote the infimum and supremum, respectively, of the function $\int_{t_1}^s \frac{u^\Delta}{u^\alpha}\Delta\xi$ for $s \in [t_1,t]$. Letting $F(u) = \int_\varepsilon^u \frac{1}{v^\alpha}dv$ and $G(\xi) = u(\xi)$, by Lemma 2.3, for all $\varepsilon > 0$, we have

$$^\Delta = \Big[\int_\varepsilon^{u(\xi)} \frac{1}{v^\alpha}dv\Big]^\Delta \le \frac{u^\Delta}{u^\alpha}. \tag{4.13}$$

Integrating the above equation from t_1 to s, we obtain

$$\begin{aligned}\int_{t_1}^s \frac{u^\Delta}{u^\alpha}\Delta\xi &\ge \int_\varepsilon^{u(s)} \frac{1}{v^\alpha}dv - \int_\varepsilon^{u(t_1)} \frac{1}{v^\alpha}dv\\ &= (1-\alpha)u^{(1-\alpha)}(s) - (1-\alpha)u^{(1-\alpha)}(t_1)\\ &> -(1-\alpha)u^{(1-\alpha)}(t_1).\end{aligned} \tag{4.14}$$

So, from (4.14), we have

$$\Lambda(t) \ge m_x = -(1-\alpha)u^{(1-\alpha)}(t_1) \tag{4.15}$$

for $t \in [t_1, \infty)$. From (4.12) and (4.15), we have

$$\frac{r\phi u^\Delta}{u^\alpha} \le \phi^\alpha(t_1)\omega(t_1) - \int_{t_1}^t \phi^{\alpha^\sigma} p\Delta s + (1-\alpha)r(t_1)\phi^\Delta(t_1)u^{(1-\alpha)}(t_1). \tag{4.16}$$

Since $\int_{t_0}^\infty \phi^\alpha(\sigma(s))p(s)\Delta s = \infty$, from (4.16), there exists a sufficiently large $t_2 \ge t_1$ such that

$$\frac{r\phi u^\Delta}{u^\alpha} < -1, \quad \text{for } t \ge t_2. \tag{4.17}$$

Multiplying both sides of (4.17) by $\frac{1}{r\phi}$, we obtain

$$\frac{u^\Delta}{u^\alpha} < -\frac{1}{r\phi}.$$

Integrating the above equation from t_2 to t, by Lemma 2.3 and (4.13), we have

$$-(1-\alpha)u^{(1-\alpha)}(t_2) \le \int_{t_2}^t \frac{u^\Delta}{u^\alpha}\Delta\xi < -\int_{t_2}^t \frac{1}{r(s)\phi(s)}\Delta s \to -\infty, \quad \text{as} \quad t \to \infty,$$

which is a contradiction. This completes the proof. □

Let $\phi(s) = s^\beta$. As a consequence of Theorem 4.1, we obtain the following theorem which is the main result of Jia et al [10] on a regular time scale when $r(t) = 1$.

Theorem 4.2. *If there exists a real number β satisfying $0 < \beta \le 1$, such that $\int_{t_0}^\infty \sigma^{\alpha\beta}(s)p(s)\Delta t = \infty$. Then (4.1) with $r(t) = 1$ is oscillatory.*

5. Some examples

Let us consider the following examples to better understand our results.

Example 5.1. Consider the difference equation

$$\Delta^2 x(n) + [\frac{a}{\ln^{b+1}(n)} + \frac{c(-1)^n}{\ln^b(n)}]|x(n+1)|^\alpha \operatorname{sgn} x(n+1) = 0, \tag{5.1}$$

for $\alpha > 0$ and $n > 1$, where $a > 0$, $0 < b \le 1$ and c is any real number. Letting $\phi(n) = \ln^b(n-1)$, from (3.1), equation (5.1) is oscillatory when $\alpha > 1$, since

$$\sum_{n=1}^\infty \ln^b(n)[\frac{a}{\ln^{b+1}(n)} + \frac{c(-1)^n}{\ln^b(n)}] = \sum_{n=1}^\infty [\ln(n) + c(-1)^n] = \infty.$$

Specifically, all bounded solutions of (5.1) are oscillatory when $\alpha > 0$.

Example 5.2. Consider the second-order sublinear difference equation

$$\Delta^2 x(n) + [\frac{a}{\ln^{b+1}(n)} + \frac{c(-1)^n}{\ln^b(n)}]|x(n+1)|^\alpha \operatorname{sgn} x(n+1) = 0, \tag{5.2}$$

for $0 < \alpha < 1$ and $n > 1$, where $a > 0$, $0 < b \le \alpha$ and c is any real number. Letting $\phi(n) = \ln(n-1)$, from (4.2), the equation (5.2) is oscillatory, since

$$\sum_{n=1}^\infty \ln^\alpha(n)[\frac{a}{\ln^{b+1}(n)} + \frac{c(-1)^n}{\ln^b(n)}] \ge \sum_{n=1}^\infty [\ln(n) + c(-1)^n] = \infty.$$

Example 5.3. Consider the second-order dynamic equation

$$x^{\Delta\Delta}(t)+p(t)|x(\sigma(t))|^{\alpha-1}x(\sigma(t))=0,\quad \alpha>1,\ t\geq\frac{3\pi}{2}. \tag{5.3}$$

Let

$$p(t)=\begin{cases}\frac{1}{t^{1+\beta}}+\frac{\sin t}{t^{\beta}}, & t\in[2k\pi-\frac{\pi}{2},\,2k\pi+\frac{\pi}{2}],\\ \frac{1}{k^{1+\beta}}+\frac{(-1)^k}{k^{\beta}}, & t=2k\pi+\pi,\end{cases}$$

where $0<\beta\leq 1$, $k\geq 1$. Here we take $\phi(t)=t^{\beta}$. It is easy to see that

$$\sigma^{\beta}(t)p(t)=\begin{cases}\frac{1}{t}+\sin t, & t\in[2k\pi-\frac{\pi}{2},\,2k\pi+\frac{\pi}{2}),\\ [2k\pi+\pi]^{\beta}\big[[2k\pi+\frac{\pi}{2}]^{-1-\beta}+[2k\pi+\frac{\pi}{2}]^{-\beta}\big], & t=2k\pi+\frac{\pi}{2},\\ [2k\pi+\frac{3\pi}{2}]^{\beta}[k^{-1-\beta}+(-1)^k k^{-\beta}], & t=2k\pi+\pi.\end{cases}$$

From Theorem 3.1, we obtain

$$\int_{t_0}^{\infty}\frac{1}{\phi(s)r(s)}\Delta s=\int_{t_0}^{\infty}\frac{1}{s^{\beta}}\Delta s=\infty,$$

and

$$\begin{aligned}
&\int_{t_0}^{\infty}\phi^{\sigma}(s)p(s)\Delta s\\
&=\int_{\frac{3\pi}{2}}^{\infty}\sigma^{\beta}(s)p(s)\Delta s\\
&=\sum_{k=1}^{\infty}\int_{2k\pi-\frac{\pi}{2}}^{2k\pi+\frac{\pi}{2}}[\frac{1}{s}+\sin s]\Delta s+\sum_{k=1}^{\infty}[2k\pi+\pi]^{\beta}\big[[2k\pi+\frac{\pi}{2}]^{-1-\beta}+[2k\pi+\frac{\pi}{2}]^{-\beta}\big]\\
&\quad+\sum_{k=1}^{\infty}[2k\pi+\frac{3\pi}{2}]^{\beta}[k^{-1-\beta}+(-1)^k k^{-\beta}]\\
&\geq\sum_{k=1}^{\infty}\int_{2k\pi-\frac{\pi}{2}}^{2k\pi+\frac{\pi}{2}}[\frac{1}{s}+\sin s]\Delta s+\sum_{k=1}^{\infty}[2k\pi+\frac{3\pi}{2}]^{\beta}[k^{-1-\beta}+(-1)^k k^{-\beta}]\\
&=\sum_{k=1}^{\infty}\ln\frac{2k\pi+\frac{\pi}{2}}{2k\pi-\frac{\pi}{2}}+\sum_{k=1}^{\infty}\big[\frac{1}{k}[2\pi+\frac{3\pi}{2k}]^{\beta}+(-1)^k[2\pi+\frac{3\pi}{2k}]^{\beta}\big]\\
&\geq\sum_{k=1}^{\infty}\ln[1+\frac{\pi}{2k\pi-\frac{\pi}{2}}]+\sum_{k=1}^{\infty}(-1)^k[2\pi+\frac{3\pi}{2k}]^{\beta}\\
&=\sum_{k=1}^{\infty}\ln[1+\frac{\pi}{2k\pi-\frac{\pi}{2}}]+[2\pi]^{\beta}\sum_{k=1}^{\infty}(-1)^k[1+\frac{3\beta}{4k}+O\big(\frac{1}{k^2}\big)]^{\beta}.
\end{aligned}$$

Since

$$\sum_{k=1}^{\infty}\frac{\pi}{2k\pi-\frac{\pi}{2}}=\infty,\quad \big|\sum_{k=1}^{m}(-1)^k\big|\leq 1,\quad \sum_{k=1}^{\infty}(-1)^k\frac{3\beta}{4k}<\infty,$$

from, it is easy to see that

$$\int_{t_0}^{\infty}\phi^{\sigma}(s)p(s)\Delta s=\int_{\frac{3\pi}{2}}^{\infty}\sigma^{\beta}(s)p(s)\Delta s=\infty.$$

From (3.1), the equation (5.3) is oscillatory. We note that the results of Jia et al [9, 10] do not handled since the time scale is irregular; however, our results are established for arbitrary time scales.

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