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EXISTENCE OF SOLUTIONS TO HEMIVARIATIONAL INEQUALITIES INVOLVING THE $p(x)$-BIHARMONIC OPERATOR

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Abstract. This article concerns the existence of solutions to boundary-value problems involving the $p(x)$-biharmonic operator. Our technical approach is the variational-hemivariational inequality on bounded domains by using the mountain pass theorem and the critical point theory for Motreanu-Panagiotopoulos type functionals.

1. Introduction

It is well known that the mathematical modeling of equations in different fields of researches, such as mechanical engineering, Micro Electro-Mechanical systems, economics, computer science, electro-rheological fluids (cf. [24]) and many others, leads naturally to the consideration of nonlinear differential problems. It also appears in nonlinear elasticity petroleum extraction and in the theory of quasi-regular and quasi-conformal mappings. Analysis of solutions of specific problems is of considerable importance in the theory of partial differential equations. In recent years there has been an increased interest in differential problems governed by higher order operators, like the polyharmonic operator, like the $p(x)$-Laplacian. The $p(x)$-Laplace operator $\Delta_{p(x)}u = div(|\nabla u|^{p(x)-2}\nabla u)$ is a natural generalization of the $p-$Laplacian operator $\Delta_p u = div(|\nabla u|^{p-2}\nabla u)$ where $p > 1$ is a real constant. The main difference between them is that $p-$Laplacian operator is $(p-1)$-homogenous, but the $p(x)$-Laplacian operator, when $p(x)$ is not constant, is not homogeneous. For $p(x)$-Laplacian operator, we refer the readers to [11, 14, 15, 17, 20] and references cited therein.

Many authors consider the existence of nontrivial solutions for some fourth order problems such as [9, 10]. This is a generalization of the classical p-biharmonic operator $\Delta(|\Delta u|^{p-2}\Delta u)$ obtained in the case that p is a positive constant. Recently, many researchers pay their attention to impulsive differential equations by variational method and critical point theory, and we refer the readers to [2, 25, 26]. The study of differential equations and variational problems with $p(x)$-growth conditions was an interesting topic, which arises from nonlinear elastic mechanics. Some

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of these problems come from different areas of applied mathematics and physics such as Micro Electro-Mechanical systems, surface diffusion on solids. The $p(x)$-biharmonic operator possesses more complicated nonlinearities than $p-$biharmonic. Recently, Ayoujil and El Amrouss. [3] studied the spectrum of a fourth order elliptic equation with variable exponent. Ge-Xue [16] and Qian-Shen [23], considered some differential inclusions involving $p(x)$-Laplacian and Clarke subdifferential with Dirichlet boundary condition.

The purpose of this article is to study the nonlinear, nonsmooth, boundary value problem involving the $p(x)$-biharmonic operator

$$\begin{gathered} -\Delta^2_{p(x)} - a(x)|u|^{p(x)-2}u \in -\partial F(x,u) \quad \text{in } \Omega \\ u \geq 0 \quad \text{in } \Omega \\ u = \Delta u = 0 \quad \text{on } \partial\Omega \end{gathered} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^{\mathbb{N}}$ with smooth boundary $\partial\Omega$, $N \geq 1$ and $\Delta^2_{p(x)} = \Delta(|\Delta u|^{p(x)-2}\Delta u)$ is the $p(x)$-biharmonic operator of fourth order, with $p \in C_+(\overline{\Omega}) = \{p \in C(\overline{\Omega}) : p(x) > 1\}$, $a \in L^\infty(\Omega)$ such that $\inf_{x\in\Omega} a(x) = a^- > 0$, $\sup_{x\in\Omega} a(x) = a^+ > 0$.

To formulate our problem, we shall consider a Carathéodory function $F : \Omega\times\mathbb{R} \to \mathbb{R}$ which is locally Lipschitz in the second variable and satisfies some conditions (F1)–(F5), presented in section 3. By $\partial F(x,u)$ we denote the subdifferential with respect to the u variable in the sense of Clarke [4].

For the $p(x)$-operators the natural setting is described by the variable exponent Sobolev spaces $W^{L,p(\cdot)}(\Omega)$. We will study a class of problem for hemivariational inequalities on some domains of the type $\mathcal{Z}$ which is a nonempty, closed, convex cone of $W^{2,p(x)}(\Omega) \cap W^{1,p(x)}_0(\Omega)$. In fact, our purpose is to study the following variational-hemivariational inequality problem: Find $u \in \mathcal{Z}$ as a weak solution of problem (1.1) such that

$$\begin{aligned} &\int_\Omega |\Delta u|^{p(x)-2}\Delta u\Delta(v-u)dx + \int_\Omega a(x)|u|^{p(x)-2}u(v-u)dx \\ &+ \int_\Omega F^0(x,u(x),-v(x)+u(x))dx \geq 0, \end{aligned} \tag{1.2}$$

for all $v \in \mathcal{Z}$.

Our method is more direct and is based on the critical point theory for nonsmooth Lipschitz functionals developed by Motreanu and Panagiotopoulos [21]. To investigate the existence of solution of (1.2), we shall construct a functional $\mathcal{I}(u)$ associated to (1.2). For the convenience of the reader, in the next section we briefly present the basic notion and facts from the theory, which will be used in the study of problem (1.2).

The article is organized as follows. First, we introduce the basic definitions and properties in the framework of the generalized Lebesgue and Sobolev spaces $L^{p(\cdot)}(\Omega), W^{L,p(\cdot)}(\Omega)$, and refer the reader to [5, 7, 8, 12, 13] more details. Then we show basic notions about generalized directional derivative, hypotheses on F, and facts about the mountain pass theorem. Finally, whose an existence results for a $p(x)$-biharmonic problem under Dirichlet boundary conditions, by using the symmetric mountain pass theorem by Motreanu and Panagiotopoulos. This means we will show that $u \in W^{2,p(x)}(\Omega) \cap W^{1,p(x)}_0(\Omega)$ is a critical point of $\mathcal{I}(u)$ in the sense of Motreanu-Panagiotopoulos is a solution of (1.2).

2. Preliminaries

To discuss problem (1.2), we need to state some properties of the spaces $L^{p(\cdot)}(\Omega)$ and $W^{L,p(\cdot)}(\Omega)$ which we call the generalized Lebesgue-Sobolev spaces. For $p \in C_+(\bar\Omega)$, denote by $1 < p^- = \min_{x\in\bar\Omega} p(x) \le p^+ = \max_{x\in\bar\Omega} p(x) < +\infty$, the following result holds. The variable exponent Lebesgue space $L^{p(\cdot)}(\Omega)$ is defined by

$$\{u : \Omega \to \mathbb{R} : \int_\Omega |u(x)|^{p(x)} dx < \infty\}.$$

The space $L^{p(x)}(\Omega)$ is endowed by the Luxemburg norm

$$\|u\|_{p(\cdot)} = \inf\{\lambda > 0 : \int_\Omega |\frac{u(x)}{\lambda}|^{p(x)} dx\} \le 1\}.$$

Note that, when p is constant the Luxemburg norm $\|\cdot\|_{p(\cdot)}$ coincide with the standard norm $\|\cdot\|_p$ of the Lebesgue space $L^p(\Omega)$. Then $(L^{p(x)}(\Omega), \|\cdot\|_{p(\cdot)})$ is a Banach space [18].

Let p' be the function obtained by conjugating the exponent p pointwise, that is $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$ for all $x \in \bar\Omega$, then p' belongs to $C_+(\bar\Omega)$.

Proposition 2.1 ([18]). *The space $L^{p(\cdot)}(\Omega)$ is separable, reflexive, and Banach; $Lp'(\cdot)(\Omega)$ is its dual space.*

Proposition 2.2 ([12]).
(i) *For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$, the following Hölder type inequality valid,*

$$\int_\Omega |u(x)v(x)|dx \le \big(\frac{1}{p^-} + \frac{1}{p'^-}\big)\|u\|_{p(x)}\|v\|_{p'(x)}.$$

(ii) *If $p, q \in C(\bar\Omega)$ and $1 \le p \le q$ in Ω, then the embedding $L^{q(\cdot)} \hookrightarrow L^{p(\cdot)}$ is continuous.*

Proposition 2.3 ([12]). *Let $p \in C_+(\overline{\Omega})$, and let $\varphi_{p(\cdot)}(u) = \int_\Omega |u(x)|^{p(x)}dx$. If $u, (u_n)_n$ are in $L^{p(\cdot)}(\Omega)$, when $1 \le p_- \le p_+ \le \infty$, then the following relations hold:*
(i) $\|u\|_{p(\cdot)} \ge 1 \Rightarrow \|u\|_{p(\cdot)}^{p_-} \le \varphi_{p(\cdot)} \le \|u\|_{p(\cdot)}^{p_+}$,
(ii) $\|u\|_{p(\cdot)} \le 1 \Rightarrow \|u\|_{p(\cdot)}^{p_+} \le \varphi_{p(\cdot)} \le \|u\|_{p(\cdot)}^{p_-}$.

The generalized Lebesgue-Sobolev space $W^{L,p(x)}(\Omega)$ for $L = 1, 2, \dots$ is defined as

$$W^{L,p(\cdot)}(\Omega) = \{u \in L^{p(\cdot)}(\Omega) : D^\alpha u \in L^{p(\cdot)}(\Omega), |\alpha| \le L\},$$

where $D^\alpha u = \frac{\partial^{|\alpha|}}{\partial^{\alpha_1}x_1\dots\partial^{\alpha_n}x_n}$ for $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_N)$ which is a multi-index and $|\alpha| = \Sigma_{i=1}^N \alpha_i$. The space $W^{L,p(x)}(\Omega)$ with the norm

$$\|u\|_{W^{L,p(\cdot)}}(\Omega) = \sum_{|\alpha|\le L} \|D^\alpha u\|_{p(\cdot)},$$

is a separable and reflexive Banach space.

The space $W_0^{L,p(x)}(\Omega)$ is the closure in $W^{L,p(\cdot)}(\Omega)$ of the set of all $W^{L,p(\cdot)}(\Omega)$ functions with compact support.

Proposition 2.4 ([6]). *$W_0^{L,p(\cdot)}(\Omega)$ is a separable, uniformly convex and reflexive Banach space.*

For every $u \in W_0^{L,p(\cdot)}(\Omega)$ the Poincaré inequality holds, i.e., there exists a positive constant C_p in which

$$\|u\|_{L^{p(\cdot)}(\Omega)} \leq C_p \|\nabla u\|_{L^{p(\cdot)}(\Omega)}.$$

(see [14]). Hence, an equivalent norm for the space $W_0^{L,p(\cdot)}(\Omega)$ is

$$\|u\|_{W_0^{L,p(\cdot)}(\Omega)} = \sum_{|\alpha|=L} \|D^\alpha u\|_{p(\cdot)}.$$

Let p_L^* denote the critical variable exponent related to p, defined on $\bar{\Omega}$ by

$$p_L^*(x) = \begin{cases} \frac{Np(x)}{N-Lp(x)} & Lp(x) < N, \\ +\infty & Lp(x) \geq N. \end{cases} \tag{2.1}$$

Proposition 2.5 ([12, 18]). *For $p, q \in C_+(\overline{\Omega})$ in which $q(x) \leq p_L^*(x)$ for each $x \in \overline{\Omega}$, there is a continuous embedding*

$$W^{L,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega).$$

This embedding is compact if $q(x) < p_L^*(x)$ for each $x \in \bar{\Omega}$.

Remark 2.6. (i) $(W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega), \|\cdot\|)$ is a separable and reflexive Banach space. By proposition (2.5) there is a continuous and compact embedding of $W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega)$ into $L^{q(x)}$, where $q(x) < p_2^*(x)$ for $x \in \overline{\Omega}$.

(ii) Define

$$\|u\| = \inf\{\lambda > 0 : \int_\Omega [|\frac{\Delta u}{\lambda}|^{p(x)} + a(x)|\frac{u}{\lambda}|^{p(x)}]dx \leq 1\},$$

for all $u \in W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega)$, then $\|u\|$ is a norm on $W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega)$. According to [27], the norm $\|\cdot\|_{2,p(x)}$ is equivalent to the norm $|\Delta\cdot|_{p(x)}$ in the space X. Consequently, the norms $\|\cdot\|_{2,p(x)}, \|\cdot\|$ and $|\Delta\cdot|_{p(x)}$ are equivalent.

In this article, we denote $X = W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega)$ and $X^\star$ its dual space.

Proposition 2.7. *Define $\Phi(u) = \int_\Omega [|\triangle u|^{p(x)} + a(x)|u(x)|^{p(x)}dx]$. For $u, u_n \in X$,*

(i) $\|u\| < (=;>)1 \Leftrightarrow \Phi(u) < (=;>)1$,
(ii) $\|u\| \leq 1 \Rightarrow \|u\|^{p^+} \leq \Phi(u) \leq \|u\|^{p^-}$,
(iii) $\|u\| \geq 1 \Rightarrow \|u\|^{p^-} \leq \Phi(u) \leq \|u\|^{p^+}$,
(iv) $\|u_n\| \to 0 \Leftrightarrow \Phi(u_n) \to 0$,
(v) $\|u_n\| \to \infty \Leftrightarrow \Phi(u_n) \to \infty$.

The proof of the above proposition is similar to the proof in [12]; we omit it.

Proposition 2.8 ([1, 6]). *Let h be of class $C(\bar{\Omega})$. If $p^+ < N/L$ and $1 \leq h(x) \leq p_L^*(x)$ for each $x \in \Omega$, then there exists $\mathcal{C}_{h^+} = \mathcal{C}_{h^+}(N, p, L, \Omega) > 0$ such that*

$$\|u\|_{h(\cdot)} \leq \mathcal{C}_{h^+} \|u\|_{(W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega))}, \quad \forall u \in (W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega)). \tag{2.2}$$

Proposition 2.9. *Let $u \in L^{p(x)}(\Omega)$. Then*

(i) $|u|^{p(x)-1} \in L^{p'(x)}$, *where* $p'(x) = \frac{p(x)}{p(x)-1}$,
(ii) $\||u|^{p(x)-1}\|_{p'(x)} \leq 1 + \|u\|_{p(x)}^{p^+}$.

Proof. (a) is clear. To show (b), if $\| |u|^{p(x)-1}\|_{p'(x)} \le 1$, then the inequality in (b) is obvious. So, we presume that $\| |u|^{p(x)-1}\|_{p'(x)} > 1$.
If $\| |u|\|_{p(x)} > 1$, using Proposition 2.3(i)

$$\| |u|^{p(x)-1}\|_{p'(x)}^{p'^-} \le \int_\Omega |u|^{(p(x)-1)p'(x)} = \int_\Omega |u|^{p(x)} \le \|u\|_{p(x)}^{p^+}.$$

Hence, $\||u|^{p(x)-1}\|_{p'(x)} \le 1 + \|u\|_{p(x)}^{p^+}$. In a similar way, if $\||u|\|_{p(x)} < 1$, then $\| |u|^{p(x)-1}\|_{p'(x)} \le 1 + \|u\|_{p(x)}^{p^+}$. $\square$

Now, we review some definitions and basic properties of the theory of generalized differentiation for locally Lipschitz functions. Let X be a Banach space and $X^\star$ its topological dual. By $\|\cdot\|$ we will denote the norm in X and by $\langle\cdot,\cdot\rangle$ the duality brackets for the pair $(X, X^\star)$. A function $h : X \to \mathbb{R}$ is said to be locally Lipschitz, if for every $x \in X$ there exists a neighbourhood U of x and a constant $K > 0$ depending on U such that $|h(y) - h(z)| \le K\|y - z\|$ for all $y, z \in U$. For a locally Lipschitz function $h : X \to \mathbb{R}$ we define the generalized directional derivative of h at $u \in X$ in the direction $\gamma \in X$ by

$$h^0(u;\gamma) = \limsup_{w\to u, t\to 0^+} \frac{h(w + t\gamma) - h(w)}{t}.$$

The generalized gradient of h at $u \in X$ is defined by

$$\partial h(u) = \{x^\star \in X^\star : \langle x^\star, \gamma\rangle_X \le h^0(u;\gamma),\ \forall\gamma \in X\},$$

which is a nonempty, convex and $w^\star$-compact subset of $X^\star$, where $\langle\cdot,\cdot\rangle_X$ is the duality pairing between $X^\star$ and X.

Proposition 2.10 ([4]). *Let $h, g : X \to \mathbb{R}$ be locally Lipschitz functions. Then*

(i) $h^0(u;\cdot)$ *is subadditive, positively homogeneous.*
(ii) $(-h)^0(u;v) = h^0(u;-v)$ *for all* $u, v \in X$.
(iii) $h^0(u;v) = \max\{\langle\xi, v\rangle : \xi \in \partial h(u)\}$ *for all* $u, v \in X$.
(iv) $(h + g)^0(u;v) \le h^0(u;v) + g^0(u;v)$ *for all* $u, v \in X$.

Definition 2.11 ([22]). Let X be a Banach space. $\mathcal{I} : X \to (-\infty, +\infty]$ is a Motreanu-Panagiotopoulos-type functional, where $\mathcal{I} = h + \chi$ in which $h : X \to \mathbb{R}$ is locally Lipschitz and $\chi : X \to (-\infty, +\infty]$ is convex, proper and lower semicontinuous.

Definition 2.12 ([21]). An element $u \in X$ is said to be a critical point of $\mathcal{I} = h+\chi$ if

$$h^0(u; v - u) + \chi(v) - \chi(u) \ge 0, \quad \forall v \in X.$$

Let $h : X \to \mathbb{R}$ be a locally Lipschitz functional, and assume the functional $\chi : X \to \mathbb{R} \cup \{+\infty\}$ is convex, lower semicontinuous and proper, whose restriction to $\mathrm{dom}(\chi) = \{x \in X : \chi(u) < \infty\}$ is continuous. Then $h + \chi$ is a Motreanu-Panagiotopoulos functional.

Definition 2.13. Let $h : X \to \mathbb{R}$ be locally Lipschitz and $\mathcal{Z}$ be a nonempty, closed, convex subset of X. The indicator of $\mathcal{Z}$ is the function $\chi_{\mathcal{Z}} : X \to \mathbb{R}\cup\{+\infty\}$ defined by putting for every $u \in X$,

$$\chi_{\mathcal{Z}} = \begin{cases} 0 & u \in \mathcal{Z} \\ +\infty & u \notin \mathcal{Z}\,. \end{cases} \tag{2.3}$$

It is easily seen that $\chi_{\mathcal{Z}}$ is proper, convex and lower semicontinuous, while its restriction to $\mathrm{dom}(\chi_{\mathcal{Z}}) = \mathcal{Z}$ is the constant 0. Clearly $u \in X$ is a critical point for the Motreanu-Panagiotopoulos functional $h + \chi_{\mathcal{Z}}$ if and only if $u \in \mathcal{Z}$ and the following condition holds

$$h^0(u; v - u) \geq 0, \quad \forall v \in \mathcal{Z}.$$

Definition 2.14 ([21]). Let X be a Banach space and $\mathcal{I} : X \to (-\infty, +\infty]$, $\mathcal{I} = h + \chi$ Motreanu-Panagiotopoulos type functional. It is said to satisfy the Palais-Smale condition at level $c \in \mathbb{R}$ (for short (PS)c), if every sequence $\{u_n\}$ in X satisfying $\mathcal{I}(u_n) \to c$ and

$$h^0(u; v - u_n) + \chi(v) - \chi(u_n) \geq -\epsilon_n \|v - u_n\|, \quad \forall v \in X,$$

for a sequence ϵ_n in $[0, \infty)$ tends to zero, contains a convergent subsequence.

The next theorem is due to Motreanu and Panagiotopoulos [21] and extends to a nonsmooth setting the well known "mountain pass theorem".

Theorem 2.15 ([21]). *Assume that the functional $\mathcal{I} : X \to (-\infty, +\infty]$ defined by $\mathcal{I} = h + \chi$, satisfies (PS), $\mathcal{I}(0) = 0$, and*

(i) *there exist constants $a > 0$ and $\rho > 0$, such that $\mathcal{I}(u) \geq a$ for all $\|u\| = \rho$;*
(ii) *there exists $e \in X$, with $\|e\| > \rho$ and $\mathcal{I}(e) \leq 0$. Then*

$$c = \inf_{f \in \Gamma} \sup_{t \in [0,1]} \mathcal{I}(f(t)),$$

is a critical value of $\mathcal{I}$ for $c \geq a$, where

$$\Gamma = \{f \in C([0, 1], X) : f(0) = 0, f(1) = e\}.$$

Definition 2.16. The functional $\mathcal{I} : X \to X^\star$ satisfies the condition $(\mathcal{S}_+)$ if for any sequence $\{u_n\}_n \subset X$ which converges weakly to u in X and

$$\limsup_{n \to \infty} \langle \mathcal{I}(u_n), u_n - u \rangle \leq 0,$$

then $\{u_n\}_n$ converges strongly to u in X.

As it is customary for solving of (1.2), we consider a functional $\mathcal{I}(u) = \phi(u) + \mathcal{S}(u) - \mathcal{F}(u) + \chi(u)$ associated to (1.2) which is defined by $\mathcal{I}(u) : W_0^{L,p(\cdot)}(\Omega) \to \mathbb{R}$ such that

$$\phi(u) = \int_\Omega \frac{1}{p(x)} |\Delta u|^{p(x)} dx, \quad \forall u \in W_0^{L,p(x)}(\Omega),$$
$$\mathcal{S}(u) = \int_\Omega \frac{1}{p(x)} [a(x)|u|^{p(x)}] dx, \quad \forall u \in W_0^{L,p(x)}(\Omega),$$
$$\mathcal{F}(u) = \int_\Omega F(x, u(x)) dx, \quad \forall u \in W_0^{L,p(x)}(\Omega),$$

where $\chi(u)$ is the indicator function of $\mathcal{Z}$. Functionals $\phi, \mathcal{S}, \mathcal{F}$ are locally Lipschitz. In conclusion, $\mathcal{I} = \phi + \mathcal{S} - \mathcal{F} + \chi$ is a Motreanu-Panagiotopoulos type functional.

Proposition 2.17 ([9]). *Suppose that*

$$\phi(u) = \int_\Omega \frac{1}{p(x)} |\Delta u|^{p(x)} dx, \quad \forall u \in X.$$

Then the operator $\phi'(u) : X \to X^\star$ defined as

$$\langle \phi'(u), v\rangle = \int_\Omega |\Delta u|^{P(x)-2}\Delta u \Delta v dx, \quad \forall u, v \in X.$$

satisfies the following properties:

(i) *ϕ' is continuous, bounded and strictly monotone.*
(ii) *ϕ' is of (S_+) type.*
(iii) *ϕ' is a homeomorphism.*

Definition 2.18. Consider the function

$$\mathcal{S}(u) = \int_\Omega \frac{1}{p(x)} a(x)|u|^{p(x)}dx, \quad \forall u \in X.$$

Then the operator $\mathcal{S}'$ is the derivative operator of $\mathcal{S}$ in the weak sense, where $\mathcal{S}'(u) : X \to X^\star$, is defined by

$$\langle \mathcal{S}'(u), v\rangle = \int_\Omega a(x)|u|^{p(x)-2}uv dx, \quad \forall u, v \in X.$$

3. Main result

We assume that $F : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, which is locally Lipschitz in the second variable and satisfying the following properties:

(F1) $F(x, 0) = 0$, a.e. $x \in \Omega$ and $p, q \in C_+(\overline{\Omega})$, there exists a constant $c_1 > 0$ such that $|\xi| \le c_1(|s|^{p(x)-1} + |s|^{q(x)-1})$, whenever $\xi \in \partial F(x, s)$ with $(x, s) \in \Omega \times \mathbb{R}$.

(F2) There exists a constant $\nu \in]p, p^*_L[$ such that

$$\nu F(x, s) + F^0(x, s; -s) \le 0, \quad \forall (x, s) \in \Omega \times \mathbb{R}.$$

(F3) $\lim_{s\to 0} \max\{|\xi| : \xi \in \partial F(x, s)\}/s^{p(x)-1} = 0$ uniformly for every $x \in \Omega$.

(F4) There exists a constant $R > 0$ such that

$$c_R =: \inf\{F(x, s) : (x, |s|) \in \Omega \times [\mathbb{R}, +\infty)\} > 0.$$

(F5) There exists $u \in X\backslash\{0\}$ such that

$$C\|u\|^{p^+} \le \int_\Omega F(x, u(x))dx, \quad \text{if } \|u\| \ge 1,$$

or

$$C\|u\|^{p^-} \le \int_\Omega F(x, u(x))dx, \quad \text{if } \|u\| \le 1,$$

where $C > 1/p^-$.

Here, we denote by $\partial F(x, s)$ and $F^0(x, s; \cdot)$ the generalized gradient and the generalized directional derivative of $F(x, \cdot)$ at the point $s \in \mathbb{R}$, respectively.

Proposition 3.1 ([19]). *If $F : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies* (F1)–(F2)*, and $p, q \in C_+(\overline{\Omega})$, with $p^+ < q^- \le q^+ < (p^*_L)^-$, then for every $\epsilon > 0$ there exists $c(\epsilon) > 0$ such that*

(i) *$|\xi| \le \epsilon|s|^{p(x)-1} + c(\epsilon)|s|^{q(x)-1}$ for all $\xi \in \partial F(x, s)$ with $(x, s) \in \Omega \times \mathbb{R}$;*
(ii) *$|F(x, s)| \le \epsilon|s|^{p^+} + c(\epsilon)|s|^{q(x)}$ for all $(x, s) \in \Omega \times \mathbb{R}$.*

Define $\mathcal{F} : W_0^{L,p(x)}(\Omega) \to \mathbb{R}$ dy

$$\mathcal{F}(u) = \int_\Omega F(x, u(x))dx, \quad \forall u \in W_0^{L,p(x)}(\Omega).$$

Proposition 3.2 ([19]). *Let $F : \Omega \times \mathbb{R} \to \mathbb{R}$ be a locally Lipschitz function which satisfies* (F1). *Then $\mathcal{F}$ is well-defined and it is locally Lipschitz. Moreover,*

$$\mathcal{F}^0(u;v) \le \int_\Omega F^0(x,u(x);v(x))dx, \quad \forall u,v \in W_0^{L,p(x)}(\Omega).$$

The next lemma points out the relationship between the critical points of $\mathcal{I}(u)$ and the solutions of problem (1.2).

Lemma 3.3. *Every critical point of $\mathcal{I}$ is a solution of problem* (1.2).

Proof. Let $u \in X$ be a critical point of $\mathcal{I}(u) = \phi(u) + \mathcal{S}(u) - \mathcal{F}(u) + \chi(u)$. Then $u \in \mathcal{Z}$ and by definition (2.13)

$$\langle \phi' u, v-u\rangle + \langle \mathcal{S}' u, v-u\rangle - \mathcal{F}^0(u; v-u) \ge 0, \quad \forall v \in X.$$

Using Proposition (3.2) and the property (ii) from Proposition (2.10), we obtain the desired inequality. $\square$

Lemma 3.4. *Assume that $F : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies* (F1)–(F3). *Then $\mathcal{I}$ satisfies the (PS)c condition for each $c \in \mathbb{R}$.*

Proof. Fix $c \in \mathbb{R}$ and let $\{u_n\}$ be a sequence in X in which

$$\mathcal{I}(u_n) = \phi(u_n) + \mathcal{S}(u_n) - \mathcal{F}(u_n) + \chi(u_n) \to c, \tag{3.1}$$

$$\begin{aligned}&\langle \phi' u_n, v-u_n\rangle + \langle \mathcal{S}' u_n, v-u_n\rangle - \mathcal{F}^0(u_n; v-u_n) + \chi(v) - \chi(u_n)\\ &\ge -\epsilon_n \|v-u_n\|_{p(\cdot)}, \quad \forall v \in X,\end{aligned} \tag{3.2}$$

where ϵ_n is a sequence in $[0,+\infty)$ converges to zero. According to (3.1), one concludes that the sequence $\{u_n\}$ belongs entirely to $\mathcal{Z}$. Setting $v = 2u_n$ in (3.2),

$$\int_\Omega |\triangle u_n|^{p(x)}dx + \int_\Omega a(x)|u_n|^{p(x)}dx + \int_\Omega F^0(x,u_n;-u_n)dx \ge -\epsilon_n\|u_n\|_{p(\cdot)}. \tag{3.3}$$

We infer from (3.1) that for enough large $n \in \mathbb{N}$,

$$c + 1 \ge \phi(u_n) + \mathcal{S}(u_n) - \mathcal{F}(u_n). \tag{3.4}$$

Multiplying (3.3) by ν^{-1}, adding this one to (3.4) and using the condition (F2), for enough large $n \in \mathbb{N}$,

$$\begin{aligned}\frac{\epsilon_n}{\nu}\|u_n\|_{p(\cdot)} + c + 1 &\ge \big(\frac{1}{p(x)} - \frac{1}{\nu}\big)\int_\Omega |\triangle u_n|^{p(x)}dx + \big(\frac{1}{p(x)} - \frac{1}{\nu}\big)\int_\Omega a(x)|u_n|^{p(x)}dx\\ &\quad - \int_\Omega [F(x,u_n(x)) + \frac{1}{\nu}F^0(x,u_n(x);-u_n(x)]dx\\ &\ge \big(\frac{1}{p(x)} - \frac{1}{\nu}\big)\Big[\int_\Omega |\triangle u_n|^{p(x)}dx + \int_\Omega a(x)|u_n|^{p(x)}dx\Big]\\ &\ge \big(\frac{1}{p^+} - \frac{1}{\nu}\big)\|u\|_X\end{aligned}$$

This estimate ensures that the sequence $\{u_n\}$ is bounded in $\mathcal{Z}$. Since X is a reflexive Banach space, it follows that there exists an element $u \in \mathcal{Z}$ in which $\{u_n\}$ has a weakly convergent subsequence (denoted also by $\{u_n\}$) to u in X. X is compactly embedded in $L^{q(x)}(\Omega)$, so $u_n \to u$ in $L^{q(x)}(\Omega)$; i.e.,

$$\|u_n - u\|_{q(x)} \to 0, \quad \text{as } n \to \infty. \tag{3.5}$$

Setting $v = u$ in (3.2),

$$
\begin{aligned}
&\langle \phi' u_n, u - u_n\rangle + \int_\Omega a(x)|u_n|^{p(x)-2}u_n(u-u_n)dx + \int_\Omega F^0(x, u_n(x); (u_n - u)(x)dx \\
&\geq -\epsilon_n \|u - u_n\|_{p(x)}.
\end{aligned}
\tag{3.6}
$$

Using (3.6), the Proposition (3.1)(i), for any $\epsilon > 0$,

$$
\begin{aligned}
\langle \phi' u_n, u_n - u\rangle &\leq \int_\Omega a(x)|u_n|^{p(x)-2}u_n(u-u_n)dx + \int_\Omega F^0(x, u_n; (u_n - u)dx \\
&\quad + \epsilon_n \|u - u_n\|_{p(x)} \\
&\leq a^+ \int_\Omega |u_n|^{p(x)-1}(u-u_n)dx + \int_\Omega \epsilon |u_n|^{p(x)-1}(u_n - u)dx \\
&\quad + \int_\Omega c(\epsilon)|u_n|^{q(x)-1}(u_n - u)dx + \epsilon_n \|u - u_n\|_{p(x)} \\
&\leq (a^+ - \epsilon) \int_\Omega |u_n|^{p(x)-1}(u-u_n)dx \\
&\quad + \int_\Omega c(\epsilon)|u_n|^{q(x)-1}(u_n - u)dx + \epsilon_n \|u - u_n\|_{p(x)}.
\end{aligned}
$$

Since $\{u_n\} \subseteq L^{p(x)}$, by the compactly embedded X into $L^{q(x)}$ for the second part of above estimate and by using Hölder's inequality,

$$
\begin{aligned}
\langle \phi' u_n, u_n - u\rangle &\leq (a^+ - \epsilon)(\frac{1}{p^-} + \frac{1}{p'^-})\| |u_n|^{p(x)-1}\|_{p'(x)} \|u - u_n\|_{p(x)} \\
&\quad + c(\epsilon)(\frac{1}{q^-} + \frac{1}{q'^-})\| |u_n|^{q(x)-1}\|_{q'(x)} \|u_n - u\|_{q(x)} + \epsilon_n \|u - u_n\|_{p(x)}.
\end{aligned}
$$

From the condition $1 \leq p \leq q$, it follows that the embedding $L^{q(\cdot)} \hookrightarrow L^{p(\cdot)}$ is continuous.

By compact embedding X into $L^{q(x)}$, in view of proposition (2.9) and by the fact that

$$
\| |u_n|^{q(x)-1}\|_{q'(x)} \leq \max\{\|u_n\|_{q(x)}^{q^- - 1}, \|u_n\|_{q(x)}^{q^+ - 1}\}
$$

for all n, it results that for the arbitrariness of $\epsilon > 0$ and $\epsilon_n \to 0$, then

$$
\limsup_{n\to\infty} < \phi' u_n, u_n - u > \leq 0.
$$

Taking into account that the operator ϕ' has the (S_+) property, so $\{u_n\}$ converges strongly to u in X. This completes the proof. □

Now we state the main result of this paper for obtaining nontrivial solution of (1.2).

Theorem 3.5. *Assume that the function $F : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies* (F1)–(F5). *Then Problem* (1.2) *has a nontrivial solution.*

Proof. According to Lemma (3.3), it is sufficient to prove the existence of a critical point of functional $\mathcal{I}$. For this, we check that $\mathcal{I}$ satisfies in the conditions of the Mountain Pass Theorem.

Lemma (3.4), guarantees that $\mathcal{I}$ satisfies the $(PS)_c$ condition for every $c \in \mathbb{R}$. By Proposition (3.1) (ii), let $\epsilon \in (0, \frac{1}{\mathcal{C}_{p^+}^{p^+} p^+})$ be fixed, where $\mathcal{C}_{p^+}$ is the Sobolev constant given in (2.2) and $h_+ = p_+$. Put $k_\epsilon = (\frac{1}{p^+} - \epsilon\mathcal{C}_{p^+}^{p^+}) > 0$, and $C_\epsilon = \max\{\mathcal{C}_{q(\cdot)}^{q^+}, \mathcal{C}_{q(\cdot)}^{q^-}\}$, where $\mathcal{C}_{q^+}$ is the Sobolev constant given in (2.2) for $h_+ = q_+$. Take $r \in (0,1]$ be so small that $r^{p^+-q^-} > \frac{C_\epsilon}{k_\epsilon}$. Then, for each $u \in X$, in which $\|u\| = r$,

$$\begin{aligned}
\mathcal{I}(u) &= \int_\Omega [\frac{|\triangle u|^{p(x)}}{p(x)} + \frac{a(x)|u(x)|^{p(x)}}{p(x)} - F(x,u(x))]dx \\
&\geq \frac{1}{p^+}\Phi(u) - \epsilon\|u\|_{p^+}^{p^+} - c(\epsilon)\varphi_{q(\cdot)}(u) \\
&\geq \frac{1}{p^+}\Phi(u) - \epsilon\|u\|_{p^+}^{p^+} - c(\epsilon)\max\{\|u\|_{q(\cdot)}^{q^+}, \|u\|_{q(\cdot)}^{q^-}\} \\
&\geq \frac{1}{p^+}r^{p^+} - \epsilon r^{p^+}\mathcal{C}_{p^+}^{p^+} - c(\epsilon)\max\{\mathcal{C}_{q^+}^{q^+}, \mathcal{C}_{q^+}^{q^-}\}r^{q^-} \\
&\geq r^{q^-}[(\frac{1}{p^+} - \epsilon\mathcal{C}_{p^+}^{p^+})r^{p^+-q^-} - C_\epsilon] \\
&\geq r^{q^-}(k_\epsilon r^{p^+-q^-} - C_\epsilon)
\end{aligned}$$

Therefore, $\mathcal{I}(u) \geq a$, where $a = r^{q^-}(k_\epsilon r^{p^+-q^-} - C_\epsilon) > 0$ for each $u \in X$, $\|u\| = r$.

To use the Mountain-Pass Theorem it remains to show that there exists an $e \in X$ with $\|e\| > \rho$ and $\mathcal{I}(e) \leq 0$. Let us fix $u \in \mathcal{Z}$ with $\|u\| \geq 1$. Using proposition (2.7) (iii) and hypothesis (F5), it follows that

$$\begin{aligned}
\mathcal{I}(u) &= \int_\Omega \Big([\frac{|\triangle u|^{p(x)}}{p(x)} + \frac{a(x)|u|^{p(x)}}{p(x)}] - F(x,u(x))\Big)dx \\
&\leq \frac{1}{p^-}\int_\Omega (|\triangle u|^{p(x)} + a(x)|u|^{p(x)}) - \int_\Omega F(x,u(x))dx \\
&\leq \frac{1}{p^-}\Phi(u) - \int_\Omega F(x,u(x))dx \\
&\leq (\frac{1}{p^-} - C)\|u\|^{p^+},
\end{aligned} \tag{3.7}$$

where $C > 1/p^-$. Thus, $\mathcal{I}(u) \leq 0$. Fix arbitrary $u_0 \in \mathcal{Z}\backslash\{0\}$, consider $u = tu_0$ ($t > 0$) in (3.7), then $\mathcal{I}(tu_0) \leq 0$. Put $e = tu_0$, so $\|e\| > \rho$ and $\mathcal{I}(e) \leq 0$. This completes the proof. $\square$

Conclusion. Lemma (3.4) ensures that the functional $\mathcal{I}$ satisfies $(PS)_c$ and $\mathcal{I}(0) = 0$. By Theorem (3.5), it follows that there are constants $a, \rho > 0$ and $e \in X$ such that $\mathcal{I}$ fulfills the properties (i) and (ii) from Theorem (2.15). Hence, the number $c = \inf_{f\in\Gamma}\sup_{t\in[0,1]}\mathcal{I}(f(t))$, is a critical value of $\mathcal{I}$ with $c \geq a > 0$, where $\Gamma = \{f \in C([0,1],X) : f(0) = 0, f(1) = e\}$. It is obvious that the critical point $u \in X$ which is correspond to c cannot be trivial since $\mathcal{I}(u) = c > 0 = \mathcal{I}(0)$. According to the Lemma (3.3) which concludes that u is an element of $\mathcal{Z}$ and it is a solution of (1.2).

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