Electronic Journal of Differential Equations, Vol. 2015 (2015), No. 83, pp. 1–9.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

UNIQUENESS OF SELF-SIMILAR VERY SINGULAR SOLUTION FOR NON-NEWTONIAN POLYTROPIC FILTRATION EQUATIONS WITH GRADIENT ABSORPTION

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Abstract. Uniqueness of self-similar very singular solutions with compact support are proved for the non-Newtonian polytropic filtration equation with gradient absorption

$$\frac{\partial u}{\partial t} = \operatorname{div}(|\nabla u^m|^{p-2}\nabla u^m) - |\nabla u|^q, \quad x \in \mathbb{R}^N, \quad t > 0,$$

where $m > 0$, $p > 1$, $m(p-1) > 1$ and $q > 1$.

1. Introduction

This article concerns the non-Newtonian polytropic filtration equation with gradient absorption

$$\frac{\partial u}{\partial t} = \operatorname{div}(|\nabla u^m|^{p-2}\nabla u^m) - |\nabla u|^q, \qquad x \in \mathbb{R}^N, \quad t > 0, \tag{1.1}$$

where $m > 0, p > 1$, $m(p-1) > 1$ and $q > 1$.

Such an equation, especially the case $m = 1$ and $p = 2$, appears as the viscosity approximation to the well-known Hamilton-Jacobi equation, in the stochastic control theory, as well as in a number of interesting and different physical considerations. For more details, see [3, 9, 10] and the references therein.

In this article, we pay attention to self-similar very singular solutions of (1.1). Due to the possible degeneracy and singularity, it is necessary to clarify the concept of weak solutions of (1.1). A non-negative function u is said to be a weak solution of (1.1), if $u \in C_{\rm loc}(0,\infty; L^2(\mathbb{R}^N))$, $u^m \in L^p_{\rm loc}(0,\infty; W^{1,p}_{loc}(\mathbb{R}^N))$, $|\nabla u| \in L^q_{\rm loc}(\mathbb{R}^N \times (0,\infty))$ and u satisfies (1.1) in the sense of distributions in $\mathbb{R}^N \times (0,\infty)$. Further, by a very singular solution u, we mean a weak solution with $u \in C(\mathbb{R}^N \times [0,\infty)\backslash\{(0,0)\})$ satisfying

$$\lim_{t\to 0} \sup_{|x|>\varepsilon} u(x,t) = 0 \tag{1.2}$$

and

$$\lim_{t\to 0} \int_{|x|<\varepsilon} u(x,t)\,\mathrm{d}x = \infty \tag{1.3}$$

2000 *Mathematics Subject Classification.* 35K65, 35K92, 35K15.
Key words and phrases. Polytropic filtration; gradient absorption; uniqueness; self-similar; very singular.

Submitted November 20, 2014. Published April 7, 2015.

for any $\varepsilon > 0$.

In 1986, Brezis et al [4] investigated the semilinear heat equation with concentration absorption

$$\frac{\partial u}{\partial t} = \Delta u - u^q;$$

they proved the existence and uniqueness of self-similar very singular solutions when $1 < q < 1 + 2/N$. Since that time the self-similar very singular solutions of diffusion equations with concentration absorption have been studied extensively, see for example, [13, 5, 11, 12, 8]. Recently, the equations with gradient absorption have attracted much attention. In 2001, Qi et al [14] and Benachour et al [2, 1] independently obtained the existence and uniqueness of self-similar very singular solutions of viscous Hamilton-Jacobi equation

$$\frac{\partial u}{\partial t} = \Delta u - |\nabla u|^q \tag{1.4}$$

by two different methods. Afterwards, these previous results on (1.4) were extended to the p-Laplacian with gradient absorption

$$\frac{\partial u}{\partial t} = \operatorname{div}(|\nabla u|^{p-2}\nabla u) - |\nabla u|^q. \tag{1.5}$$

For $1 < p < 2$, i.e., fast diffusion case, Shi [15] proved the existence and uniqueness of self-similar very singular solutions of (1.5) when $1 < q < p - \frac{N}{N+1}$. After that, Iagara et al [7, 6] generalized the corresponding results restricted to $q > 1$. More precisely, they proved that there exists a unique self-similar very singular solution of (1.5) when $\frac{2N}{N+1} < p < 2$ and $\frac{p}{2} < q < p - \frac{N}{N+1}$. On the other hand, for slow diffusion case, i.e., $p > 2$, Shi [16] obtained existence of self-similar very singular solutions with compact support of (1.5) when $p - 1 < q < p - \frac{N}{N+1}$. And soon, the corresponding existence result in [16] was extended to the equation (1.1) in [17].

As far as we know, however, the uniqueness of self-similar very singular solutions of (1.1) has not been obtained. In the present paper, we shall show the uniqueness of self-similar very singular solutions of (1.1), which not only extends the corresponding results in [2, 1, 14], but completes the investigations in [17]. Our main result is the following:

Theorem 1.1. *The equation* (1.1) *has at most one self-similar very singular solution with compact support.*

Remark 1.2. According to the main result in [17], there exists a (forward) self-similar very singular solution with compact support of (1.1) if and only if $m(p-1) < q < \frac{p+Nm(p-1)}{N+1}$, and so, under which the uniqueness will be discussed in what follows.

This article is organized as follows. In Section 2, we derive some properties of self-similar very singular solutions of (1.1). In particular, we prove the monotonicity of self-similar solutions with respect to initial data in the sense that two positive orbits do not intersect each other. Finally, the proof of Theorem 1.1 is given in Section 3.

2. Preliminaries

In this section, we derive some properties of self-similar very singular solutions of (1.1) which are important for the proof of Theorem 1.1. Owing to the homogeneity of (1.1), we actually look for a (forward) self-similar very singular solution u to (1.1) of the form

$$u(x,t) = (\frac{\alpha}{t})^{\alpha} f(r) \tag{2.1}$$

where $r = |x|(\frac{\alpha}{t})^{\alpha\beta}$, for some profile f and exponents α and β to be determined. Inserting this setting in (1.1) gives the vales of α and β

$$\alpha = \frac{p-q}{p(q-1) - q(m(p-1)-1)} > 0, \quad \beta = \frac{q - m(p-1)}{p-q} > 0,$$

and implies that the profile f is a solution of the ordinary differential equation

$$\big(|(f^m)'|^{p-2}(f^m)'\big)' + \frac{n-1}{r}|(f^m)'|^{p-2}(f^m)' + \beta r f' + f - |f'|^q = 0,\ r > 0 \tag{2.2}$$

with

$$(f^m)'(0) = 0, \quad f(0) = a, \tag{2.3}$$

where a is a positive constant to be determined. Note that condition (1.2) is equivalent to, if u is given by (2.1),

$$\lim_{r\to\infty} r^{1/\beta} f(r) = 0. \tag{2.4}$$

In addition, it is easy to see that if $N\beta < 1$ (i.e. $q < \frac{p+Nm(p-1)}{N+1}$) and the solution f of (2.2) satisfies (2.3)–(2.4), then u given explicitly by (2.1) satisfies (1.3) automatically. According to [17, Lemma 3.1], however, the condition (2.4) does not hold if $N\beta \ge 1$, that is, there is no self-similar singular solution.

Let $z = f^m, a^m = b$, then the problem (2.2)–(2.3) is replaced by the following problem with respect to z,

$$\begin{gathered} (|z'|^{p-2}z')' + \frac{n-1}{r}|z'|^{p-2}z' + \beta r(z^{1/m})' + z^{1/m} - |(z^{1/m})'|^q = 0, \quad r > 0, \\ z(0) = b > 0, \quad z'(0) = 0 \end{gathered} \tag{2.5}$$

and the condition (2.4) is replaced by

$$\lim_{r\to\infty} r^{1/\beta} z^{1/m}(r) = 0. \tag{2.6}$$

By the standard theory of ordinary differential equations, the local existence and uniqueness of solution for (2.5) is easy to be obtained. Let $z(\cdot;b)$ be the solution of (2.5) and define

$$R(b) := \sup\{r_0 > 0 : z(r;b) > 0, \quad r \in [0, r_0)\}.$$

In the sequel, where there is no confusion, we will omit b and let $z = z(\cdot;b)$.

Before going further, we present some basic properties of z which have already been proved in [17].

Lemma 2.1. *Assume that $\alpha > 0$, $\beta > 0$ and $b > 0$. Let z be a solution to (2.5) with support $[0, R(b))$. Then*

(i) $z'(r) < 0$ *in* $(0, R(b))$;
(ii) $\lim_{r\to R(b)^-} z(r) = 0$;
(iii) $\lim_{r\to R(b)^-} z'(r) = 0$ *when* $R(b) = \infty$.

Next, we prove the monotonicity of solutions of (2.5) with respect to b in the sense that two positive orbits do not intersect each other.

Lemma 2.2. *Assume that $\alpha,\beta>0$, z_i are solutions of (2.5) on $[0,R_i)$ with initial data $z_i(0)=b_i, i=1,2$ and $\min\{R_1,R_2\}<\infty$, where $[0,R_i)$ denotes the maximal existence interval of z_i and the $R_i>0$ are possibly infinity. If $b_1<b_2$, then*
$$z_1(r)<z_2(r),\quad \textit{for all} 0\le r\le R:=\min\{R_1,R_2\}.$$

Proof. Suppose contrarily that there exists $R_0\in[0,R]$ such that $z_1(r)<z_2(r)$ for $r\in[0,R_0)$ and $z_1(R_0)=z_2(R_0)$. We define
$$g_k(r):=k^{-mp/(m(p-1)-1)}z_1(kr),\quad r\in[0,R_1/k)$$
for $k>0$ and then $g_k(r)$ solves
$$\begin{aligned}&(|g_k'|^{p-2}g_k')'+\frac{N-1}{r}|g_k'|^{p-2}g_k'+\beta r(g_k^{1/m})'\\&+g_k^{1/m}-k^{\frac{q(p+1-m(p-1))-p}{m(p-1)-1}}|(g_k^{1/m})'|^q=0\end{aligned}\tag{2.7}$$
Note that g_k is strictly decreasing with respect to k, and $\lim_{k\to0}g_k(r)=+\infty$ for any $r\in[0,R]$, then there exists a small $k_0>0$ such that
$$z_2(r)<g_k(r)\quad\text{for any } r\in[0,R] \text{ and } k\in[0,k_0].$$
Define
$$\tau:=\sup\big\{k_0>0; z_2(r)<g_k(r) \text{ for } r\in[0,R_0] \text{ and } k\in[0,k_0]\big\},$$
we see that $\tau<1$, $g_\tau(r)\ge z_2(r)$ and there exists $r_0\in[0,R_0]$ such that $g_\tau(r_0)=z_2(r_0)$.

If $r_0=R_0$, then
$$g_\tau(R_0)=\tau^{-mp/(m(p-1)-1)}z_1(\tau R_0)=z_2(R_0).$$
Since $z_1(R_0)=z_2(R_0)$ and g_τ is strictly decreasing with respect to τ, we conclude that $\tau=1$ and this contradicts to the hypothesis; while if $r_0\in(0,R_0)$, we have
$$g_\tau(r_0)=z_2(r_0),\quad g_\tau'(r_0)=z_2'(r_0),\quad g_\tau''(r_0)\ge z_2''(r_0).$$
Since that $\alpha>0$, that is, $p>q>\frac{p}{p+1-m(p-1)}$, we deduce from (2.5) that
$$\begin{aligned}&(|g_\tau'|^{p-2}g_\tau')'(r_0)-(|z_2'|^{p-2}z_2')'(r_0)\\&=(\tau^{\frac{q(p+1-m(p-1))-p}{m(p-1)-1}}-1)|(z_2^{1/m})'|^q\\&<0\end{aligned}$$
which contradicts $g_\tau''(r_0)\ge z_2''(r_0)$.

Thus, $r_0=0$ and $g_\tau(r)>z_2(r)$ for $r\in(0,R_0]$. Then we have
$$g_\tau(0)=z_2(0),$$
$$\lim_{r\to0^+}g_\tau'(r)=\lim_{r\to0^+}z_2'(r)=0,$$
and
$$\lim_{r\to0^+}(|g_\tau'|^{p-2}g_\tau')'(r)=\lim_{r\to0^+}(|z_2'|^{p-2}z_2')'(r)=-\frac{b_2^{1/m}}{N}<0.$$
By continuity there exists $\varepsilon>0$ such that
$$g_\tau(r)>z_2(r)>0\quad\text{and}\quad 0>g_\tau'(r)>z_2'(r)$$
for $r\in(0,\varepsilon)$. Further, we can choose $\varepsilon>0$ small enough such that the following inequalities hold for $r\in(0,\varepsilon)$,
$$(|g_\tau'|^{p-2}g_\tau')'(r)-(|z_2'|^{p-2}z_2')'(r)>0,$$

$$(g_\tau^{1/m})'(r) - (z_2^{1/m})'(r) > 0,$$
$$|(z_2^{1/m})'|^q - \tau^{\frac{q(p+1-m(p-1))-p}{m(p-1)-1}} |(g_\tau^{1/m})'|^q > 0.$$

Thus, we obtain that

$$\begin{aligned}
0 &= \Big((|g_\tau'|^{p-2}g_\tau')' - (|z_2'|^{p-2}z_2')'\Big)(r) + \frac{N-1}{r}\Big(|g_\tau'|^{p-2}g_\tau' - |z_2'|^{p-2}z_2'\Big)(r) \\
&\quad + \beta r\Big((g_\tau^{1/m})' - (z_2^{1/m})'\Big)(r) + \Big((g_\tau^{1/m}) - (z_2^{1/m})\Big)(r) \\
&\quad + \Big(|(z_2^{1/m})'|^q - \tau^{\frac{q(p+1-m(p-1))-p}{m(p-1)-1}} |(g_\tau^{1/m})'|^q\Big)(r) > 0
\end{aligned}$$

for $r \in (0,\varepsilon)$, which is impossible. Summing up, we completed the proof of Lemma 2.2. □

According to Lemma 2.2, we can define three sets for every $b > 0$,

$$\mathcal{A} = \{b > 0; R(b) < \infty \text{ and } z'(R(b)) < 0\},$$
$$\mathcal{B} = \{b > 0; R(b) < \infty \text{ and } z'(R(b)) = 0\},$$
$$\mathcal{C} = \{b > 0; R(b) = \infty \text{ and } z(r) > 0, r \geq 0\}.$$

Obviously, these sets are disjoint and $\mathcal{A} \cup \mathcal{B} \cup \mathcal{C} = (0,\infty)$. From [17, Theorem 1.1], we have the following lemma.

Lemma 2.3. *Assume that $N\beta < 1$, then*

(i) *set $\mathcal{A}$ is nonempty and open;*
(ii) *set $\mathcal{B}$ is nonempty and closed, and the interface relation*
$$\lim_{r\to R(b)^-} \Big(z^{\frac{m(p-1)-1}{m(p-1)}}\Big)'(r;b) = -\frac{m(p-1)-1}{m(p-1)}(\beta R(b))^{1/(p-1)} \tag{2.8}$$
holds if $b \in \mathcal{B}$;
(iii) *set $\mathcal{C}$ is nonempty and open, and $\lim_{r\to\infty} r^{1/\beta} z^{1/m}(r;b) > 0$ if $b \in \mathcal{C}$.*

Remark 2.4. By Lemma 2.3, it is easy to see that the solution $z(\cdot;b)$ of the problem (2.5) satisfies (2.6) if and only if $b \in \mathcal{B}$. That is to say, to obtain the uniqueness of self-similar very singular solution of (1.1), it is suffice to show that the set $\mathcal{B}$ consists only one element.

3. Proof of the Theorem 1.1

We need an auxiliary lemma. Let $z(\cdot;b)$ be a solution of (2.5) satisfying $b \in \mathcal{B}$, then $R(b) < \infty$ and (2.6) holds. Denote $\xi_0 = R(b)$ and define

$$U(x,t) = k^{1/m}(\frac{\alpha}{t})^\alpha z^{1/m}(\xi),$$

where $\xi = k^{-\gamma}|x|(\frac{\alpha}{t})^{\alpha\beta}$ and $\gamma = \frac{m(p-1)-1}{mp}$, then

$$\operatorname{supp} U = \big\{(x,t) \in \mathbb{R}^N \times (0,\infty); |x| \leq \xi_0 k^\gamma (\frac{\alpha}{t})^{-\alpha\beta}\big\}.$$

Lemma 3.1. *For $t > 0$ fixed and $\delta > 0$ small enough there exists $\theta = \theta(\delta) \in (0,1)$ such that $U(x,t) < U(x,t+\delta)$ for*

$$\theta\xi_0 \leq k^{-\gamma}|x|(\frac{\alpha}{t})^{\alpha\beta} \leq \xi_0.$$

Moreover, we have

$$\lim_{\delta\to 0}\theta(\delta)=\theta_0\in(0,1).$$

Proof. It suffices to prove the existence of $\xi_1\in(0,\xi_0)$ such that

$$\Big(\frac{\alpha}{t}\Big)^{\alpha}z^{1/m}(\xi)<\Big(\frac{\alpha}{t+\delta}\Big)^{\alpha}z^{1/m}\Big(\xi\big(1+\frac{\delta}{t}\big)^{-\alpha\beta}\Big),\quad \xi_1\le\xi\le\xi_0. \tag{3.1}$$

That is,

$$z^{\lambda}(\xi)<\big(1+\frac{\delta}{t}\big)^{-\alpha m\lambda}z^{\lambda}\Big(\xi\big(1+\frac{\delta}{t}\big)^{-\alpha\beta}\Big),\quad \xi_1\le\xi\le\xi_0,$$

where $\lambda=\frac{m(p-1)-1}{m(p-1)}$. Denote $\varepsilon=\delta/t$, then we need prove that there exists the smallest $\xi_1\le\xi_0$ such that

$$z^{\lambda}(\xi)<(1+\varepsilon)^{-\alpha m\lambda}z^{\lambda}\big(\xi(1+\varepsilon)^{-\alpha\beta}\big) \tag{3.2}$$

holds on $[\xi_1,\xi_0]$. Note that

$$\begin{aligned}&(1+\varepsilon)^{-\alpha m\lambda}z^{\lambda}(\xi(1+\varepsilon)^{-\alpha\beta})\\&=z^{\lambda}(\xi)-\alpha\lambda m\varepsilon z^{\lambda}(\xi)-\alpha\beta\varepsilon\xi(z^{\lambda})'(\xi)+O(\varepsilon^2),\end{aligned}$$

so (3.2) reads

$$z^{\lambda}(\xi)<-\frac{\beta\xi}{m\lambda}(z^{\lambda})'(\xi)+O(\varepsilon). \tag{3.3}$$

Recalling (2.6), the set of $\eta\in(0,\xi_0)$ such that

$$z^{\lambda}(\eta)=-\frac{\beta\xi}{m\lambda}(z^{\lambda})'(\eta)$$

is not empty. Let $\tilde\xi$ be the least upper bound of this set, then $0<\tilde\xi<\xi_0$ and

$$z^{\lambda}(\xi)=-\frac{\beta\xi}{m\lambda}(z^{\lambda})'(\xi)$$

for $\tilde\xi<\xi<\xi_0$. For $\varepsilon>0$ small enough we can deduce the existence of $\xi_1\in(\tilde\xi,\xi_0)$ such that (3.3) hold on $[\xi_1,\xi_0]$. Denote $\theta=\theta(\delta)=\xi_1/\xi_0$ and $\theta_0=\tilde\xi/\xi_0$, it is obvious that

$$\lim_{\delta\to 0}\xi_1=\tilde\xi\quad\text{and}\quad\lim_{\delta\to 0}\theta(\delta)=\theta_0\in(0,1).$$

The proof is complete. □

Now we give the proof of the main result.

Proof of Theorem 1.1. By Remark 2.4, it is suffice to show that the set $\mathcal{B}$ consists only one element. We give the proof by contradiction. Without loss of generality, assume that z and Z are two solutions of (2.5) satisfying $z(0),Z(0)\in\mathcal{B}$ and $z(0)<Z(0)$. Denote

$$R_1:=\inf\{r\ge 0:z(r)=0\},\quad R_2:=\inf\{r\ge 0:Z(r)=0\}.$$

By Lemma 2.2, we obtain $R_1<R_2$ and $z(r)<Z(r)$ for $r\in[0,R_1]$. We define

$$z_k(r)=kz(k^{-\gamma}r),$$

where $\gamma=\frac{m(p-1)-1}{mp}$. Then z_k will be larger than Z on $[0,R_2]$ for sufficiently large k. We now define

$$\tau=\inf\left\{k\ge 1;z_k(r)\ge Z(r),r\in[0,R_2]\right\}.$$

Obviously, if $\tau \le 1$, then $z(r) = z_1(r) \ge Z(r)$ for $r \in [0, R_2]$, which contradicts the hypothesis. Thus, we suppose that $\tau > 1$ in the following proof. By the definition of τ, $z_\tau(r)$ must touch $Z(r)$ at $r_0 \in [0, R_2]$ from the above, so we divide the next proof into two cases: $r_0 \in [0, R_2)$ and $r_0 = R_2$.

Case (i). If $z_\tau(r)$ touch $Z(r)$ at $r_0 \in [0, R_2)$, by the similar proof to that of Proposition 2.2, we will derive a contradiction, so $z_\tau(r)$ can not touch $Z(r)$ at $r_0 \in [0, R_2)$.

Case (ii). We firstly define the functions u, U_τ corresponding to Z and z_τ by

$$u(x,t) := (\frac{\alpha}{t})^\alpha Z^{1/m}(r),$$
$$U_\tau(x,t) := (\frac{\alpha}{t})^\alpha z_\tau^{1/m}(r) = \tau^{1/m}(\frac{\alpha}{t})^\alpha z^{1/m}(\tau^{-\gamma} r).$$

Then u is a solution of (1.1) and U_τ is a supersolution. Indeed, a straightforward computation shows that

$$\frac{\partial U_\tau}{\partial t} - \operatorname{div}(|\nabla U_\tau^m|^{p-2}\nabla U_\tau^m) + |\nabla U_\tau|^q = (1 - \tau^{\frac{q(m(p-1)-p-1)+p}{mp}}|\nabla U_\tau|^q) \ge 0.$$

By Lemma 3.1, for sufficiently small $\delta > 0$, there exist $\theta_0, \theta(\delta) \in (0,1)$ such that

$$U_\tau(x,1) < U_\tau(x, 1+\delta)$$

for $\theta(\delta)R_2\tau^\gamma \le |x| < R_2\tau^\gamma(1+\delta)^\beta$ and

$$\lim_{\delta\to 0}\theta(\delta) = \theta_0.$$

Combining this with $z_\tau(r) \ge Z(r)$ for $r \in [0, R_2]$, we obtain that

$$u(x,1) < U_\tau(x, 1+\delta) \tag{3.4}$$

for $\theta(\delta)R_2\tau^\gamma \le |x| < R_2\tau^\gamma(1+\delta)^\beta$.

On the other hand, as previously proved, $z_\tau(r)$ can not touch $Z(r)$ at $r_0 \in [0, R_2)$, which implies for any fixed $\varepsilon_1 > 0$, there exists $\kappa \in (0,1)$ such that

$$Z(|x|) < \kappa z_\tau(|x|), \quad |x| < (1-\varepsilon_1)R_2\tau^\gamma;$$

that is,

$$u(x,1) < \kappa U_\tau(x,1), \quad |x| < (1-\varepsilon_1)R_2\tau^\gamma. \tag{3.5}$$

Now we choose sufficiently small $\varepsilon_1 > 0$ and $\delta_0 > 0$ such that

$$\theta(\delta) < 1 - \varepsilon_1$$

for $\delta \in (0, \delta_0)$ and

$$\theta_0 < 1 - \varepsilon_1.$$

So we obtain that

$$\theta(\delta)R_2\tau^\gamma < (1-\varepsilon_1)R_2\tau^\gamma, \quad \delta \in [0, \delta_0). \tag{3.6}$$

By continuity of U_τ, there exists $\delta_1 \in (0, \delta_0)$ such that

$$\kappa U_\tau(x,1) \le U_\tau(x, 1+\delta)$$

for $\delta \in (0, \delta_1)$ and $|x| < (1-\varepsilon_1)R_2\tau^\gamma$. Combining with (3.5), we have

$$u(x,1) < U_\tau(x, 1+\delta) \tag{3.7}$$

for $\delta \in (0, \delta_1)$ and $|x| < (1-\varepsilon_1)R_2\tau^\gamma$. Thus, combining (3.4), (3.6) and (3.7), for any $x \in \mathbb{R}^N$ we have

$$u(x,1) < U_\tau(x, 1+\delta) = \tau^{1/m}(\frac{\alpha}{t+\delta})^\alpha z^{1/m}(\tau^{-\gamma}|x|(\frac{\alpha}{t+\delta})^{\alpha\beta})$$

Furthermore, from the continuity with respect to τ, there exists $\tau_1 \in (0, \tau)$ such that

$$u(x,1) \le U_{\tau_1}(x, 1+\delta)$$

for $x \in \mathbb{R}^N$. By comparison we obtain $u(x,t) \le U_{\tau_1}(x, t+\delta)$; that is,

$$(\frac{\alpha}{t})^\alpha Z^{1/m}\Big(|x|(\frac{\alpha}{t})^{\alpha\beta}\Big) \le \tau_1^{1/m}\big(\frac{\alpha}{t+\delta}\big)^\alpha z^{1/m} Big(\tau_1^{-\gamma}|x|(\frac{\alpha}{t+\delta})^{\alpha\beta}\Big) \tag{3.8}$$

for any $(x,t) \in \mathbb{R}^N \times [1, \infty)$. Rewriting (3.8) in the form

$$Z^{1/m}(r) \le \tau_1^{1/m}\big(\frac{t}{t+\delta}\big)^\alpha z^{1/m}\Big(\tau_1^{-\gamma} r\big(\frac{t}{t+\delta}\big)^{\alpha\beta}\Big)$$

and letting $t \to \infty$, we have

$$Z(r) \le \tau_1 z(\tau_1^{-\gamma} r) = z_{\tau_1}(r)$$

which contradicts the fact that τ is the smallest constant with that property. Thus, z_τ does not reach Z at $r_0 = R_0$ and we may conclude that $\tau \le 1$ but it is impossible. Summing up, we completed the proof of Theorem 1.1. □

Acknowledgments. This work is Supported by NNSFC (No. 11071099), and by the Scientific Research Foundation of Graduate School of South China Normal University.

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